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Lipschitzian semigroups and abstract functional differential equations

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ABSTRACT

The paper is devoted to studying an abstract functional differential equation by a nonlinear semigroup approach. We first prove in details the equivalence of the well posedness of an abstract functional differential equation and an associated abstract Cauchy problem in the sense of strong solutions. Secondly, a sufficient condition is derived for well posedness of the abstract functional differential equation. Thirdly, we present principles of linearized stability for the abstract functional differential equation. Finally, the results obtained are applied to a reaction–diffusion equation with delays.

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1. Introduction

By nonlinear semigroup theory, we investigate abstract functional differential equation in the paper

$$\text{(FDE)} \qquad \begin{cases} u'(t) = Au(t) + \Phi u_t, & t > 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where A is a closed and densely defined linear operator on Banach space X and Φ is a nonlinear Lipschitz continuous operator from $L^p([-1, 0], X)$ to X.

A. Bátkai and S. Piazzera discussed in [1] the linear version of FDE where Φ is a bounded linear operator from $W^{1,p}([-1, 0], X)$ to X. Concretely, they proved the equivalence of the well posedness of an abstract functional differential equation and an associated abstract Cauchy problem. Moreover, they derived sufficient conditions for the well posedness, exponential stability and norm continuity of the solutions. However, they anticipated that it seemed to be impossible at the present time to extend their approach to nonlinear equations owing to the incompleteness of the theory of nonlinear semigroups. Motivated by this, we attempt to study FDE in the framework of nonlinear semigroups of Lipschitz operators.

Different nonlinear types of FDE have been investigated in many literatures. Travis and Webb discussed in [2] the existence and stability of mild solution of FDE when A generates a strongly continuous semigroup of linear bounded operators on X and Φ is globally Lipschitz continuous from $C = C([-1, 0], X)$ to X. Concretely, the mild solution of FDE was researched as a nonlinear semigroup on C and a sufficient condition was derived for asymptotic stability of mild solutions. Parrott in [3] studied FDE where A generates a strongly continuous semigroup of linear bounded operators on X and $\Phi : C([-r_0, 0], X) \to X (r_0 > 0)$ is nonlinear continuously Fréchet-differentiable. More specifically, he gave a principle of linearized stability for FDE and investigated the semigroup associated with solutions of linearized equation of FDE by the strong positivity property of irreducibility. For more details, we can see references therein and papers [4–8].

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To the best of the authors' knowledge, the existing work almost deals with FDE in the space of continuous initial functions [9–11,3,2,8]. Compared with the space of continuous functions, the space L^p allows for more general initial functions for FDE, which provides a very tractable space for the study of solutions of FDE. From the point of view of applications(especially, control theory [12]), it turns out to be a better choice to study FDE in the product space $X \times L^p([-1, 0], X)$. Based on this, our research develops in the product space.

The paper is arranged as follows. In Section 2, we first recall some notions, symbols and results of Lipschitzian semigroups on X. Secondly, we show that the abstract Cauchy problem (ACP) induced by nonlinear operator A is well posed in the sense of strong solutions if and only if A generates an exponentially bounded Lipschtizian semigroup. Section 3 is devoted to proving the equivalence of the well posedness of FDE and the abstract Cauchy problem $(\mathcal{ACP})$ induced by nonlinear operator $\mathcal{A}$ in the product space $X \times L^p([-1, 0], X)$. Section 4 gives a sufficient condition for $\mathcal{A}$ generating an exponentially bounded Lipschitzian semigroup. Principles of linearized stability are presented for FDE in Section 5. Moreover, the results derived are applied to a reaction–diffusion equation with delays.

2. Preliminaries

Throughout this paper, let X and Y be Banach spaces over the same coefficient field $\mathbb{K}\,(= \mathbb{R}$ or $\mathbb{C})$. Let $C \subset X$ and $D \subset Y$ be their respective subsets. A mapping $T : C \to D$ is called Lipschitz (or Lipschitz continuous) if there exists a real constant $M > 0$ such that $\|Tx - Ty\| \le M\|x - y\|$ for all $x, y \in C$, where the constant M is usually said to be Lipschitz constant of T on C. The minimum Lipschitz constant of T on C, denoted by $L(T)$, can be computed by

$$L(T) = \sup_{x,y\in C, x\neq y} \frac{\|Tx - Ty\|}{\|x - y\|}.$$

It is easy to check that the nonnegative functional $L(\cdot)$ is a seminorm of the space $Lip(C, D)$ of all Lipschitz operators from C into D.

