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A NOVEL DUAL APPROACH TO NONLINEAR SEMIGROUPS OF LIPSCHITZ OPERATORS

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Abstract. Lipschitzian semigroup refers to a one-parameter semigroup of Lipschitz operators that is strongly continuous in the parameter. It contains C_0-semigroup, nonlinear semigroup of contractions and uniformly k-Lipschitzian semigroup as special cases. In this paper, through developing a series of Lipschitz dual notions, we establish an analysis approach to Lipschitzian semigroup. It is mainly proved that a (nonlinear) Lipschitzian semigroup can be isometrically embedded into a certain C_0-semigroup. As application results, two representation formulas of Lipschitzian semigroup are established, and many asymptotic properties of C_0-semigroup are generalized to Lipschitzian semigroup.

1. Introduction

Let X and Y be Banach spaces over the same coefficient field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), and let $C \subset X$ and $D \subset Y$ be their subsets. A mapping T from C into D is called *Lipschitz operator* if there exists a real constant $M > 0$ such that

$$\| Tx - Ty \| \leq M \| x - y \|, \forall x, y \in C, \tag{1.1}$$

where the constant M is commonly referred to as a Lipschitz constant of T. Clearly, if the mapping T is a constant (that is, there exists a vector $y_0 \in D$ such that $Tx = y_0$ for all $x \in C$), then T is a Lipschitz operator.

Let $Lip(C, D)$ denote the space of Lipschitz operators from C into D, and for every $T \in Lip(C, D)$ let $L(T)$ denote the minimum Lipschitz constant of T, i.e.,

$$L(T) = \sup_{x,y \in C, x \neq y} \frac{\| Tx - Ty \|}{\| x - y \|}. \tag{1.2}$$

Then, it can be shown that the nonnegative functional $L(\cdot)$ is a seminorm of $Lip(C, D)$ and hence $(Lip(C, D), L(\cdot))$ is a seminormed linear space. From the definition (1.2) it is seen that $L(T) = 0$ if and only if T is a constant. Hence, if Q stands for the subspace of constant operators, then the quotient space $Lip(C, D)/Q$ is a Banach space. Moreover, let $x_0 \in C$ and let

$$Lip_{x_0}(C, D) = \{T \in Lip(C, D) : Tx_0 = 0\},$$

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then, it can be shown that $Lip_{x_0}(C, D)$ is isometrically isomorphic to $Lip(C, D)/Q$ and hence is a Banach space [12].

Obviously, a bounded linear operator T from X into Y belongs to $Lip_0(X, Y)$ and $L(T)$ is equal to its operator norm $\|T\|$. Moreover, it is easy to show that for any pair of $T \in Lip(C, D_1)$ and $S \in Lip(D_1, D)$, the inequality $L(ST) \leq L(T)L(S)$ holds. Hence, the seminorm $L(\cdot)$ can be viewed as a nonlinear generalization of operator norm of bounded linear operator.

Definition 1. A one-parameter family $\{T_t\}_{t\geq 0}$ of Lipschitz operators from C into itself is called a Lipschitzian semigroup on C, if it possesses the following two properties: (i) $T_0 = I$ (the identity operator of C), $T_tT_s = T_{t+s}$ for all $t, s \geq 0$; and (ii) for every $x \in C$ the mapping $t \mapsto T_tx$ is continuous at $t = 0$.

Furthermore, if $\lim_{t\to 0^+} L(T_t - I) = 0$, then the Lipschitzian semigroup $\{T_t\}_{t\geq 0}$ is said to be uniformly continuous.

Remark 1. (i) From Definition 1, if $\{T_t\}_{t\geq 0}$ is a Lipschitzian semigroup, then for all $x \in C$ the mapping $t \mapsto T_tx$ is continuous in the domain interval $[0, +\infty)$.

(ii) C_0-semigroup [5, 11], semigroups of contractions [2, 10], semigroups of ω-type [10], and uniformly k-Lipschitzian semigroups [7] are all special examples of Lipschitzian semigroups. So, Lipschitzian semigroup is a more general type of operator semigroup.

Definition 2. Let $\{T_t\}_{t\geq 0}$ be a Lipschitzian semigroup on C, and let

$$D(A) = \big\{x \in C : \text{ the limit } \lim_{t\to 0^+} t^{-1}(T_tx - x) \text{ exists in } X\big\}.$$

If $D(A)$ is nonempty, then we say that $\{T_t\}_{t\geq 0}$ possesses a generator A, which is defined by

$$A : D(A) \subset C \to X,\ Ax = \lim_{t\to 0^+} \frac{T_tx - x}{t}.$$

It should be noted that, unlike semigroups of linear operators, a nonlinear Lipschitzian semigroup may have no generator. In fact, an example of a semigroup of contractions without generator was given in [2].

Operator semigroup is an important research subject in operator theory and its application fields, since the solution propagator of abstract system $x'(t) = Ax(t)$ often corresponds to a type of operator semigroup. The semigroup is linear if A is linear, and it is nonlinear if A is not linear. The theory of linear semigroup has now secured its position as important area in the field of functional analysis since the foundation work of Hille and Yosida [8]. However, to the author's knowledge, the systemic research on nonlinear semigroups has been mainly focused on semigroup of contractions [2, 10]. Up to now, there has not been a systemic analysis work on general Lipschitzian semigroup. Recently, Dorroh and Neuberger [6] studied the semigroup of continuous transformations and really suggested a heuristic way to analyze nonlinear semigroup in terms of Lie generators.

In this paper our purpose is to develop a new approach to Lipschitzian semigroup. This approach is based on the Lipschitz dual ideas suggested in [12]. In Section 2 we introduce a dual notion of Lipschitz operator, named Lipschitz dual operator, and then prove several important properties of Lipschitz operator and its Lipschitz dual operator. Based on these, in Section 3 we consider the semigroup consisting of the Lipschitz dual operators of Lipschitzian semigroup. It is shown that under a proper condition this semigroup is a C_0^*-semigroup [3] and hence corresponds to a certain C_0-semigroup. This C_0-semigroup can be viewed as a linear extension of Lipschitzian semigroup under an isometrical mapping. As application results, two representation formulas of Lipschitzian semigroup are established in Section 4 and many asymptotic properties of Lipschitzian semigroup are proved in Section 5, respectively.

The following notation will be used in this paper:

I :	The identity operator for any space.
$\mathcal{B}(E)$:	The space of bounded linear operators of Banach space E;
E^* :	The dual space of Banach space E;
$y_\sigma \stackrel{w^*}{\to} y$:	The convergence in the weak*-topology;
T^* :	The dual operator of bounded linear operator T.

2. Dual notion of Lipschitz operator

In this section we define a dual notion of Lipschitz operator, named Lipschitz dual operator, and then prove some properties of the space of Lipschitz operators on C. The set C is henceforth assumed to be closed unless otherwise stated.

If $T \in Lip(C,C)$, then it is easy to verify that for all $f \in Lip(C,\mathbb{K})$ the functional g, defined by $g(x) = f(Tx)$ for all $x \in C$, belongs to $Lip(C,\mathbb{K})$. So, for every $T \in Lip(C,C)$ we can define an operator from $Lip(C,\mathbb{K})$ into itself as follows.

Definition 3. Let $T \in Lip(C,C)$. Then, the operator $T^{l*} : Lip(C,\mathbb{K}) \to Lip(C,\mathbb{K})$ defined by

$$(T^{l*}f)(x) = f(Tx), \forall f \in Lip(C,\mathbb{K}), x \in C. \tag{2.1}$$

is called the Lipschitz dual operator of T.

It follows from the definition (2.1) that when $C = X$ and T is linear, the restriction of T^{l*} to X^* is just the dual operator T^* of T. This shows that Lipschitz dual operator is a nonlinear extension of the dual operator of bounded linear operator. Actually, Lipschitz dual operator inherits many properties of dual operator, such as these stated in the following proposition.

