Integr. equ. oper. theory 46 (2003) 489–500
0378-620X/040489-12, DOI 10.1007/s00020-001-1150-z

Integral Equations and Operator Theory

An Estimate of Growth Bound of Positive C_0-Semigroup on L^p Space and its Applications

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Abstract. Let $\{T(t)\}_{t\geq 0}$ be a positive C_0-semigroup on $L^p(\Omega)$, with infinitesimal generator A. In this paper, it is proved that if there exists a $c \in L^\infty(\Omega) \cap D(A^*)$ such that $ess \inf_{r\in\Omega} c(r) > 0$ and $b := ess \sup_{x\in\Omega} \frac{(A^*c)(x)}{c(x)} < \infty$, where A^* is the adjoint of A, then the growth bound of $T(t)$ is upper bounded by b when $p = 1$, and by $\frac{b}{p} + \frac{a}{q}$ when $1 < p < \infty$ and $c \in D(A)$, where $a = ess \sup_{x\in\Omega} \frac{(Ac)(x)}{c(x)}$. This is an operator version of a classical stability result on Z-matrix. As application examples, some new results on the asymptotic behaviours of population system and neutron transport system are obtained.

Mathematics Subject Classification (2000). Primary 47D06; Secondary 34B40.

Keywords. Z-Matrix, Positive C_0-semigroup, Infinitesimal generator, Population system, Neutron transport system.

1. Introduction

Let $\{T(t)\}_{t\geq 0}$ be C_0-semigroup on a L^p space, with infinitesimal generator A. According to L. Weis[14], if $\{T(t)\}_{t\geq 0}$ is positive, then its growth bound equals to the spectral bound of A. One of the significance of this result, for example, is that the exponential stability of linear system of the type

$$u'(t) = Au(t), t \geq 0, \tag{1.1}$$

where the coefficient operator A generates positive a C_0-semigroup on a certain L^p space, can be exactly determined by the spectra of A. However, in many applications, it is difficult to precisely compute the spectral bound of operator A because A, as a generator, is unbounded for many practical systems governed by partial differential equations. So, it is still interesting to develop some new approaches

This work was completed with the support of the Natural Science Foundation of China under the contact no. 10101019.

to estimate the growth bound of positive C_0-semigroups on L^p spaces that are frequently used as the state spaces of many practical systems.

The starting point of this paper is the following classical stability result concerning Z-matrix [4]: if $-A$ is a Z-matrix and $Ac < 0$ for some positive vector $c \in \mathbb{R}^n$, then $\lim_{t\to+\infty} u(t) = 0$ for every solution $u(t) = e^{At}u_0$ of the equation $u'(t) = Au(t)$ with initial value $u(0) = u_0$. An $n \times n$ real matrix B is said to be Z-matrix if every off-diagonal term is non-positive (see, e.g., [1, 6]). By properties of nonnegative matrix, we know that $-A$ is a Z-matrix if and only if the matrix exponential $e^{At} \geq 0$. In view of this, if $\{e^{At}\}_{t\geq 0}$ is viewed as a C_0-semigroup on the partially ordered Banach space $\mathbb{R}^n$, then it is natural to expect that the above stability result on e^{At} can be generalized to positive semigroups on some partially ordered Banach spaces. In the subsequent section, we will give such a generalization in L^p spaces. As application examples, the asymptotic behaviors of population system and neutron transport system are considered in Section 3.

Throughout this paper, let $(\Omega, \mathcal{A}, m)$ be a measure space, $L^p(\Omega)$ the usual Banach space constructed on $(\Omega, \mathcal{A}, m)$. Denote by $L^p_+(\Omega)$ the positive cone of nonnegative functions in $L^p(\Omega)$, and by q the conjugate exponent partner of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ when $1 < p < \infty$, and $q = \infty$ when $p = 1$. A^*, if exists, always denotes the adjoint of operator A.

The linear system (1.1) is said to be *exponentially stable* if there exists a positive real number b such that $\lim_{t\to+\infty} e^{bt}u(t) = 0$ for all mild solution $u(t)$, and to be *unstable* if $\lim_{t\to+\infty} u(t) \neq 0$ for some mild solution $u(t)$.

For general materials about positive operator semigroups, we will refer to [3] and [9].