Definition 1 (*[13]*). A one-parameter family $(T(t))_{t\ge 0}$ of Lipschitz operators from C into itself is called Lipschitzian semigroup on C if it possesses the following two properties: (i) $T(0) = I$ (the identity operator on C), $T(t)T(s) = T(t + s)$ for all $t, s \ge 0$; and (ii) the mapping $t \mapsto T(t)x$ is continuous at $t = 0$ for every $x \in C$.

Moreover, Lipschitzian semigroup $(T(t))_{t\ge 0}$ is called exponentially bounded if there exist two constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that $L(T(t)) \le Me^{\omega t}$ for all $t \ge 0$.

From [13], we have already learned that Lipschitzian semigroup is a more general type of C_0-semigroup [9], semigroups of contractions [14,15], semigroups of ω-type [15] and uniformly k-Lipschitzian semigroups [16]. Unlike linear strongly continuous semigroups, it is unknown whether every Lipschitzian semigroup is exponentially bounded. However, we can give an equivalent characterization for exponential boundedness of Lipschitzian semigroups.

Lemma 1 (*[13]*). *Let $(T(t))_{t\ge 0}$ be a Lipshitzian semigroup on C. Then, $(T(t))_{t\ge 0}$ is exponentially bounded if and only if* $\limsup_{t\to 0^+} L(T(t)) < \infty$.

Definition 2 (*[13]*). Let $(T(t))_{t\ge 0}$ be a Lipschitzian semigroup on C, and let

$$D(A) = \left\{ x \in C : \text{ the limit } \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}.$$

If $D(A)$ is nonempty, then we say that $(T(t))_{t\ge 0}$ possesses a generator A, which is defined by

$$A : D(A) \subset C \longrightarrow X, \quad Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

We begin to investigate abstract Cauchy problem

$$(ACP) \qquad \begin{cases} u'(t) = Au(t), & t > 0 \\ u(0) = x, \end{cases}$$

where $A : D(A) \subseteq X \to X$ is a nonlinear operator. By virtue of [14] we recall that a function $u(\cdot) \in C([0, \infty), X)$ is called a strong solution of (ACP) if

(i) $u(\cdot)$ is Lipschitz continuous on every compact interval of $[0, \infty)$;

(ii) $u(t)$ is differentiable for almost every$t \in (0, \infty)$;

(iii) $u'(t) = Au(t)$, *a.e.* $t > 0$; $u(0) = x$.

In fact, $u(\cdot)$ is a strong solution of (ACP) if and only if

$$u(t) \in D(A), \quad a.e.\ t > 0; \quad u(t) = x + \int_0^t Au(r)\,\mathrm{d}r, \quad t \ge 0.$$

Definition 3. The abstract Cauchy problem (ACP) is well posed if for arbitrary $x \in D(A)$, (ACP) has a strong solution $u(t, x)$ with the initial value x and there exists a locally bounded function $p(t) > 0$ such that strong solutions depend on the initial values by the following relation

$$\|u(t,x)-u(t,y)\| \le p(t)\|x-y\| \tag{2.1}$$

where $u(t,x)$ and $u(t,y)$ are strong solutions of ACP with initial values x and y, respectively.

Remark 1. From the definition of strong solution, we can easily derive that ACP cannot have a strong solution for $x \notin \overline{D(A)}$. Accordingly, the condition $x \in D(A)$ is necessary in the definition of well posedness.

Definition 4 (*[17]*). X is said to enjoy the Radon–Nikodym property if every Lipschitz continuous function $F : \mathbb{R} \to X$ is almost everywhere differentiable.