Proposition 1 ([12])**.** *Let $T \in Lip(C,C)$. Then, the Lipschitz dual operator T^{l*} of T is linear and bounded, and the operator norm $\| T^{l*} \|$ of T^{l*} is equal to $L(T)$.*

Proof. The linearity of T^{l*} is clear. Since we have that, for all $f \in Lip(C,\mathbb{K})$,

$$\begin{aligned} L(T^{l*}f) &= \sup_{x,y\in X, x\neq y} \frac{|(T^{l*}f)(x) - (T^{l*}f)(y)|}{\|x-y\|} \\ &= \sup_{x,y\in X, x\neq y} \frac{|f(Tx) - f(Ty)|}{\|x-y\|} \\ &\leq \sup_{x,y\in X, x\neq y} \frac{L(f)\cdot\|Tx-Ty\|}{\|x-y\|} \\ &\leq L(T)\cdot L(f), \end{aligned}$$

we know that T^{l*} is a linear and bounded operator of $Lip(C,\mathbb{K})$ with $\|T^{l*}\| \leq L(T)$. Next, we prove the converse inequality $L(T) \leq \|T^{l*}\|$. For each $f \in X^*$ let f_C

denote the restriction of f to the subset C. Then, we know that f_C belongs to $Lip(C,\mathbb{K})$ and $L(f_C) \leq \|f\|$. With this in hand, we can show that, $\forall x, y \in C$,

$$\begin{aligned} \|Tx - Ty\| &= \sup_{f\in X^*,\, \|f\|\leq 1} |f(Tx) - f(Ty)| \\ &= \sup_{f\in X^*,\, \|f\|\leq 1} |f_C(Tx) - f_C(Ty)| \\ &\leq \sup_{f\in X^*,\, L(f_C)\leq 1} |(T^{l*}f_C)(x) - (T^{l*}f_C)(y)| \\ &\leq \sup_{f\in Lip(C,\mathbb{K}),\, L(f)\leq 1} L(T^{l*}f)\|x-y\| \\ &\leq \|T^{l*}\|\,\|x-y\|. \end{aligned}$$

Which indicates that $L(T) \leq \|T^{l*}\|$. Therefore, the proof is completed. □

Let $x_0 \in C$. It is easy to verify that every $x \in C$ corresponds to a bounded linear functional, v_x, of $Lip_{x_0}(C,\mathbb{K})$ in such a manner that $v_x(f) = f(x)$ for all $f \in Lip_{x_0}(C,\mathbb{K})$. Let j_{x_0} denote the mapping $x \mapsto v_x$ from C into the dual space $Lip_{x_0}(C,\mathbb{K})^*$ of $Lip_{x_0}(C,\mathbb{K})$. Then, it follows that $j_{x_0}(x_0) = 0$ and

$$\big(j_{x_0}(x)\big)(f) = v_x(f) = f(x),\ \forall x \in C, f \in Lip_{x_0}(C,\mathbb{K}). \tag{2.2}$$

It can be further shown that j_{x_0} is an isometry, that is, $\|j_{x_0}(x) - j_{x_0}(y)\| = \|x-y\|$ for all $x, y \in C$. Therefore, C is isometrically embedded into $Lip_{x_0}(C,\mathbb{K})^*$ with j_{x_0} being the isometrical mapping (see also [1]).

Let $G_{x_0} = \overline{span}\{j_{x_0}(x) : x \in C\}$, i.e., let G_{x_0} be the closure of the subspace of $Lip_{x_0}(C,\mathbb{K})^*$ spanned linearly by the set $\{j_{x_0}(x) : x \in C\}$. Then, we can prove the following important result.

Proposition 2. *Let $x_0 \in C$. Then, the Banach space $Lip_{x_0}(C,\mathbb{K})$ is isometrically isomorphic to the dual space $G^*_{x_0}$ of G_{x_0}.*

Proof. Since G_{x_0} is a subspace of the dual space $Lip_{x_0}(C,\mathbb{K})^*$ of $Lip_{x_0}(C,\mathbb{K})$, then by [9, Theorem 4.6.2] $G^*_{x_0}$ is isometrically isomorphic to the quotient space $Lip_{x_0}(C,\mathbb{K})^{**}/G^{\perp}_{x_0}$, where $Lip_{x_0}(C,\mathbb{K})^{**}$ is the bidual dual space of $Lip_{x_0}(C,\mathbb{K})$ and $G^{\perp}_{x_0} \subset Lip_{x_0}(C,\mathbb{K})^{**}$ is the annihilator of G_{x_0}. Hence, to complete the proof we only show that $Lip_{x_0}(C,\mathbb{K})$ is isometrically isomorphic to $Lip_{x_0}(C,\mathbb{K})^{**}/G^{\perp}_{x_0}$.

For each $A \in Lip_{x_0}(C,\mathbb{K})^{**}$ let f_A denote the functional on C defined by

$$f_A(x) = A(j_{x_0}(x)),\ \forall x \in C. \tag{2.3}$$

Obviously, $f_A(x_0) = 0$ because $j_{x_0}(x_0) = 0$. And, by the isometry property of j_{x_0} we have that, for all $x, y \in C$,

$$\begin{aligned} |f_A(x) - f_A(y)| &= |A(j_{x_0}(x)) - A(j_{x_0}(y))| \\ &\leq \|A\| \cdot \|j_{x_0}(x) - j_{x_0}(y)\| \\ &\leq \|A\| \cdot \|x-y\|. \end{aligned} \tag{2.4}$$

Hence, f_A belongs to $Lip_{x_0}(C,\mathbb{K})$ with $L(f_A) \leq \|A\|$.

Let Γ denote the mapping $A \mapsto f_A$ from $Lip_{x_0}(C,\mathbb{K})^{**}$ into $Lip_{x_0}(C,\mathbb{K})$. Then, the formula (2.3) and the inequality (2.4) imply that Γ is linear and bounded with $\|\Gamma\| \leq 1$. Moreover, from (2.3) it is seen that $\Gamma(A) = 0$ for all $A \in G^{\perp}_{x_0}$. Hence, if $\widetilde{A}, \widetilde{B} \in Lip(C,\mathbb{K})^{**}/G^{\perp}_{x_0}$ satisfy that $\widetilde{A} = \widetilde{B}$, then, $\Gamma(B) = \Gamma(A)$ for all $B \in \widetilde{B}$ and $A \in \widetilde{A}$.

Let $\Phi : Lip_{x_0}(C,\mathbb{K})^{**}/G_{x_0}^{\perp} \to Lip_{x_0}(C,\mathbb{K})$ be defined as

$$\Phi(\widetilde{A}) = \Gamma(A), \forall \widetilde{A} \in Lip_{x_0}(C,\mathbb{K})^{**}/G_{x_0}^{\perp}, \tag{2.5}$$

where $\widetilde{A}$ represents the equivalent class with respect to $A \in Lip_{x_0}(C,\mathbb{K})^{**}$. Then, it is seen that if Γ is surjective and $G_{x_0}^{\perp}$ is identical with the null space of Γ, then Φ is an isomorphism. Noticing this, we divide the remainder of the proof into three steps.

Step 1: Prove that Γ is a surjection.

Let J be the natural embedding mapping from $Lip_{x_0}(C,\mathbb{K})$ into its bidual space $Lip_{x_0}(C,\mathbb{K})^{**}$. Then, it follows from formulas (2.2) and (2.3) that, for all $f \in Lip_{x_0}(C,\mathbb{K})$,

$$\Gamma(J(f))(x) = \big(J(f)\big)(j_{x_0}(x)) = \big(j_{x_0}(x)\big)(f) = f(x),\ \forall x \in C, \tag{2.6}$$

that is, $\Gamma(J(f)) = f$. Hence, Γ is surjective because $f \in Lip_{x_0}(C,\mathbb{K})$ is arbitrary.

Step 2: Prove that the null space $N(\Gamma)$ of Γ is identified with $G_{x_0}^{\perp}$.