2. Main Results

In this section, we generalize the mentioned stability result concerning Z-matrix to positive C_0-semigroups on $L^p(\Omega)$ spaces. To this end, we first prove the following key lemma.

Lemma 2.1. *Suppose that T is a positive operator on $L^p(\Omega)$ with $1 < p < \infty$. If $f, g \in L^p_+(\Omega)$ such that $f^p, g^q \in L^p_+(\Omega)$, then*

$$T(f \cdot g) \leq (T(f^p))^{\frac{1}{p}} \cdot (T(g^q))^{\frac{1}{q}} \tag{2.1}$$

where $(f \cdot g)(x) = f(x) \cdot g(x)$ for almost every $x \in \Omega$.

Proof. It follows from the famous Young's inequality that

$$\begin{aligned}(f \cdot g)(x) &= (\alpha f \cdot \frac{1}{\alpha} g)(x) \\ &\leq \frac{1}{p}(\alpha f)^p(x) + \frac{1}{q}(\frac{g}{\alpha})^q(x)\end{aligned}$$

for all $\alpha > 0$ and almost all $x \in \Omega$. So, by the positivity of T we have

$$0 \leq T(f \cdot g) \leq \frac{\alpha^p}{p} T(f^p) + \frac{\alpha^{-q}}{q} T(g^q). \tag{2.2}$$

For all $x \in \{x \in \Omega : T(f^p)(x) \neq 0 \text{ and } T(g^q)(x) \neq 0\}$, let in (2.2)

$$\alpha = \left(T(g^q)(x) \cdot \frac{1}{T(f^p)(x)} \right)^{\frac{1}{pq}},$$

then we have

$$T(f \cdot g)(x) \leq \Big(T(f^p) \Big)^{\frac{1}{p}}(x) \cdot \Big(T(g^q) \Big)^{\frac{1}{q}}(x). \tag{2.3}$$

And, for all $x \in \{x \in \Omega : T(f^p)(x) = 0 \text{ or } T(g^q)(x) = 0\}$ let $\alpha \to +\infty$ or $\alpha \to 0^+$ in (2.2), then we have $T(f \cdot g)(x) = 0$. That is, the inequality (2.3) also holds for all $x \in \{x \in \Omega : T(f^p)(x) = 0 \text{ or } T(g^q)(x) = 0\}$. Therefore, the inequality (2.1) holds. The proof is completed. □

Remark 2.2. In [10], a similar result was proved and some novel estimates of growth bound of positive semigroups on L^p spaces with $1 < p < \infty$ were obtained.

In the following we frequently utilize the clear fact that $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$ whenever $p_1 > p_2$ and the total measure $m(\Omega) < \infty$.

Theorem 2.3. *Suppose that A is the generator of positive C_0-semigroup $\{T(t)\}_{t \geq 0}$ on $L^p(\Omega)$ space, and that there exists a function $c \in D(A^*) \bigcap L^\infty(\Omega)$ such that $d := ess \inf_{x \in \Omega} c(x) > 0$ and $b := ess \sup_{x \in \Omega} \frac{(A^* c)(x)}{c(x)} < \infty$.*

(i) If $p = 1$, then, for all $t \geq 0$,

$$\| T(t) \|_1 \leq \frac{k}{d} \cdot e^{bt} \tag{2.4}$$

(ii) If $1 < p < \infty, m(\Omega) < \infty$ and $c \in D(A)$ such that $a := ess \sup_{x \in \Omega} \frac{(Ac)(x)}{c(x)} < \infty$, then, for all $t \geq 0$,

$$\| T(t) \|_p \leq \left(\frac{k}{d} \right)^{1+p^{-1}} \cdot e^{(\frac{a}{q} + \frac{b}{p})t}. \tag{2.5}$$

In both (i) and (ii), $k = \| c \|_\infty$, the norm of $c \in L^\infty(\Omega)$.