Lemma 2. *Let $(T(t))_{t\ge 0}$ be an exponentially bounded Lipschitzian semigroup on X with its generator A. Then, $x \in D_s$ if and only if for arbitrary $T > 0$, the function $t \mapsto T(t)x$ is Lipschitz continuous in $[0,T]$, where*

$$D_s = \left\{ x \in X : \liminf_{t\to 0^+} \frac{\|T(t)x - x\|}{t} < \infty \right\}.$$

We omit the proof of Lemma 2 because Lemma 2 is a direct generalization of the result of nonlinear contraction semigroups in [14].

Theorem 1. *Assume X to possess the Radon–Nikodym property and C to be a closed subset of X. If A is the generator of some exponentially bounded Lipschtizian semigroup on C, ACP is well posed.*

Conversely, if ACP is well posed, there exists an exponentially bounded Lipschitzian semigroup $(T(t))_{t\ge 0}$ on $\overline{D(A)}$ with its generator A_1 such that $AT(t)x = A_1T(t)x$ a.e. $t > 0$ for $x \in D(A)$.

Proof. Assume A to be the generator of Lipschitzian semigroup $(T(t))_{t\ge 0}$ with $L(T(t)) \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$. Let $u(t,x) = T(t)x$ for every $x \in D(A)$. It is evident that $u(\cdot,x)$ is continuous in $[0,\infty)$. The combination of $D(A) \subset D_s$ and Lemma 2 implies that $u(\cdot,x)$ is Lipschitz continuous in every compact interval of $[0,\infty)$ for $x \in D(A)$. From the Radon–Nikodym property of X, we can easily conclude that $u(t,x)$ is differentiable for almost every $t \in (0,\infty)$. According to the definition of generator, we have $u'(t,x) = Au(t,x)$ a.e. $t > 0$. Accordingly, $u(\cdot,x)$ is a strong solution of ACP. Moreover, let $x, y \in D(A)$, we have

$$\|u(t,x)-u(t,y)\| = \|T(t)x - T(t)y\| \le M\mathrm{e}^{\omega t}\|x-y\|. \tag{2.2}$$

Let $p(t) = M\mathrm{e}^{\omega t}$ and it is obvious that $p(t)$ is locally bounded. Hence ACP is well posed.

Conversely, ACP is assumed to be well posed. Let $T(t)x = u(t,x)$ for all $x \in D(A)$, where $u(t,x)$ is a strong solution of ACP with the initial value x. Evidently, $T(0)x = x$ and $T(\cdot)x$ is continuous in $[0,\infty)$. Let $s > 0$ such that $T(s)x = u(s,x) \in D(A)$. Then, $u(t,x) := u(t,T(s)x)$ is a strong solution of ACP with the initial value $T(s)x$. Moreover, let $v(t) = u(t+s,x)$. Then, $v(\cdot)$ is continuous in $[0,\infty)$ and differentiable for almost every $t \in (0,\infty)$ and

$$v'(t) = u'(t+s,x) = Au(t+s,x) = Av(t), \quad \text{a.e. } t > 0.$$

This means that $v(t)$ is also a strong solution of ACP with the initial value $T(s)x = u(s,x)$. From the uniqueness of the strong solution, we derive that $v(t) \equiv u(t)$, i.e.,

$$T(t)T(s)x = T(t+s)x,\ t \in [0,\infty), \quad \text{a.e. } s > 0. \tag{2.3}$$

For every $x \in \overline{D(A)}$, there exists $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to x$ as $n \to \infty$. From the local boundedness of $p(t)$, we derive that the following inequality

$$\|T(t)x_n - T(t)x_m\| \le p(t)\|x_n - x_m\|$$

implies that $\{T(t)x_n\}_{n\in\mathbb{N}}$ is a uniform Cauchy sequence in X for t in every compact of $[0,\infty)$. Let $y_t = \lim_{n\to\infty} T(t)x_n$. Then, $y_t \in \overline{D(A)}$ and without loss of generality, take $y_t = T(t)x$. It is evident that $T(\cdot)x$ is continuous on $[0,\infty)$ in the light of the uniformity of the limit for t. By virtue of (2.3), $(T(t))_{t\ge 0}$ is a Lipschitzian semigroup on $\overline{D(A)}$. Moveover, owing to $L(T(t)) \le p(t)$, the local boundedness of $p(t)$ and Lemma 1 imply that $(T(t))_{t\ge 0}$ is exponentially bounded.