Let $A \in G_{x_0}^{\perp}$. Then, $A(m) = 0$ for all $m \in G_{x_0}$ and particularly $A(j_{x_0}(x)) = 0$ for all $x \in C$. Hence, by formula (2.3) we obtain that

$$\Gamma(A)(x) = f_A(x) = A(j_{x_0}(x)) = 0,\ \forall x \in C.$$

That is, $\Gamma(A) = 0$. Hence, $G_{x_0}^{\perp} \subset N(\Gamma)$ because of the arbitrariness of $A \in G_{x_0}^{\perp}$.

Conversely, if $A \in N(\Gamma)$, then $A(j_{x_0}(x)) = f_A(x) = \Gamma(A)(x) = 0$ for all $x \in C$. That is, $A(m) = 0$ for all $m \in \{j_{x_0}(x) :\ x \in C\}$. Now that $G_{x_0} = \overline{span}\{j_{x_0}(x) : x \in C\}$, we have that $A(m) = 0$ for all $m \in G_{x_0}$. Hence, $A \in G_{x_0}^{\perp}$ and then $N(\Gamma) \subseteq G_{x_0}^{\perp}$ since $A \in N(\Gamma)$ is arbitrary. Therefore, $N(\Gamma) = G_{x_0}^{\perp}$, as claimed.

Step 3: Prove that Φ is an isometry.

Let $\widetilde{A} \in Lip_{x_0}(C,\mathbb{K})^{**}/G_{x_0}^{\perp}$, which is induced by $A \in Lip_{x_0}(C,\mathbb{K})^{**}$. Since $G_{x_0}^{\perp} = N(\Gamma)$, we have that $\Gamma(A) = \Gamma(B)$ for all $B \in \widetilde{A}$. Hence, noticing that $\|\Gamma\| \le 1$, we obtain that

$$\|\Phi(\widetilde{A})\| = \|\Gamma(A)\| = \inf_{B\in\widetilde{A}} \|\Gamma(B)\| \le \inf_{B\in\widetilde{A}} \|B\| = \|\widetilde{A}\|. \tag{2.7}$$

On the other hand, substituting $f = \Gamma(A)$ into equality (2.6) yields that

$$\Gamma(J(\Gamma(A))) = \Gamma(A). \tag{2.8}$$

Since $N(\Gamma) = G_{x_0}^{\perp}$, it follows from (2.8) that $\widetilde{J(\Gamma(A))} = \widetilde{A}$. Thus, by using the isometry property of J we obtain that

$$\|\widetilde{A}\| = \|\widetilde{J(\Gamma(A))}\| \le \|J(\Gamma(A))\| = \|\Gamma(A)\| = \|\Phi(\widetilde{A})\|. \tag{2.9}$$

This, together with inequality (2.7), implies that $\|\Phi(\widetilde{A})\| = \|\widetilde{A}\|$. Therefore, Φ is an isometry from $Lip_{x_0}(C,\mathbb{K})^{**}/G_{x_0}^{\perp}$ onto $Lip_{x_0}(C,\mathbb{R})$. □

Remark 2. Since $G_{x_0} \subset Lip_{x_0}(C,\mathbb{K})^{*}$, it follows from Proposition 2 that the weak*-topology of $Lip_{x_0}(C,\mathbb{K})$ is really induced by the bilinear form $\langle\cdot,\cdot\rangle_1$ defined by

$$\langle m, f\rangle_1 = m(f),\ \forall f \in Lip_{x_0}(C,\mathbb{K}), m \in G_{x_0}, \tag{2.10}$$

where $m(f)$ represents the value of the functional m at f.

Let Q be the subspace of constant functionals on C, and let $Lip(C,\mathbb{K})/Q$ be the quotient space of $Lip(C,\mathbb{K})$ with respect to Q. In the following, by using Proposition 2 we will show that $Lip(C,\mathbb{K})/Q$ is a dual space and hence can be equipped with a weak*-topology. To this end, we first prove the lemma below.

Lemma 1. *Let q be the quotient mapping from $Lip(C,\mathbb{K})$ onto $Lip(C,\mathbb{K})/Q$. Then, (i) for all $f \in Lip(C,\mathbb{K})$ it holds that $\|q(f)\| = L(f)$ and; (ii) for all $T \in Lip(C,C)$, $q(T^{l*}f) = q(T^{l*}g)$ whenever $q(f) = q(g)$.*

Proof. (i) Let $f \in Lip(C,\mathbb{K})$. Then, it follows from definition (1.2) that $L(f) = L(g)$ for all $g \in Lip(C,\mathbb{K})$ satisfying $g - f \in Q$. Since $f - g \in Q$ for all $g \in q(f)$, we have that $\|q(f)\| = \inf_{g\in q(f)} L(g) = L(f)$.

(ii) Let $T \in Lip(C,C)$. If $q(f) = q(g)$, then, $f - g \in Q$. And, since $(T^{l*}f)(x) - (T^{l*}g)(x) = f(Tx) - g(Tx)$ for all $x \in C$, we thus have that $T^{l*}f - T^{l*}g \in Q$, i.e., $q(T^{l*}f) = q(T^{l*}g)$. □

In the following, we will let q denote the quotient mapping from $Lip(C,\mathbb{K})$ onto $Lip(C,\mathbb{K})/Q$, and for every $f \in Lip(C,\mathbb{K})$ let $f_{x_0} \in Lip_{x_0}(C,\mathbb{K})$ be defined by $f_{x_0}(x) = f(x) - f(x_0)$ for all $x \in C$.

Corollary 1. *Let $x_0 \in C$. Then, $Lip(C,\mathbb{K})/Q$ is isometrically isomorphic to the dual space $G^*_{x_0}$ of G_{x_0}, and the weak*-topology of $Lip(C,\mathbb{K})/Q$ is characterized by G_{x_0} as follows: Let $\{f_\sigma\}$ be a net of $Lip(C,\mathbb{K})$, $f \in Lip(C,\mathbb{K})$, then, $q(f_\sigma) \xrightarrow{w^*} q(f)$ if and only if*

$$\langle m, (f_\sigma)_{x_0}\rangle_1 \to \langle m, f_{x_0}\rangle_1,\ \forall m \in G_{x_0} \tag{2.11}$$

where $\langle\cdot,\cdot\rangle_1$ is defined as in Remark 2. Particularly, if the net $\{f_\sigma\}$ is bounded, then $q(f_\sigma) \xrightarrow{w^*} q(f)$ if and only if $(f_\sigma)_{x_0}(x) \to f_{x_0}(x)$ for all $x \in C$.

Proof. Let $x_0 \in C$, and let $\Theta : Lip(C,\mathbb{K})/Q \to Lip_{x_0}(C,\mathbb{K})$ be defined by

$$\big(\Theta(q(f))\big)(x) = f(x) - f(x_0),\ \forall f \in Lip(C,\mathbb{K}),\ x \in C. \tag{2.12}$$

Then, it is not hard to show that Θ is an isometrical isomorphic mapping from $Lip(C,\mathbb{K})/Q$ onto $Lip_{x_0}(C,\mathbb{K})$. Hence, by Proposition 2 we know that $Lip(C,\mathbb{K})/Q$ is isometrically isomorphic to $G^*_{x_0}$.