Proof. (i) Let $p = 1$. Since $c \in D(A^*)$, it follows that the adjoint semigroup $T^*(t)$ on $L^\infty(\Omega)$ satisfies

$$0 \leq e^{-bt} T^*(t) c = c + \int_0^t T^*(r)(A^* c - bc)\, dr \leq c, \ t \geq 0. \tag{2.6}$$

So, for all $f \in L^1_+(\Omega)$ and $t \geq 0$, we have

$$\begin{aligned}
\| T(t)f \|_1 &= \int_\Omega (T(t)f)(x)\, dm(x) \\
&= \int_\Omega \frac{(T(t)f)(x)}{c(x)} \cdot c(x)\, dm(x) \\
&\leq \frac{1}{d} \int_\Omega (T(t)f)(x) \cdot c(x)\, dm(x) \\
&= \frac{1}{d} \int_\Omega f(x) \cdot (T^*(t)c)(x)\, dm(x) \\
&\leq \frac{1}{d} \int_\Omega f(x) \cdot e^{bt} c(x)\, dm(x) \\
&\leq \frac{k}{d} e^{bt} \| f \|_1 .
\end{aligned}$$

That is, $\| T(t) \|_1 \leq \frac{k}{d} \cdot e^{bt}$, as expected.

(ii) It is clear by the definition of a that $Ac \leq a\,c$. So, by the semigroup property of $T(t)$, we have

$$0 \leq e^{-at} T(t)c = c + \int_0^t e^{-ar} T(r)(Ac - ac)\, dr \leq c,\ t \geq 0. \tag{2.7}$$

Let $f \in L^p_+(\Omega)$ such that $f^p \in L^p_+(\Omega)$, then by the positivity of $T(t)$ we have

$$T(t)(c \cdot f^p) \leq T(t)(ess \sup_{x \in \Omega} c(x) \cdot f^p) = k \cdot T(t)(f^p).$$

Hence, by Lemma 1 we can further prove that, for all $t \geq 0$,

$$\begin{aligned}
0 &\leq T(t)(c \cdot f) = T(t)\big((c^{\frac{1}{p}} f) \cdot c^{\frac{1}{q}}\big) \\
&\leq \Big(T(t)(c \cdot f^p)\Big)^{\frac{1}{p}} \cdot \Big(T(t)(c)\Big)^{\frac{1}{q}} \\
&\leq k^{-p} \cdot \Big(T(t)(f^p)\Big)^{\frac{1}{p}} \cdot \Big(e^{at}\, c\Big)^{\frac{1}{q}}.
\end{aligned} \tag{2.8}$$

Let $\{T^*(t)\}_{t \geq 0}$ be the positive operator semigroup generated by A^* on $L^q(\Omega)$. Then, similar to (2.6) we have

$$0 \leq e^{-bt}\, T^*(t)c \leq c,\ t \geq 0.$$

So, by the fact that $L^p(\Omega) \subset L^1(\Omega)$, it can be shown that, for all $f \in L^p_+(\Omega)$ and all $t \geq 0$,

$$\begin{aligned} \| T(t)f \|_1 &= \int_\Omega \frac{\big(T(t)f\big)(x)}{c(x)} \cdot c(x)\, dm(x) \\ &\leq \frac{1}{d} \int_\Omega (T(t)f)(x) \cdot c(x)\, dm(x) \\ &= \frac{1}{d} \int_\Omega f(x) \cdot (T^*(t)c)(x)\, dm(x) \\ &\leq \frac{1}{d} \int_\Omega f(x) \cdot e^{bt} c(x)\, dm(x) \\ &\leq \frac{k}{d} \cdot e^{bt} \cdot \| f \|_1 . \end{aligned} \tag{2.9}$$

Thus, the density of $L^p(\Omega)$ in $L^1(\Omega)$ implies that $T(t)$ has an extension, denoted by $T_1(t)$, to $L^1(\Omega)$, which satisfies

$$\| T_1(t) \|_1 \leq \frac{k}{d} \cdot e^{bt},\ t \geq 0.$$

Now, let $f \in L^p_+(\Omega)$ such that $f^p \in L^p_+(\Omega)$. Since $d \cdot f(x) \leq (c \cdot f)(x)$, the positivity of $T(t)$ yields $d \cdot T(t)f \leq T(t)(c \cdot f)$. Combining (2.8) with (2.9), we thus get