Let A_1 be the generator of $(T(t))_{t\ge 0}$ and $x \in D(A)$. If $T(\cdot)x$ is differentiable at t, then $T(t)x \in D(A_1)$ and $T'(t)x = A_1T(t)x$ in the light of the definition of the generator. In addition, $T(t)x$ is a strong solution of ACP, which implies that $T(t)x$ is differentiable for almost every $t \in (0,\infty)$ and $T'(t)x = AT(t)x$ at differential point t. Consequently, $AT(t)x = A_1T(t)x$ for $x \in D(A)$ and a.e. $t > 0$. $\square$

Definition 5 (*[14]*). Let $(T(t))_{t\ge 0}$ be an exponentially bounded Lipschitzian semigroup on the closed subset C of X. $(T(t))_{t\ge 0}$ is generated by A if the following conditions hold:

(1) $\overline{D(A)} = C$;
(2) for each $x \in D(A)$, $u(t,x) = T(t)x$ is a strong solution of (ACP).

Remark 2. Definition 2 is equivalent to Definition 5 for strongly continuous semigroup of bounded linear operators [9]. For a nonlinear exponentially bounded Lipschitzian semigroup $(T(t))_{t\ge 0}$, however, Definitions 2 and 5 cannot imply each other in general. First, Definition 2 does not imply Definition 5 in a general Banach space. Indeed, when A is the generator of $(T(t))_{t\geq 0}$ and $x \in D(A)$, Definition 2 does not guarantee the differentiability of the map $t \mapsto T(t)x$ at almost all $t > 0$, that is, $T(t)x$ may not be a strong solution of ACP for $x \in D(A)$. Moreover, the generator of a nonlinear Lipschitzian semigroup may not be densely defined and even some nonlinear Lipschitzian semigroup may not possess generator [14]. Secondly, it is still open whether Definition 5 implies Definition 2, although it can be shown by Theorem 1 that the coefficient operator A of ACP is contained in the generator A_1 of an exponentially bounded Lipschitz semigroup.

It can be shown by Theorem 1 that Definition 2 implies Definition 5 in the special case that the generator is densely defined and X has the Radon–Nikodym property.

Corollary 1. *Let X enjoy the Radon–Nikodym property. ACP is well posed if and only if A generates an exponentially bounded Lipschitzian semigroup on $\overline{D(A)}$.*

3. The semigroup framework

By means of results obtained in Section 2, we investigate the functional differential equation

$$\text{(FDE)}\qquad \begin{cases} u'(t) = Au(t) + \Phi u_t, & t > 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where

- $A : D(A) \subseteq X \to X$ is a closed and densely defined linear operator;
- $f \in L^p([-1, 0], X)$, $1 < p < \infty$;
- $\Phi : L^p([-1, 0], X) \to X$ is a Lipschitz continuous operator;
- $u : [-1, \infty) \to X$ and $u_t : [-1, 0] \to X$ is defined by $u_t(\sigma) = u(t + \sigma)$ for $\sigma \in [-1, 0]$.

A function $u(\cdot) \in C([-1, \infty), X)$ is said to be a strong solution of FDE if

(i) $u(t)$ and u_t are Lipschitz continuous on every compact interval of $[0, \infty)$;
(ii) $u(t)$ are differentiable and $u(t) \in D(A)$, $u_t(\cdot) \in W^{1,p}([-1, 0], X)$ for almost every $t \in (0, \infty)$;
(iii) $u(0) = x$, $u_0 = f$ and $u'(t) = Au(t) + \Phi u_t$ holds for almost all $t \geq 0$.

In the sequel, we adopt the Banach space $\mathcal{X} = X \times L^p([-1, 0], X)$ with the norm $\|\cdot\|_{\mathcal{X}}$ defined by

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{X}} = \max\left\{ \|x\|, \|f\|_{L^p} \right\} \quad \text{for all } \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{X}$$

and define the operator $\mathcal{A}$ in $\mathcal{X}$ by

$$\mathcal{A} = \begin{pmatrix} A & \Phi \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \tag{3.1}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1, 0], X) : f(0) = x \right\}. \tag{3.2}$$

Obviously, the operator $\mathcal{A}$ is closed densely defined in $\mathcal{X}$ and a necessary condition for FDE to have a strong solution is that $u_0 = f \in W^{1,p}([-1, 0], X)$ and $u(0) = x \in D(A)$, i.e., $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{A}$.