Let $f_\sigma, f \in Lip(C,\mathbb{K})$. Since Θ is an isomorphic mapping, the weak* convergence of the net $\{q(f_\sigma)\}$ in $Lip(C,\mathbb{K})/Q$ is really equivalent to the weak* convergence of the net $\{\Theta(q(f_\sigma))\}$ in $Lip_{x_0}(C,\mathbb{K})$. Hence, by Proposition 2 and Remark 2, $q(f_\sigma) \xrightarrow{w^*} q(f)$ iff

$$\langle m, \Theta(q(f_\sigma))\rangle_1 \to \langle m, \Theta(q(f))\rangle_1,\ \forall m \in G_{x_0}. \tag{2.13}$$

From the formula (2.12) it is seen that the relation (2.13) is really equivalent to the relation (2.11). That is, $q(f_\sigma) \xrightarrow{w^*} q(f)$ iff

$$\langle m, (f_\sigma)_{x_0}\rangle_1 \to \langle m, f_{x_0}\rangle_1,\ \forall m \in G_{x_0}. \tag{2.14}$$

If $\{f_\sigma\}$ is bounded, then, noticing that $G_{x_0} = \overline{span}\{j_{x_0}(x) : x \in C\}$, we find that the relation (2.14) is really equivalent to

$$(f_\sigma)_{x_0}(x) = \langle j_{x_0}(x), (f_\sigma)_{x_0}\rangle_1 \to \langle j_{x_0}(x), f_{x_0}\rangle_1 = f_{x_0}(x),\ \forall x \in C. \tag{2.15}$$

That is, $q(f_\sigma) \xrightarrow{w^*} q(f)$ if and only if $(f_\sigma)_{x_0}(x) \to f_{x_0}(x)$ for all $x \in C$. Therefore, the proof is completed. □

Remark 3. It follows from formula (2.11) that the weak*-topology of $Lip(C,\mathbb{K})/Q$ is really induced by the bilinear form $\langle\cdot,\cdot\rangle_2$ defined by

$$\langle q(f), m\rangle_2 = \langle m, f_{x_0}\rangle_1,\ \forall f \in Lip(C,\mathbb{K}), m \in G_{x_0}, \tag{2.16}$$

where $\langle \cdot, \cdot \rangle_1$ is defined as in (2.10). We see that

$$\langle q(f), j_{x_0}(x) \rangle_2 = f(x) - f(x_0), \ \forall f \in Lip(C, \mathbb{K}), x \in C. \tag{2.17}$$

3. Lipschitz dual semigroup of Lipschitzian semigroup

Let $\{T_t\}_{t \geq 0}$ be a Lipschitzian semigroup on C, and for every $t \geq 0$ let D_t denote the Lipschitz dual operator of T_t (i.e., $D_t = T_t^{l*}$). Then, by Proposition 1 we know that each D_t is a bounded linear operator of $Lip(C, \mathbb{K})$. Moreover, it is not hard to verify that $\{D_t\}_{t \geq 0}$ satisfies the semigroup property: $D_0 = I$ and $D_t D_s = D_{t+s}$ for all $t, s \geq 0$. Therefore, the one-parameter class $\{D_t\}_{t \geq 0}$ is a semigroup of bounded linear operators of the seminormed linear space $Lip(C, \mathbb{K})$.

In this section, through analysis on the semigroup $\{D_t\}_{t \geq 0}$, we will derive a C_0-semigroup into which $\{T_t\}_{t \geq 0}$ is isometrically embedded.

Proposition 3. *Let q be the quotient mapping from $Lip(C, \mathbb{K})$ onto $Lip(C, \mathbb{K})/Q$. Define for every $t \geq 0$ the mapping $\widetilde{D_t} : Lip(C, \mathbb{K})/Q \to Lip(C, \mathbb{K})/Q$ as follows:*

$$\widetilde{D_t}(q(f)) = q(D_t f), \ \forall f \in Lip(C, \mathbb{K}). \tag{3.1}$$

Then, the set $\{\widetilde{D_t}\}_{t \geq 0}$ possesses the following properties (a)-(d):

(*a*) *For all $t \geq 0$, $\widetilde{D_t}$ is linear and bounded, and $\|\widetilde{D_t}\| = L(T_t)$.*

(*b*) *$\{\widetilde{D_t}\}_{t \geq 0}$ is an operator semigroup on $Lip(C, \mathbb{K})/Q$, that is, $\widetilde{D_0} = I$ and $\widetilde{D_{t+s}} = \widetilde{D_t}\widetilde{D_s}$ for all $t, s \geq 0$.*

(*c*) *For all $t \geq 0$, $\widetilde{D_t}$ is weak*-weak* continuous, that is, $\widetilde{D_t}(q(f_\sigma)) \xrightarrow{w^*} \widetilde{D_t}(q(f))$ whenever $q(f_\sigma) \xrightarrow{w^*} q(f)$.*

(*d*) *For all $f \in Lip(C, \mathbb{K})$, the mapping $t \mapsto \widetilde{D_t}(q(f))$ is continuous in the weak*-topology of $Lip(C, \mathbb{K})/Q$ in the open interval $(0, \infty)$.*

Proof. By Lemma 1 we see that (3.1) is well defined for all $t \geq 0$, that is, if $q(f) = q(g)$, then $q(D_t f) = q(D_t g)$ for all $t \geq 0$.

(*a*) The linearity of $\widetilde{D_t}$ is clear. By Lemma 1 and Proposition 1 we have that

$$\begin{aligned} \|\widetilde{D_t}\| &= \sup_{\|q(f)\| \leq 1} \|\widetilde{D_t}(q(f))\| = \sup_{\|q(f)\| \leq 1} \|q(D_t f)\| \\ &= \sup_{\|q(f)\| \leq 1} L(D_t f) = \sup_{L(f) \leq 1} L(D_t f) = \|D_t\| \\ &= L(T_t). \end{aligned}$$

Hence, $\widetilde{D_t}$ is bounded and $\|\widetilde{D_t}\| = L(T_t)$.

(*b*) It follows directly from the formula (3.1) that $\widetilde{D_0} = I$. Let $t, s \geq 0$. Since by the semigroup property of $\{D_t\}_{t \geq 0}$ it holds that, $\forall f \in Lip(C, \mathbb{K})$,

$$\widetilde{D_{t+s}}(q(f)) = q(D_{t+s} f) = q(D_t D_s f) = \widetilde{D_t}(q(D_s f)) = \widetilde{D_t}\widetilde{D_s}(q(f)),$$

we have that $\widetilde{D_t}\widetilde{D_s} = \widetilde{D_{t+s}}$. Therefore, $\{\widetilde{D_t}\}_{t \geq 0}$ is a semigroup of $Lip(C, \mathbb{K})/Q$.

(*c*) Let $t > 0$ and let $q(f_\sigma) \xrightarrow{w^*} q(f)$. Then, the net $\{f_\sigma\} \subset Lip(C, \mathbb{K})$ is bounded and by Corollary 1 it holds that $f_\sigma(x) - f_\sigma(x_0) \to f(x) - f(x_0)$ for all $x \in C$. Hence, the net $\{D_t f_\sigma\}$ is bounded, and for all $x \in C$,

$$\begin{aligned} (D_t f_\sigma)(x) - (D_t f_\sigma)(x_0) &= f_\sigma(T_t x) - f_\sigma(T_t x_0) \\ &= f_\sigma(T_t x) - f_\sigma(x_0) - (f_\sigma(T_t x_0) - f_\sigma(x_0)) \\ &\to f(T_t x) - f(T_t x_0) = (D_t f)(x) - (D_t f)(x_0). \end{aligned}$$

Therefore, applying Corollary 1 to the net $\{q(D_t f_\sigma)\}$ we obtain that $q(D_t f_\sigma) \overset{w^*}{\to} q(D_t f)$, that is, $\widetilde{D_t}(q(f_\sigma)) \overset{w^*}{\to} \widetilde{D_t}(q(f))$ according to (3.1).

(*d*) Let $t_0 > 0$ and $f \in Lip(C, \mathbb{K})$. We shall prove that $\widetilde{D_t}(q(f)) \overset{w^*}{\to} \widetilde{D_{t_0}}(q(f))$ as $t \to t_0$ as follows.