$$\begin{aligned} \| T(t)f \|_p^p &= \int_\Omega ((T(t)f)(x))^p\, dm(x) \\ &\leq \frac{1}{d^p} \int_\Omega \big(\big(T(t)(c \cdot f)\big)(x)\big)^p\, dm(x) \\ &\leq \frac{k}{d^p} \int_\Omega (T(t)f^p)(x) \cdot \Big(e^{at} c(x)\Big)^{\frac{p}{q}}\, dm(x) \\ &\leq \frac{1}{d^p} \cdot k^{1+\frac{p}{q}} \cdot e^{\frac{apt}{q}} \int_\Omega (T(t)f^p)(x)\, dm(x) \\ &= \frac{1}{d^p} \cdot k^p \cdot e^{\frac{apt}{q}} \cdot \| T(t)f^p \|_1 \\ &\leq \frac{1}{d^p} \cdot k^p \cdot e^{\frac{apt}{q}} \cdot \frac{k}{d} e^{bt} \| f^p \|_1 \\ &= \left(\frac{k}{d}\right)^{1+p} \cdot e^{\frac{apt}{q}+bt} \| f \|_p^p . \end{aligned}$$

This, associated with the density of $L^\infty(\Omega)$ in $L^p(\Omega)$, implies that

$$\| T(t) \|_p \leq \left(\frac{k}{d}\right)^{\frac{1}{p}+1} \cdot e^{(\frac{a}{q}+\frac{b}{p})t}, t \geq 0,$$

as claimed. Therefore, the proof is completed. □

Corollary 2.4. *Suppose that A generates a positive C_0-semigroup on $L^p(\Omega)$, and that there exists $c \in L^\infty_+(\Omega) \bigcap D(A^*)$.*

If $d = ess \inf_{x\in\Omega} c(x) > 0$ *and* $b = ess \sup_{x\in\Omega} \frac{(A^*c)(x)}{c(x)} < \infty$, *then*

(i) *in the case that* $p = 1$, *the corresponding linear system* (1.1) *is exponentially stable if* $b < 0$; *and*

(ii) *in the case that* $1 < p < \infty$ *and* $m(\Omega) < \infty$, *the corresponding linear system* (1.1) *is exponentially stable if* $c \in D(A)$ *and* $\frac{b}{p} + \frac{a}{q} < 0$ *(particularly,* $a < 0$ *and* $b < 0$*), where* $a = ess \sup_{x\in\Omega} \frac{(Ac)(x)}{c(x)}$.

If c *does not vanish almost everywhere in* Ω *(i.e.,* $c \neq 0$ *a.e) and* $A^*c \geq 0$, *then the corresponding system* (1) *is unstable for all* $1 \leq p < \infty$.

Proof. Let $\{T(t)\}_{t\geq 0}$ be the operator semigroup generated by A. Then, every solution $u(t)$ with initial value x can be identified with $T(t)x$ for all $t \geq 0$. So, by Theorem 1, if $d > 0$ and $b < \infty$, then (i) and (ii) hold in respective cases.

Now, suppose that $c \neq 0$ and $A^*c \geq 0$. Let $T^*(t)$ be the adjoint of $T(t)$. Then

$$T^*(t)c = \int_0^t T^*(r)A^*c\,dr + c, t \geq 0.$$

Hence, there holds that $T^*(t)c \geq c$ for all $t \geq 0$ since $A^*c \geq 0$. Therefore,

$$\begin{aligned} 0 &< \| c \|_2^2 = \int_\Omega c^2(x)\,dm(x) \\ &\leq \int_\Omega (T^*(t)c)(x)c(x)\,dm(x) \\ &\leq \begin{cases} \| T^*(t)c \|_q^q \cdot \| c \|_p^p & \text{if } 1 < p < \infty \\ \| T^*(t)c \|_\infty \cdot \| c \|_1 & \text{if } p = 1. \end{cases} \end{aligned}$$

Noticing that $\| T(t) \| = \| T^*(t) \|$, we conclude from the above inequalities that (1.1) is unstable. The proof is completed. □

Corollary 2.5. *Let* $-A = (-a_{ij})$ *be an* $n \times n$ Z*-matrix. Then, for all positive vector* $c = (c_1, c_2, \cdots, c_n)^T$ *of* $\mathbb{R}^n$, *the matrix exponential* e^{At} *satisfies that,*

$$\| e^{At}x \|_\infty \leq \frac{c_M}{c_m} e^{bt} \| x \|_\infty,\ t \geq 0, \tag{2.10}$$

where $\| \cdot \|_\infty$ *denotes the* l^∞*-norm on* $\mathbb{R}^n$, *and*

$$c_m = \min_{1\leq i\leq n} c_i,\ c_M = \max_{1\leq i\leq n} c_i,\ b = \max_{1\leq i\leq n} \frac{1}{c_i} \sum_{j=1}^n a_{ij}c_j.$$