Definition 6. Functional differential equation (FDE) is said to be well posed if

(i) for every $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$, there exists a unique strong solution $u(x, f)(\cdot)$ of FDE with the initial value $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ and

(ii) these strong solutions continuously depend on the initial values in the sense that there exists a locally bounded function $w(t) > 0$ such that

$$\|u(x, f)(t) - u(y, g)(t)\| \leq w(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad t \in [0, \infty) \tag{3.3}$$

where $u(x, f)(\cdot)$ and $u(y, g)(\cdot)$ are two strong solutions of FDE with initial values $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} y \\ g \end{smallmatrix}\right) \in D(\mathcal{A})$, respectively.

Let $(\mathcal{ACP})$ be the abstract Cauchy problem associated with $\mathcal{A}$, i.e.,

$$(\mathcal{ACP})\qquad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), & t > 0, \\ \mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}. \end{cases}$$

In what follows, we characterize the relations between strong solutions of FDE and $(\mathcal{ACP})$.

Proposition 1. *Let $\binom{x}{f} \in D(\mathcal{A})$ and $u(\cdot) : [-1, \infty) \to X$ be a strong solution of FDE. Then, the function*

$$\mathbb{R}_+ \ni t \longmapsto \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{X}$$

is a strong solution of $(\mathcal{ACP})$ with the initial value $\binom{x}{f}$.

The proof is directly derived from the definition of a strong solution of FDE.

Proposition 2. *Let $\binom{x}{f} \in D(\mathcal{A})$ and $\mathcal{U}(\cdot) : [0, \infty) \to \mathcal{X}$, $\mathcal{U}(t) = \binom{z(t)}{v(t)}$ be a strong solution of $(\mathcal{ACP})$ with the initial value $\binom{x}{f}$. Let $u : [-1, \infty) \to X$ be the function defined by*

$$u(t) = \begin{cases} z(t), & t \geq 0, \\ f(t), & t \in [-1, 0). \end{cases} \tag{3.4}$$

Then, $u_t = v(t)$ for almost every $t \geq 0$ and u is a strong solution of FDE.

Proof. Since $\mathcal{U}$ is a strong solution of $(\mathcal{ACP})$, $v \in C([0, \infty), L^p([-1, 0], X))$ and $v(t)$ is differentiable for almost every $t \in (0, \infty)$. Without loss of generality, we assume that $v(t)$ is differentiable at $t = s$, and then $v'(s)$ is a linear bounded operator from $[0, \infty)$ to $L^p([-1, 0], X)$. Furthermore, $v(t)$ solves the Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \frac{\mathrm{d}}{\mathrm{d}\sigma} v(t), \quad a.e.\ t > 0, \\ &v(t)(0) = z(t), \quad t \geq 0, \\ &v(0) = f, \end{aligned} \tag{3.5}$$

in the space $L^p([-1, 0], X)$. It should be noted that the map $\mathbb{R}_+ \ni t \longmapsto u_t \in L^p([-1, 0], X)$ solves (3.5) in the space $L^p([-1, 0], X)$. Therefore, let $q(t) = u_t - v(t)$ for $t \geq 0$. Then q is a strong solution of the abstract Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} q(t) = \frac{\mathrm{d}}{\mathrm{d}\sigma} q(t), \quad t > 0, \\ &q(t)(0) = 0, \quad t \geq 0, \\ &q(0) = 0. \end{aligned} \tag{3.6}$$

Since (3.6) is the abstract Cauchy problem associated with the generator of nilpotent left transform semigroup on $L^p([-1, 0], X)$ with the initial value 0, we derive that $q(t) = 0$ for almost every $t > 0$. Consequently, $u(\cdot) : [-1, \infty) \to X$ defined by (3.4) is a strong solution of FDE. □

Let $P_1 : \mathcal{X} \longrightarrow X$ be the projection onto the first component of $\mathcal{X}$, i.e.,

$$P_1 \begin{pmatrix} x \\ f \end{pmatrix} = x \quad \text{for all } \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{X}.$$