Using the semigroup property of $\{D_t\}_{t\geq 0}$, we can easily show that the real function $a(t) = \log \|D_t\|$ is a subadditive in the interval $[0, +\infty)$. So, by [8, Th. 7.4.1] we know that the function $\|D_t\|$ is bounded in any closed subinterval of $(0, +\infty)$. Particularly, the net $\{D_t f : 0.5t_0 \leq t \leq 1.5t_0\}$ is bounded. Since for all $x \in C$,

$$(D_t f)(x) = f(T_t x) \to f(T_{t_0} x) = (D_{t_0} f)(x) \text{ as } t \to t_0,$$

then by Corollary 1 we have that $q(D_t f) \overset{w^*}{\to} q(D_{t_0} f)$ as $t \to t_0$. That is, $\widetilde{D_t}(q(f)) \overset{w^*}{\to} \widetilde{D_{t_0}}(q(f))$ as $t \to t_0$ in terms of (3.1). Therefore, the proof is completed. □

Corollary 2. *Let $x_0 \in C$. Then, there exists a semigroup $\{S_t\}_{t\geq 0}$ of bounded linear operators of Banach space G_{x_0} such that*

(*a*) *$\{\widetilde{D_t}\}_{t\geq 0}$ is the dual semigroup of $\{S_t\}_{t\geq 0}$, i.e., $\widetilde{D_t} = S_t^*$ for all $t \geq 0$;*

(*b*) *$\langle q(f), S_t(j_{x_0}(x))\rangle_2 = f(T_t x) - f(T_t x_0)$ for all $t \geq 0, f \in Lip(C, \mathbb{K})$ and $x \in C$, where $\langle \cdot, \cdot \rangle_2$ is defined as in (2.16); and*

(*c*) *for all $m \in G_{x_0}$ the mapping $t \mapsto S_t m$ is continuous in the strong topology of G_{x_0} in the open interval $(0, +\infty)$.*

Proof. Let $t \geq 0$. Since the operator $\widetilde{D_t}$ is weak*-weak* continuous, then by the famous representation theorem [9] we know that there exists a bounded linear operator S_t on G_{x_0} such that $\widetilde{D_t}$ is the dual operator of S_t, i.e.,

$$\langle q(f), S_t m\rangle_2 = \langle \widetilde{D_t}(q(f)), m\rangle_2, \ \forall f \in Lip(C, \mathbb{K}), m \in G_{x_0}.$$

Particularly, for all $x \in C, f \in Lip(C, \mathbb{K})$ it holds that

$$\begin{aligned} \langle q(f), S_t(j_{x_0}(x))\rangle_2 &= \langle \widetilde{D_t}(q(f)), j_{x_0}(x)\rangle_2 = \langle q(D_t f), j_{x_0}(x)\rangle_2 \\ &= (D_t f)(x) - (D_t f)(x_0) = f(T_t x) - f(T_t x_0). \end{aligned} \tag{3.2}$$

Therefore, the statements (*a*) and (*b*) are derived.

Let $s > 0$. Then, it follows from the last equality (3.2) that, for all $x \in C$,

$$\begin{aligned} & \|S_t(j_{x_0}(x)) - S_s(j_{x_0}(x))\| \\ = & \sup_{\|q(f)\|=1} \left|\langle q(f), S_t(j_{x_0}(x)) - S_s(j_{x_0}(x))\rangle_2\right| \\ \leq & \sup_{L(f)=1} \left(|f(T_t x) - f(T_s x)| + |f(T_t x_0) - f(T_s x_0)|\right) \\ \leq & \|T_t x - T_s x\| + \|T_t x_0 - T_s x_0\|. \end{aligned}$$

Hence, $S_t(j_{x_0}(x)) \to S_s(j_{x_0}(x))$ as $t \to s$ since $\{T_t\}_{t\geq 0}$ is a Lipschitzian semigroup on C. Equivalently, $S_t m \to S_s m$ for all $m \in span\{j_{x_0}(x) : \ x \in C\}$. Now, let $m \in G_{x_0}$. Then, for all $\epsilon > 0$ there exists an $m' \in span\{j_{x_0}(x) : \ x \in C\}$ such that $\|m - m'\| < \frac{\epsilon}{2}$. So, it is deduced that

$$\begin{aligned} & \limsup_{t\to s} \|S_t m - S_s m\| \\ \leq & \limsup_{t\to s} \left(\|S_t m' - S_s m'\| + \|(S_t - S_s)(m - m')\|\right) \\ \leq & \epsilon \cdot \sup_{0.5s\leq t\leq 1.5s} \|S_t\|. \end{aligned}$$

Since $\{S_t\}_{\geq 0}$ is an operator semigroup, by [8, Th. 7.4.1] we know that $t \mapsto \|S_t\|$ is bounded in any compact set of $(0, +\infty)$. Particularly, $t \to \|S_t\|$ is bounded in $[0.5s, 1.5s]$. Therefore, by the arbitrariness of ϵ, we conclude from the last inequality that $S_t m \to S_s m$ as $t \to s$. That is, the mapping $t \mapsto S_t m$ is strongly continuous in $(0, +\infty)$. The statement (c) is proved. □

Recall from [3] that a semigroup $\{W_t\}_{t\geq 0}$ of bounded linear operators on the dual space E^* is called C_0^*-semigroup if it satisfies that for each $t > 0$ the operator W_t is weak*-weak* continuous, and that for all $y \in E^*$ the mapping $t \mapsto W_t y$ is continuous at $t = 0$ in the weak*-topology of E^*. Bratteli and Robinson [3] proved that a C_0^*-semigroup is really a dual semigroup of a certain C_0-semigroup. We see from Proposition 3 and its corollary that $\{\widetilde{D_t}\}_{t\geq 0}$ would be a C_0^*-semigroup of $Lip(C, \mathbb{K})/Q$ and then $\{S_t\}_{t\geq 0}$ obtained in Corollary 2 would be a C_0-semigroup, as long as the weak*-continuity of $\widetilde{D_t}(q(f))$ at $t = 0$ is proved for all $f \in Lip(C, \mathbb{K})$.

Corollary 3. *If* $\limsup_{t\to 0^+} L(T_t) < +\infty$*, then* $\{\widetilde{D_t}\}_{t\geq 0}$ *is a* C_0^**-semigroup on* $Lip(C, \mathbb{K})/Q$*, and* $\{S_t\}_{t\geq 0}$ *obtained in Corollary 2 is a* C_0*-semigroup on* G_{x_0}*.*

Proof. Since $\limsup_{t\to 0^+} L(T_t) < +\infty$, there exists a constant $\delta > 0$ such that the net $\{L(T_t)\}_{0<t<\delta}$ is bounded. Hence, the net $\{D_t\}_{0<t<\delta}$ is bounded because $\|D_t\| = L(T_t)$ for all $t \geq 0$.

Let $f \in Lip(C, \mathbb{K})$. Then, the net $\{D_t f\}_{0<t<\delta} \subset Lip(C, \mathbb{K})$ is bounded and satisfies that, for all $x \in C$,

$$(D_t f)(x) = f(T_t x) \to f(x) = (D_0 f)(x), \text{ as } t \to 0^+.$$

So, by Corollary 1 we know that $\widetilde{D_t}(q(f)) = q(D_t f) \xrightarrow{w^*} q(D_0 f) = \widetilde{D_0}(q(f))$ as $t \to 0^+$. That is, the mapping $t \mapsto \widetilde{D_t}(q(f))$ is continuous at $t = 0$ in the weak*-topology of $Lip(C, \mathbb{K})/C$. Therefore, $\{\widetilde{D_t}\}_{t\geq 0}$ is a C_0^*-semigroup of $Lip(C, \mathbb{K})/Q$, and the corresponding $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on G_{x_0}. □

Remark 4. (i) From (b) of Corollary 2, it follows that, for all $x, y \in C$,

$$\begin{aligned}
\big\|S_t(j_{x_0}(x)) - S_t(j_{x_0}(y))\big\| &= \sup_{\|q(f)\|\leq 1} \big|\langle q(f), S_t\big(j_{x_0}(x) - j_{x_0}(y)\big)\rangle_2\big| \\
&= \sup_{L(f)\leq 1} \big|f(T_t x) - f(T_t y)\big| \\
&= \|T_t x - T_t y\|.
\end{aligned}$$

This shows that a Lipschitzian semigroup $\{T_t\}_{t\geq 0}$ satisfying the condition given in Corollary 3 can be isometrically embedded into a C_0-semigroup.