Proof. It is well known that when endowed with l^1-norm, $\mathbb{R}^n$ can be identified with $L^1(\Omega)$, where $\Omega=\{1, 2, \cdots, n\}$. Let A^T denote the transpose of A, then $-A^T$ is also Z-matrix and hence the matrix exponential e^{A^Tt} is positive. So, by Theorem 1, the positive semigroup $\{e^{A^Tt}\}_{t\geq 0}$ on $\mathbb{R}^n$ satisfies

$$\| e^{A^Tt} \|_1 \leq \frac{k}{d} \cdot e^{bt},\ t \geq 0, \tag{2.11}$$

where k, d and b are respectively determined by

$$k = \|c\|_1 = \max_{1\le i\le n} c_i,\ d = \min_{1\le i\le n} c_i$$

and

$$b = \max_{1\le i\le n} \frac{((A^T)^* c)_i}{c_i} = \max_{1\le i\le n} \frac{(Ac)_i}{c_i} = \max_{1\le i\le n} \frac{1}{c_i}\sum_{j=1}^{n} a_{ij} c_j.$$

Noticing that that $(A^T)^* = A$, we thus have that

$$\| e^{At} \|_\infty = \| e^{A^T t} \|_1 \le \frac{c_M}{c_m} \cdot e^{bt},\ t \ge 0. \tag{2.12}$$

Therefore, the proof is completed. □

Remark 2.6. It is seen that if $Ac < 0$ for some positive vector c, then the corresponding b is negative, and hence the matrix exponential e^{At} will exponentially converge to zero as the time t goes to infinity. Due to this, Corollary 2.5 can be viewed as a generalization of the mentioned stability result concerning Z-matrix.

3. Application Examples

In this section, we apply the stability results established in Section 2 to stability analysis of population system and neutron transport system.

Example. Consider the population system governed by the equations:

$$\begin{cases} \frac{\partial p(t,r)}{\partial t} + \frac{\partial p(t,r)}{\partial r} = -\mu(r)p(t,r),\ t>0,\ r\in[0,r_m] \\ p(t,0) = \beta\int_{r_1}^{r_2} k(r)h(r)p(t,r)\,dr,\ t>0 \\ p(0,r) = p_0(r),\ r\in[0,r_m], \end{cases} \tag{3.1}$$

where, t is time, r is age, $p(t,r)$ represents the population distribution, $\mu(r)$ the age-specific mortality modulus, $k(r)$ the proportion of female, and $h(r)$ the birth mode of female; r_1 and r_2 represent the minimum and maximum birth age of female, respectively, r_m is the maximum age of population; β is a positive constant which closely depends on the government policy on population.

According to physics meanings, $0 < r_m < \infty, 0 < r_1 < r_2 < r_m$, these functions $\mu(\cdot), h(\cdot)$ and $k(\cdot)$ are nonnegative and integrable in $[0, r_m)$ (in the sense of Lebesgue), and satisfy

(*i*) $0 < \mu_1 \le \mu(r) < \infty$ for all $r < r_m$, and $\int_0^{r_m} \mu(r)\,dr = \infty$;

(*ii*) $h(r) = 0$ for all $r \in [0, r_m] \setminus [r_1, r_2]$, and $\int_0^{r_m} h(r)\,dr = 1$; and

(*iii*) $k(r) \le 1$ for all $r \in [0, r_m]$, and $0 < k_0 \le k(r) < 1$ for all $r \in [r_1, r_2]$,

where both μ_1 and k_0 are positive constants.

Here we refer to [5, 7, 9, 11, 13] for more details.