Corollary 2. *Assume X to enjoy the Radon–Nikodym property. Let $\mathcal{A}$ generate an exponentially bounded Lipschitzian semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{X}$. Then, FDE enjoys a unique strong solution $u(\cdot)$ for every $\binom{x}{f} \in D(\mathcal{A})$, which is determined by*

$$u(t) = \begin{cases} P_1 \left(\mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix} \right), & t \geq 0, \\ f(t), & t \in [-1, 0). \end{cases} \tag{3.7}$$

Proof. $L^p([-1, 0], X)$ $(1 < p < \infty)$ enjoys the Radon–Nikodym property because $L^p([-1, 0], X)$ $(1 < p < \infty)$ is reflexive and every reflexive space has the Radon–Nikodym property. Hence, the product space $\mathcal{X}$ possesses Radon–Nikodym property. Since $\mathcal{A}$ is the generator of the exponentially bounded Lipschitzian semigroup $(\mathcal{T}(t))_{t \geq 0}$, according to Corollary 1, we conclude that $(\mathcal{ACP})$ is well posed, i.e., $(\mathcal{ACP})$ possesses a unique strong solution $\mathcal{U}(\cdot)$ for all $\binom{x}{f} \in D(\mathcal{A})$ determined by

$$\mathcal{U}(t) = \mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix}, \quad t \geq 0.$$

It is evident that the map $u(\cdot)$ determined in (3.7) is a unique strong solution of FDE in the light of Proposition 2. □

Theorem 2. *Assume X to enjoy the Radon–Nikodym property. Let $\mathcal{A}$ be the operator defined by (3.1) and (3.2). Then, the following statements are equivalent:*

(i) *FDE is well posed.*
(ii) *$\mathcal{A}$ generates an exponentially bounded Lipschitzian semigroup on $\mathcal{X}$.*

Proof. $(i) \Rightarrow (ii)$ Since FDE is well posed, there exists a unique strong solution of FDE for every $\binom{x}{f} \in D(\mathcal{A})$. Then, from Propositions 1 and 2, we can derive that $(\mathcal{ACP})$ has a unique strong solution for every $\binom{x}{f} \in D(\mathcal{A})$.

Now, we begin to prove that strong solutions of $(\mathcal{ACP})$ continuously depend on their respective initial values by the relation (2.1). From (3.3), we have

$$\|u(x,f)(t) - u(y,g)(t)\| \le w(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad t \in [0,\infty) \tag{3.8}$$

where $u(x,f)(\cdot)$ and $u(y,g)(\cdot)$ are two strong solutions of FDE with initial values $\binom{x}{f}, \binom{y}{g} \in D(\mathcal{A})$, respectively, and $w(\cdot)$ is a positive locally bounded function.

For $t \ge 1$, we have

$$\begin{aligned}
\|u_t(x,f) - u_t(y,g)\|_{L^p} &= \left(\int_{-1}^{0} \|u(x,f)(t+s) - u(y,g)(t+s)\|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \\
&= \left(\int_{t-1}^{t} \|u(x,f)(r) - u(y,g)(r)\|^p \,\mathrm{d}r \right)^{\frac{1}{p}} \\
&\le \left(\int_{t-1}^{t} w(r)^p \mathrm{d}r \right)^{\frac{1}{p}} \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \\
&\le \omega(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}},
\end{aligned}$$

where $\omega(t) = \max_{-1\le s\le 0} w(t+s)$. Obviously, $\omega(t)$ is also a positive locally bounded function on $[0,\infty)$. Hence, from definition of $\|\cdot\|_{\mathcal{X}}$, we immediately derive

$$\left\| \begin{pmatrix} u(x,f)(t) \\ u_t(x,f) \end{pmatrix} - \begin{pmatrix} u(y,g)(t) \\ u_t(y,g) \end{pmatrix} \right\|_{\mathcal{X}} \le p_1(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad \text{for all } t \ge 1,$$

where $p_1(t) = \max\{\omega(t), w(t)\}$ is positive locally bounded.