(ii) It can be shown that $\liminf_{t\to 0^+} L(T_t) < +\infty$ if and only if $\{T_t\}_{t\geq 0}$ is exponentially bounded. Here, the Lipschitzian semigroup $\{T_t\}_{t\geq 0}$ is said to be exponentially bounded if there are two real constants w and M with $M \geq 1$ such that $L(T_t) \leq M\, e^{wt}$ for all $t \geq 0$.

4. Application I: Representation of Lipschitzian semigroup

In this section we apply the Lipschitz dual approach developed in previous sections to establish two representation formulas of Lipschitzian semigroups.

Let $x_0 \in C$ and $\{T_t\}_{t\geq 0}$ be a Lipschitzian semigroup on C. If $\{T_t\}_{t\geq 0}$ is exponentially bounded (see (ii) of Remark 4), then, by Corollary 3 and Remark 4 we know that the operator semigroup $\{\widetilde{D_t}\}_{t\geq 0}$ defined as in Proposition 3 is the dual semigroup of the C_0-semigroup $\{S_t\}_{t\geq 0}$ obtained in Corollary 2. In this case, $\{\widetilde{D_t}\}_{t\geq 0}$ has a w^*-generator $\widetilde{D}$ defined by

$$\widetilde{D}: D(\widetilde{D}) \to Lip(C,\mathbb{K})/Q,\ \widetilde{D}(q(f)) = w^*\text{-}\lim_{t\to 0^+} \frac{\widetilde{D_t}(q(f)) - q(f)}{t}, \tag{4.1}$$

where the domain $D(\widetilde{D})$ consists of all those $q(f) \in Lip(C,\mathbb{K})/Q$ such that the weak*-limit $w^*\text{-}\lim_{t\to 0^+} t^{-1}\big(\widetilde{D_t}(q(f)) - q(f)\big)\}$ exists in $Lip(C,\mathbb{K})/Q$, and the prefix w^*- indicates that the limit is subordinate to the weak*-topology. It is known that $D(\widetilde{D})$ is dense in $Lip(C,\mathbb{K})/Q$ in the weak*-topology (see, e.g., [5, 11]).

In this section, $\{T_t\}_{t\geq 0}$ is always assumed to be exponentially bounded (see (ii) of Remark 4). Accordingly, $\{\widetilde{D_t}\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$ are the corresponding C_0^*-semigroup and C_0-semigroup, respectively, and $\widetilde{D}$ is defined as in (4.1).

Lemma 2. *Let $B : D(B) \subset Lip(C,\mathbb{K}) \to Lip(C,\mathbb{K})$ be defined by*

$$(Bf)(x) = \lim_{t\to 0^+} \frac{f(T_t x) - f(x)}{t}, \forall f \in D(B), x \in X \tag{4.2}$$

where the domain $D(B)$ consists of all those $f \in Lip(C,\mathbb{K})$ such that there exists $g \in Lip(C,\mathbb{K})$ satisfying

$$g(x) = \lim_{t\to 0^+} \frac{f(T_t x) - f(x)}{t},\ \forall x \in C.$$

Then, $\{q(f) : f \in D(B)\} \subset D(\widetilde{D})$, and $\widetilde{D}(q(f)) = q(Bf)$ for all $f \in D(B)$.

Furthermore, if $\{T_t\}_{t\geq 0}$ possesses a fixed point (i.e., there exists an $x^* \in C$ such that $T_t x^* = x^*, \forall t \geq 0$), then $D(\widetilde{D}) = \{q(f) : f \in D(B)\}$.

Proof. Let $f \in D(B)$. Then, it follows from (4.2) that, $\forall x \in C, s \geq 0$,

$$(Bf)(T_s x) = \lim_{t\to 0^+} \frac{f(T_{t+s}x) - f(T_s x)}{t} = \frac{\mathrm{d}f(T_s x)}{\mathrm{d}s}.$$

Hence, integrating both sides yields that

$$(D_t f)(x) - f(x) = f(T_t x) - f(x) = \int_0^t (Bf)(T_s x)\mathrm{d}s,\ t > 0.$$

From this equality we derive that the net $\{t^{-1}(D_t f - f)\}_{0<t<1}$ is bounded. It also follows from (4.2) that $t^{-1}\big(D_t f - f\big)(x) = t^{-1}(f(T_t x) - f(x)) \to (Bf)(x)$ as $t \to 0^+$ for all $x \in C$. Therefore, by Corollary 1 we obtain that

$$\frac{\widetilde{D_t}(q(f)) - q(f)}{t} = \frac{q(D_t f) - q(f)}{t} \xrightarrow{w^*} q(Bf),\ t \to 0^+$$

which implies that $q(f) \in D(\widetilde{D})$ and $\widetilde{D}(q(f)) = q(Bf)$.

Assume that $\{T_t\}_{t\geq 0}$ has a fixed point $x^* \in C$. Let $x_0 = x^*$, and let $q(f) \in D(\widetilde{D})$. Then, by Remark 3 we have that, for all $x \in C$,

$$\begin{aligned} \langle \widetilde{D}(q(f)), j_{x_0}(x)\rangle_2 &= \lim_{t\to 0^+} \langle \frac{\widetilde{D_t}(q(f)) - q(f)}{t}, j_{x_0}(x)\rangle_2 \\ &= \lim_{t\to 0^+} \frac{f(T_t x) - f(x) - [f(T_t x_0) - f(x_0)]}{t} \\ &= \lim_{t\to 0^+} \frac{f(T_t x) - f(x)}{t}. \end{aligned} \tag{4.3}$$

Clearly, the functional g, defined by $g(x) = \langle \widetilde{D}(q(f)), j_{x_0}(x)\rangle_2$ for all $x \in C$, belongs to $Lip(C, \mathbb{K})$. Hence, $f \in D(B)$. The proof therefore is completed. □

Remark 5. It is seen that the linear operator B defined as in (4.2) is just the Lie generator of $\{T_t\}_{t\geq 0}$ in terms of [6].

For each $f \in Lip(C, \mathbb{K})$ let $F_e : X \to \mathbb{K}$ be defined by

$$F_e(x) = \sup\{f(y) - L(f)\cdot \|x - y\| :\ y \in C\},\ x \in X. \tag{4.4}$$

Then, similar to [4], it can be shown that $F_e(x) = f(x)$ for all $x \in C$ and $F_e \in Lip(X, \mathbb{K})$ with $L(F_e) = L(f)$. In the following we call F_e the C-Z extension of f.

Lemma 3. *Suppose $\{T_t\}_{t\geq 0}$ possesses a generator A. Let B be defined as in* (4.2)*. If A is densely defined (i.e., $\overline{D(A)} = C$), then, for all $f \in D(B)$,*

$$(Bf)(x) = \lim_{t\to 0^+} \frac{F_e(x + tAx) - f(x)}{t},\ \forall x \in D(A), \tag{4.5}$$

where F_e is the C-Z extension of f. Furthermore, if C has Radon-Nikodym property, then $f \in D(B)$ if and only if there exists a $g \in Lip(C, \mathbb{K})$ such that

$$g(x) = \lim_{t\to 0^+} \frac{F_e(x + tAx) - f(x)}{t}, \forall x \in D(A). \tag{4.6}$$

Proof. Let $f \in D(B)$. Then, it follows from (4.2) that, $\forall x \in D(A)$,

$$(Bf)(x) = \lim_{t\to 0^+} \frac{f(T_t x) - f(x)}{t} = \lim_{t\to 0^+} \frac{F_e(T_t x) - f(x)}{t}. \tag{4.7}$$

It is seen that $x \in D(A)$ iff

$$\lim_{t\to 0^+} \frac{\| T_r x - x - tAx \|}{t} = 0. \tag{4.8}$$

So, combining (4.7) with (4.8) yields that

$$(Bf)(x) = \lim_{t\to 0^+} \frac{F_e(x + tAx) - f(x)}{t},\ \forall x \in D(A).$$

This is just (4.5), as expected.