Now, let $1 \le p \le \infty$, and define a linear operator $A_p : D(A_p) \subset L^p[0, r_m] \to L^p[0, r_m]$ as follows

$$A_p\phi = -\phi' - \mu\phi, \tag{3.2}$$

where the domain $D(A_p)$ consists of all those $\phi \in L^p[0, r_m]$ that is absolutely continuous with derivatives $\phi' \in L^p[0, r_m]$ and $\phi(0) = \beta \int_{r_1}^{r_2} h(r)k(r)\phi(r)\, dr$. Then, with $L^p[0, r_m]$ as the state space, the population system can be cast as the compact form

$$u'(t) = A_p u(t),\ t > 0;\ u(0) = p_0. \tag{3.3}$$

Moreover, by [9, 11] it is known that, for all $p \in [1, \infty)$, A_p generates a positive C_0-semigroup $\{T_p(t)\}_{t\geq 0}$ on $L^p[0, r_m]$. Therefore, applying Theorem 1 to this system, we have

Proposition 3.1. *Every solution $p(t, r)$ of (3.1) with $p_0 \in L^\infty[0, r_m]$ exponentially decays to 0 as time t goes to infinity (i.e., there is a positive constant b such that $e^{bt} \int_0^{r_m} \mid p(t, r) \mid\, dr \to 0$ as $t \to +\infty$) if and only if $\lambda_1 < 0$, where λ_1 is the unique real solution of the equation:*

$$1 = \beta \int_{r_1}^{r_2} h(r)k(r)e^{-\lambda r - \int_0^r \mu(s)\, ds}\, dr. \tag{3.4}$$

Proof. Define a linear operator $A^\# : D(A^\#) \subset L^1[0, r_m] \to L^1[0, r_m]$ as follows

$$A^\#\phi = \phi' - \mu\phi + \beta\phi(0)k \cdot h$$

where the domain $D(A^\#)$ consists of all those $\phi \in L^1[0, r_m]$ that is absolutely continuous with derivatives $\phi' \in L^1[0, r_m]$ and $\phi(r_m) = 0$. Then, it can be easily shown that $(A^\#)^* = A_\infty$ and hence $A^\#$ generates a positive C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $L^1[0, r_m]$.

Let c denote the positive function on $[0, r_m]$ defined by

$$c(r) = \begin{cases} e^{-\lambda_1 r - \int_0^r \mu(s)\, ds}, & r \in [0, r_2], \\ e^{-\lambda_1 r_2 - \int_0^{r_2} \mu(s)\, ds}, & \text{otherwise}. \end{cases} \tag{3.5}$$

Then we can show that $c \in D(A_\infty) \subset L^\infty[0, r_m]$ and

$$b = ess \sup_{r\in[0,r_m]} \frac{(A_\infty c)(r)}{c(r)} \leq \max\{\lambda_1, -\mu_1\}.$$

So, by Theorem 1 we have

$$\| S(t) \|_1 \leq \frac{k}{d} \cdot e^{bt},\ t \geq 0,$$

where $k = \| c \|_\infty$ and $d = ess \inf_{r\in[0,r_m]} c(r)$.

Let $T_\infty(t)$ be the adjoint semigroup of $S(t)$. Clearly, A_p has the same analytic representation for all $1 \leq p \leq +\infty$, so does $\{T_p(t)\}_{t\geq 0}$. That is, $T_p(t)f = T_\infty(t)f$ whenever $f \in L^\infty[0, r_m]$. Hence, whenever initial value p_0 belongs to $L^\infty[0, r_m]$, the corresponding solution $p(t,r)$ of (3.1) satisfies that,

$$\begin{aligned}\int_0^{r_m} \mid p(t,r) \mid \, dr &= \| T_1(t)p_0 \|_1 = \| T_\infty(t)p_0 \|_1 \\ &\leq r_m \| T_\infty(t)p_0 \|_\infty \\ &\leq r_m \| T_\infty(t) \|_\infty \cdot \| p_0 \|_\infty \\ &= r_m \| S(t) \|_1 \cdot \| p_0 \|_\infty \\ &\leq \frac{r_m k}{d} e^{bt} \| p_0 \|_\infty \, .\end{aligned}$$

Therefore, every solution $p(t,r)$ with initial value $p_0 \in L^\infty[0,r_m]$ exponentially decays to 0 if $b < 0$. Actually, $b < 0$ since $\lambda_1 < 0$.