Using the semigroup property of $\{T_t\}_{t\geq 0}$, we can show that, for all $x \in D(A)$, the mapping $t \mapsto T_t x$ is absolutely continuous in any compact set of $[0, +\infty)$ and then is differentiable almost everywhere in $[0, +\infty)$ if C has Radon-Nikodym property. We see that if $t \mapsto T_t x$ is differentiable at $t = t_0$, then $T_{t_0} x \in D(A)$.

Now, assume C has Radon-Nikodym property, and let $f \in Lip(C, \mathbb{K})$. If there exists $g \in Lip(C, \mathbb{K})$ such that

$$g(x) = \lim_{t\to 0^+} \frac{F_e(x + tAx) - f(x)}{t},\ \forall x \in D(A),$$

then, by combining with (4.8), we get

$$g(x) = \lim_{t \to 0^+} \frac{f(T_t x) - f(x)}{t}, \ \forall x \in D(A). \tag{4.9}$$

Since, for all $x \in D(A)$, $T_t x \in D(A)$ for almost all $t \in [0, +\infty)$, it follows from (4.9) that, $\forall x \in D(A)$,

$$g(T_s x) = \lim_{t \to 0^+} \frac{f(T(t+s)x) - f(T_s x)}{t} = \frac{\mathrm{d} f(T_s x)}{\mathrm{d} s}, \ a.e.$$

Integrating both sides yields

$$f(T_t x) - f(x) = \int_0^t g(T_s x) \mathrm{d}s, \ x \in D(A), \ t > 0.$$

Since $\overline{D(A)} = C$, the above equality really holds for all $x \in C$, that is,

$$f(T_t x) - f(x) = \int_0^t g(T_s x) \mathrm{d}s, \ x \in C, \ t > 0.$$

Thus, differentiating this equality yields

$$g(x) = \lim_{t \to 0^+} \frac{f(T_t x) - f(x)}{t}, \ \forall x \in C.$$

Which implies that $f \in D(B)$ and $Bf = g$. Therefore, the proof is completed. □

Remark 6. It follows from this lemma that if $A \in Lip(C, C)$, then for all $f \in X^*$ the restriction f_C of f to C belongs to $D(B)$ and $Bf_C = f_C \circ A$, where $f_C \circ A$ is the compound mapping of f_C with A (i.e., $\big(f_C \circ A\big)(x) = f_C(Ax) = f(Ax), x \in C$).

Lemma 4. *Let B be defined as in* (4.2). *If x_0 is the fixed point of $\{T_t\}_{t \ge 0}$, then there exists a real constant r such that for all $\lambda > r$, $\lambda I - B$ is invertible from $D(B) \cap Lip_{x_0}(C, \mathbb{K})$ onto $Lip_{x_0}(C, \mathbb{K})$, $(\lambda I - B)^{-1} \in \mathcal{B}(Lip_{x_0}(C, \mathbb{K}))$, and*

$$q((\lambda I - B)^{-1} f) = (\lambda I - \widetilde{D})^{-1}(q(f)), \ \forall f \in Lip_{x_0}(C, \mathbb{K}).$$

Proof. We see from (4.2) that $Bf \in Lip_{x_0}(C, \mathbb{K})$ whenever $f \in D(B) \cap Lip_{x_0}(C, \mathbb{K})$. Since $\{\widetilde{D}_t\}_{t \ge 0}$ is the dual semigroup of C_0-semigroup $\{S_t\}_{t \ge 0}$, the w^*-generator $\widetilde{D}$ of $\{\widetilde{D}_t\}_{t \ge 0}$ is the dual operator of the infinitesimal generator S of $\{S_t\}_{t \ge 0}$, i.e., $S^* = \widetilde{D}$. Applying the famous Feller-Miyadera-Phillips theorem [5] to the C_0-semigroup $\{S_t\}_{t \ge 0}$, we can find a real constant r such that the interval $(r, +\infty)$ is contained in the resolvent set $\rho(\widetilde{D})$ of $\widetilde{D}$.

Let $\lambda > r$. If $(\lambda I - B) f = 0$ for some $f \in D(B) \cap Lip_{x_0}(C, \mathbb{K})$, then, by Lemma 2 and Remark 3 we have that, $\forall x \in C$,

$$\begin{aligned} 0 &= \big((\lambda I - B) f\big)(x) \\ &= \big((\lambda I - B) f\big)(x) - \big((\lambda I - B) f\big)(x_0) \\ &= \langle q(\lambda f - Bf), j_{x_0}(x) \rangle_2 \\ &= \langle \big(\lambda I - \widetilde{D}\big)(q(f)), j_{x_0}(x) \rangle_2 \end{aligned} \tag{4.10}$$

Hence, $(\lambda I - \widetilde{D})(q(f)) = 0$. Now that $\lambda \in \rho(\widetilde{D})$, we have $q(f) = 0$ and thus $f(x) = f(x_0) = 0$ for all $x \in C$. Therefore, $f = 0$ and hence $\lambda I - B$ is injective.

We now prove that $\lambda I - B$ is surjective. Let $g \in Lip_{x_0}(C, \mathbb{K})$, and let $f_0 \in Lip_{x_0}(C, \mathbb{K})$ be defined by

$$f_0(x) = \langle (\lambda I - \widetilde{D})^{-1}(q(g)), j_{x_0}(x) \rangle_2, \ x \in C.$$

Then, by Remark 3 we have that, for all $x \in C$,

$$\begin{aligned}\langle q(f_0), j_{x_0}(x)\rangle_2 &= f_0(x) - f_0(x_0) = f_0(x)\\ &= \langle (\lambda I - \widetilde{D})^{-1}(q(g)), j_{x_0}(x)\rangle_2.\end{aligned}$$

Hence, $q(f_0) = (\lambda I - \widetilde{D})^{-1}(q(g)) \in D(\widetilde{D})$. By Lemma 2 we know that $f_0 \in D(B)$. Similarly to (4.10) we can show that, $\forall x \in C$,

$$\big((\lambda I - B)f_0\big)(x) = \langle (\lambda I - \widetilde{D})(q(f_0)), j_{x_0}(x)\rangle_2 = \langle q(g), j_{x_0}(x)\rangle_2 = g(x). \tag{4.11}$$

That is, $(\lambda I - B)f_0 = g$. Therefore, $\lambda I - B$ is surjective because $g \in Lip_{x_0}(C, \mathbb{K})$ is arbitrary. Moreover, it follows from (4.11) that

$$q((\lambda I - B)^{-1}g) = q(f_0) = (\lambda I - \widetilde{D})^{-1}(q(g)),\ \forall g \in Lip_{x_0}(C, \mathbb{K}).$$

Therefore, we conclude this lemma. □

Proposition 4. *If $\{T_t\}_{t\geq 0}$ has a fixed point $x_0 \in C$ and satisfies $L(T_t) \leq Me^{wt}$ for all $t \geq 0$, then for all $x \in C$ and $f \in Lip_{x_0}(C, \mathbb{K})$ the following two formulas hold:*

$$(i)\quad f(T_t x) = \lim_{n\to\infty} \left(\big(I - \frac{t}{n}B\big)^{-n} f\right)(x),\ t \geq 0;$$

$$(ii)\quad f(T_t x) = e^{wt} \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1} e^{jnt}}{(j-1)!} \big((jn + w - B)^{-1} f\big)(x),\ t > 0,$$

where B is defined as in (4.2), the convergence in (i) is uniform on compact subset of $[0, +\infty)$, and the convergence in (ii) is uniform on any subinterval $(0, b]$.

Proof. Let S be the infinitesimal generator of $\{S_t\}_{t\geq 0}$. When n is sufficiently large, by Lemma 4 and Remark 3 we can show that, $\forall f \in Lip_{x_0}(C, \mathbb{K}), x \in C$,

$$\begin{aligned}\left((I - \frac{t}{n}B)^{-n} f\right)(x) &= \langle q((I - \frac{t}{n}B)^{-n} f), j_{x_0}(x)\rangle_2\\ &= \langle (I - \frac{t}{n}\widetilde{D})^{-n}(q(f)), j_{x_0}(x)\rangle_2\\ &= \langle q(f), (I - \frac{t}{n}S)^{-n}(j_{x_0}(x))\rangle_2.\end{aligned} \tag{4.12}$$

Since $\{S_t\}_{t\geq 0}$ is C_0-semigroup, the exponential formula holds [5]:

$$S_t(j_{x_0}(x)) = \lim_{n\to\infty} \big(I - \frac{t}{n}S\big)^{-n}(j_{x_0}(x)),\ x \in C,\ t \geq 0, \tag{4.13}$$

where the strong convergence is uniform on the compact subset of $[0, +\infty)$. Thus, combining (4.12) with (4.13) yields that

$$\langle q(f), S_t(j_{x_0}(x))\rangle_2 = \lim_{n\to\infty} \left((I - \frac{t}{n}B\big)^{-n} f\right)(x),\ x \in C,\ t \geq 0.$$

Therefore, by (b) of Corollary 2 the formula (i) is derived.

As for formula (ii), we first apply Theorem 3.2 of [13] to the C_0-semigroup $\{S_t\}_{t\geq 0}$ to establish the formula

$$S_t(j_{x_0}(x)) = \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{(nj+w)t}(w + nj - S)^{-1}(j_{x_0}(x)), t > 0,$$

where the convergence is uniform on any subinterval of the type $(0, b]$. Similarly to (4.12), it can be shown that

$$((k + w - B)^{-1} f)(x) = \langle q(f), (k + w - S)^{-1}(j_{x_0}(x))\rangle_2$$

when k is sufficiently large. Hence, we derive that

$$\langle q(f), S_t(j_{x_0}(x))\rangle_2 = \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{(nj+w)t}((w + nj - B)^{-1} f)(x),\ t \geq 0.$$

Therefore, by (b) of Corollary 2 the formula (ii) is deduced. □

Remark 7. It should be pointed out that formula (i) is similar to that proved by Dorroh and Neuberger [6] in terms of Lie generator.

Corollary 4. *If C has Radon-Nikodyn property and $\{T_t\}_{t\geq 0}$ possesses a densely defined generator A, then for all $x \in C$ and $f \in Lip_{x_0}(C, \mathbb{K})$ we have that*

$$(i)\quad f(T_t x) = \lim_{n\to\infty} \left(\left(I - \frac{t}{n} A_P\right)^{-n} f \right)(x),\ t \geq 0; and$$

$$(ii)\quad f(T_t x) = e^{wt} \lim_{n\to\infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1} e^{jnt}}{(j-1)!} \left((jn + w - A_P)^{-1} f\right)(x),\ t > 0,$$

where $A_P : D(A_P) \to Lip_{x_0}(C, \mathbb{K})$ is defined by

$$(A_P f)(x) = \lim_{t\to 0^+} \frac{f(x + tAx) - f(x)}{t},\ f \in D(A_P), x \in D(A),$$

where the domain $D(A_P)$ consists of all those $f \in Lip_{x_0}(C, \mathbb{K})$ such that there exists $g \in Lip_{x_0}(C, \mathbb{K})$ such that

$$g(x) = \lim_{t\to 0^+} \frac{f(x + tAx) - f(x)}{t},\ x \in D(A).$$

Proof. This corollary is directly derived from Proposition 4 and Lemma 4. □

5. Application II: Asymptotic behaviors of Lipschitzian semigroups

As another application example, in this section we consider the asymptotic behaviors of Lipschitzian semigroup. For the exponential boundedness of Lipschitzian semigroup we refer to (ii) of Remark 4.

Proposition 5. *Let $\{T_t\}_{t\geq 0}$ be an exponentially bounded Lipschitzian semigroup on C. Then, the following assertions (a)-(e) are equivalent:*

(a) $\omega =: \inf_{t>0} t^{-1} \ln L(T_t) < 0$.

(b) *There exist positive constants M and w such that* $L(T_t) \leq Me^{-wt}, t \geq 0$.

(c) $L(T_{t_0}) < 1$ *for some* $t_0 > 0$.

(d) *The limit* $\lim_{t\to+\infty} L(T_t)$ *exists and equals to* 0.

(e) $\rho =: \liminf_{n\to\infty} L(T_{t_0}^n)^{\frac{1}{n}} < 1$ *for some* $t_0 > 0$.

Proof. Let $x_0 \in C$. Since $\{T_t\}_{t\geq 0}$ is exponentially bounded, by Proposition 3 and its corollaries we know that there exists a C_0-semigroup $\{S_t\}_{t\geq 0}$ of G_{x_0} such that $L(T_t) = \|\widetilde{D_t}\| = \|S_t\|$ for all $t \geq 0$. We see that the assertions (a)-(e) are only concerned with the quantity $L(T_t)$. Hence, applying those existing results on asymptotic behaviors of C_0-semigroup to $\{S_t\}_{t\geq 0}$ (refer to [5, 11, 14] for those existing results), we obtain the equivalence among the assertions (a)-(e). □

Lemma 5. *Suppose that* $\{W_t\}_{t\geq 0}$ *is a* C_0*-semigroup on Banach space* E *and* $\Omega \subset E$ *is an invariant set of* $\{W_t\}_{t\geq 0}$ *(i.e.,* $W_t(\Omega) \subset \Omega$ *for all* $t \geq 0$*). If, for some* $p \in [1, +\infty)$*,*

$$\int_0^{+\infty} \|W_t x\|^p \, dt < \infty, \forall x \in \Omega,$$

then, for all $x \in \Omega$*,* $W_t x \to 0$ *as* $t \to +\infty$*.*

Proof. One can easily derive this lemma by performing the proof of Theorem 4.1 of [11] on the invariant subset Ω. □

Proposition 6. *Let* $\{T_t\}_{t\geq 0}$ *be an exponentially bounded Lipschitzian semigroup on* C*, and* $x_0 \in C$ *its fixed point (i.e.,* $T_t x_0 = x_0, t \geq 0$*). If, for some* $p \in [1, \infty)$*,*

$$\int_0^{+\infty} \|T_t x - x_0\|^p \, dt < \infty, x \in C, \tag{5.1}$$

then, for all $x \in C$*,* $T_t x \to x_0$ *as* $t \to +\infty$*.*

Proof. Since $\{T_t\}_{t\geq 0}$ is exponentially bounded, by Corollaries 2 and 3 and Remark 4 there exists a C_0-semigroup $\{S_t\}_{t\geq 0}$ of G_{x_0} such that $\|S_t(j_{x_0}(x))\| = \|T_t x - T_t x_0\|$ for all $x \in C$. Hence, it follows from (5.1) that

$$\int_0^{+\infty} \|S_t(j_{x_0}(x))\|^p \, dt < \infty, x \in C. \tag{5.2}$$

Let $\Omega = \{j_{x_0}(x) : x \in C\}$, and let $j_{x_0}(x) \in \Omega$. Then, by Corollary 2 and Remark 3 we have that, for all $t \geq 0$, $f \in Lip(C, \mathbb{K})$,

$$\begin{aligned} \langle q(f), S_t(j_{x_0}(x))\rangle_2 &= f(T_t x) - f(T_t x_0) \\ &= f(T_t x) - f(x_0) = \langle q(f), j_{x_0}(T_t x)\rangle_2. \end{aligned}$$

Hence, $S_t(j_{x_0}(x)) = j_{x_0}(T_t x)$ for all $t \geq 0$. That is, $S_t(j_{x_0}(x)) \in \Omega$ for all $t \geq 0$. Therefore, Ω is an invariant set of $\{S_t\}_{t\geq 0}$. Applying Lemma 5 to $\{S_t\}_{t\geq 0}$ we thus obtain that $S_t(j_{x_0}(x)) \to 0$ as $t \to +\infty$. Since $\|T_t x - x_0\| = \|T_t x - T_t x_0\| = \|S_t(j_{x_0}(x))\|$ for all $t \geq 0$, we have that $T_t x - x_0 \to 0$ as $t \to +\infty$. Therefore, the proof is completed. □

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