Conversely, suppose $\lambda_1 \geq 0$. Denote by c the following nonnegative function:

$$c(r) = e^{-\lambda_1 r - \int_0^r \mu(s)\, ds}, r \in [0, r_m],$$

then we can easily show that $c \in L_+^\infty[0,r_m] \bigcap D(A_p)$ and $A_p c = \lambda_1$ for all $p \in [1,\infty)$. From which, it immediately follows that

$$T_p(t)c = \int_0^t T_p(s) A_p c \, ds + c \geq c, \forall t \leq 0.$$

Consequently, with $p_0 = c$, the solution $p(t,r) = (T_p(t)c)(r)$ can not decay to 0. Therefore, the proof is completed. □

Remark 3.2. The asymptotic behaviours of population system have been extensively studied in this literature (see, e.g., [5, 7, 9, 13]). In most existing works, the direct spectrum approach was applied to stability analysis of (3.1). Due to the clear structure of A_p, it had been shown that the spectrum bound of A_p is closely related to the unique real solution λ_1 of Equation (3.4). For example, Webb [13] proved that the spectrum set of A_1 only consists of point spectra and the possible essential spectra, specifically, the spectrum bound $s(A_1) \leq \max\{-\mu_1, \lambda_1\}$. Accordingly, Equation (3.4) is commonly called the characteristic equation of (3.1).

It should be noted that although Proposition 1 can be found in many references (say, [7, 9, 13]), the present proof provides a direct approach, which is more practicable for many complicated systems. In the following is such a representative example.

Example. Consider the neutron transport system in a slab with perfect reflection boundary conditions [1, 8]

$$\begin{cases} \frac{\partial \phi(x,v,\mu,t)}{\partial t} = -v\mu \frac{\partial \phi(x,v,\mu,t)}{\partial x} - \sigma(x,v,\mu)\phi(x,v,\mu,t) \\ \qquad\qquad + \int_D \int_V k(x,v,v',\mu,\mu')\phi(x,v',\mu',t)\, dv' d\mu', \\ \phi(-a,v,\mu,t) = \phi(a,v,\mu,t), \\ \phi(x,v,\mu,0) = \phi_0(x,v,\mu), \\ x \in Q := [-a,a], v \in V := [v_m, v_M], \mu \in D := [-1,1]. \end{cases} \tag{3.6}$$

where, v and μ respectively are velocity and direction of neutron, $\phi(x, v, \mu, t)$ represents the neutron density at position x and at time t, and both $\sigma(x, v, \mu)$ and $k(x, v, v', \mu, \mu')$ are nonnegative bounded measurable functions.

For more detailed descriptions, we refer to [1, 8]. If necessary, we will adopt the same assumptions on functions σ and k with those used in [8].

Let $\Omega = Q \times V \times D$, and A the linear operator in $L^p(\Omega)(1 \leq p < \infty)$ defined by

$$A\phi = -v\mu\frac{\partial\phi}{\partial x} - \sigma(x, v, \mu)\phi + \int_D \int_V k(x, v, v', \mu, \mu')\phi(x, v', \mu')\, dv' d\mu' \tag{3.7}$$

with the domain $D(A)$ consisting of all those $\phi \in L^p(\Omega)$ that is absolutely continuous with respect to x in Q with the derivative $\frac{\partial\phi}{\partial x} \in L^p(\Omega)$ and $\phi(-a, v, \mu) = \phi(a, v, \mu)$ for all $v \in V$ and $\mu \in D$. Then, the system (3.6) is cast as the abstract form

$$u'(t) = Au(t), t > 0; u(0) = \phi_0 \tag{3.8}$$

of which the state space is $L^p(\Omega)$. Moreover, by [8] we know that A generates a positive C_0-semigroup $T(t)$ on $L^p(\Omega)$.

Proposition 3.3. *Let*

$$\alpha = ess \sup_{(x,v,\mu)\in\Omega} \left\{ - \sigma(x, v, \mu) + \int_D \int_V k(x, v, v', \mu, \mu')\, dv' d\mu' \right\} \tag{3.9}$$

and

$$\beta = ess \sup_{(x,v,\mu)\in\Omega} \left\{ - \sigma(x, v, \mu) + \int_D \int_V k(x, v', v, \mu', \mu)\, dv' d\mu' \right\}. \tag{3.10}$$

Then, for every solution $\phi(x, v, \mu, t)$ of (3.6) *with initial value $\phi_0 \in L^p(\Omega)$, we have that, for all $t \geq 0$, either in the case when $p = 1$,*

$$\int_Q \int_D \int_V \mid \phi(x, v, \mu, t) \mid dvd\mu dx \leq e^{\beta t} \int_Q \int_D \int_V \mid \phi_0(x, v, \mu) \mid dvd\mu dx,$$

or in the case when $1 < p < \infty$,

$$\int_Q \int_D \int_V \mid \phi(x, v, \mu, t) \mid^p \; dvd\mu dx \leq e^{(\frac{\alpha p}{q}+\beta)t} \int_Q \int_D \int_V \mid \phi_0(x, v, \mu) \mid^p \; dvd\mu dx.$$

Proof. It is easy to verify that A^*, the adjoint of A, is formulated by

$$A^*\psi = v\mu\frac{\partial\psi}{\partial x} - \sigma(x, v, \mu)\psi + \int_D \int_V k(x, v', v, \mu', \mu)\psi(x, v', \mu')\, dv' d\mu'$$

with domain $D(A^*)$ consisting of all those $\psi \in L^q(\Omega)$ that is absolutely continuous with respect to x in Q with derivative $\frac{\partial\psi}{\partial x} \in L^q(\Omega)$ and $\psi(-a, v, \mu) = \psi(a, v, \mu)$ for all $v \in V$ and $\mu \in D$. Let $c(x) \equiv 1$ for all $x \in \Omega$. Then it is a routine matter to show that $c \in L^\infty(\Omega) \cap D(A) \cap D(A^*)$, $d = k = 1$, $a \leq \alpha$ and $b \leq \beta$, where k, d, a and b are the corresponding constants defined as in Theorem 1. Hence, for all $\phi_0 \in L^p(\Omega)$, we by Corollary 2.4 conclude that $\|T(t)\phi_0\|_1 \leq e^{\beta t}\|\phi_0\|_1$ in the case when $p = 1$, and that $\|T(t)\phi_0\|_p \leq e^{(\frac{\alpha}{q}+\frac{\beta}{p})t}\|\phi_0\|_p$ in the case when $1 < p < \infty$.

Now, noticing that $\phi(x, v, \mu, t) = (T(t)\phi_0)(x, v, \mu)$ for every solution $\phi(x, v, \mu, t)$ with initial value $\phi_0 \in L^p(\Omega)$, we close the proof. □

It should be pointed out that there are many research works concerning the stability of neutron transport system (see, e.g., [1, 8, 12] and the references therein). In those existing researches, the commonly adopted approach is the so-called spectrum perturbation method. Specifically, let

$$B : D(A) \to L^p(\Omega), B\phi = -v\mu\frac{\partial\phi}{\partial x} - \sigma(x, v, \mu)\phi$$

and

$$K : L^p(\Omega) \to L^p(\Omega),\ K\phi = \int_D\int_V k(x, v, v', \mu, \mu')\phi(x, v', \mu')\, dv'd\mu'.$$

Then, A can be viewed as the perturbed B by the bounded linear operator K. Due to the clear structure of B, it is not hard to show that the spectra of B are all positioned in the left half-plane $\{\lambda \in \mathbb{C} : \Re\lambda \leq -ess \inf_{v\in V,\mu\in D} \sigma(v, \mu)\}$. Hence, by perturbation property of linear operator[15], the spectrum set of A should be included in the transported left half-plane

$$\{\lambda \in \mathbb{C} : \Re\lambda \leq -ess \inf_{v\in V,\mu\in D} \sigma(v, \mu)+ \parallel K \parallel\}.$$

Therefore, it can be concluded that the neutron transport system (3.6) is exponentially stable if $-ess \inf_{v\in V,\mu\in D} \sigma(v, \mu)+ \parallel K \parallel< 0$ (see, e.g., [1, 8, 12]).

In practice, it is difficult to compute the norm $\parallel K \parallel$. Hence, the practicability of the existing spectrum perturbation results is restricted. However, with Proposition 3.2 at hand, we know that the exponential stability of (3.6) can be determined by the present quantity β or $\frac{\alpha}{q} + \frac{\beta}{p}$, which is clearly easier to be computed than $\parallel K \parallel$.

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Acknowledgement

The authors are very grateful to the reviewers for their helpful suggestions and insightful comments.

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Submitted: March 1, 2001
Revised: August 28, 2002