For $0 \le t \le 1$, we have

$$\begin{aligned}
\|u_t(x,f) - u_t(y,g)\|_{L^p} &= \left(\int_{-1}^{0} \|u(x,f)(t+s) - u(y,g)(t+s)\|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \\
&= \left(\int_{-1}^{-t} \|f(t+s) - g(t+s)\|^p \,\mathrm{d}s + \int_{-t}^{0} \|u(x,f)(t+s) - u(y,g)(t+s)\|^p \,\mathrm{d}s \right)^{\frac{1}{p}} \\
&= \left(\int_{t-1}^{0} \|f(\theta) - g(\theta)\|^p \,\mathrm{d}\theta + \int_{0}^{t} \|u(x,f)(\theta) - u(y,g)(\theta)\|^p \mathrm{d}\theta \right)^{\frac{1}{p}} \\
&\le \left(\int_{-1}^{0} \|f(\theta) - g(\theta)\|^p \,\mathrm{d}\theta + \int_{0}^{t} \|u(x,f)(\theta) - u(y,g)(\theta)\|^p \mathrm{d}\theta \right)^{\frac{1}{p}} \\
&= \left(\|f-g\|_{L^p}^p + \int_{0}^{t} \|u(x,f)(\theta) - u(y,g)(\theta)\|^p \mathrm{d}\theta \right)^{\frac{1}{p}} \\
&\le 2^{\frac{1}{p}} \left(\|f-g\|_{L^p} + \left(\int_{0}^{t} \|u(x,f)(\theta) - u(y,g)(\theta)\|^p \mathrm{d}\theta \right)^{\frac{1}{p}} \right) \\
&\le 2^{\frac{1}{p}} \left(1 + \tilde{\omega}(t)\right) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}},
\end{aligned}$$

where $\tilde{\omega}(t) = t^{\frac{1}{p}} \max_{0\le s\le t} w(s)$. Consequently, we derive

$$\left\| \begin{pmatrix} u(x,f)(t) \\ u_t(x,f) \end{pmatrix} - \begin{pmatrix} u(y,g)(t) \\ u_t(y,g) \end{pmatrix} \right\|_{\mathcal{X}} \le p_2(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad \text{for all } 0 \le t \le 1,$$

where $p_2(t) = \max\{w(t), 2^{\frac{1}{p}}(1 + \tilde{\omega}(t))\}$ is positive locally bounded.

On all accounts, we have

$$\left\| \begin{pmatrix} u(x,f)(t) \\ u_t(x,f) \end{pmatrix} - \begin{pmatrix} u(y,g)(t) \\ u_t(y,g) \end{pmatrix} \right\|_{\mathcal{X}} \le p(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad \text{for all } t \ge 0,$$

where $p(t) = \max\{p_1(t), p_2(t)\}$ is positive locally bounded.

Accordingly, $(\mathcal{ACP})$ is well posed. Since $\mathcal{X}$ has the Radon–Nikodym property and $\mathcal{A}$ is closed densely defined, according to Corollary 1, we conclude that $\mathcal{A}$ generates an exponentially bounded Lipshictzian semigroup on $\mathcal{X}$.

(ii) $\Rightarrow$ *(i)* Without loss of generality, we assume that $(\mathcal{T}(t))_{t\ge 0}$ is the exponentially bounded Lipschitzian semigroup on $\mathcal{X}$ generated by $\mathcal{A}$. Since $\mathcal{X}$ has the Radon–Nikodym property, we conclude by Corollary 2 that FDE have a unique strong solution given by (3.7) for every initial value $\binom{x}{f} \in D(\mathcal{A})$. Furthermore, we have

$$\begin{aligned} \|u(x,f)(t) - u(y,g)(t)\| &= \left\| P_1\left(\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix}\right) - P_1\left(\mathcal{T}(t)\begin{pmatrix} y \\ g \end{pmatrix}\right) \right\| \\ &\le \|P_1\| L(\mathcal{T}(t)) \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} y \\ g \end{pmatrix} \right\|_{\mathcal{X}} \quad \text{for } \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \in D(\mathcal{A}) \end{aligned}$$

i.e., FDE is well posed. □

4. The generation condition

In the preceding section, we have formulated the solvable problem of the FDE as the abstract Cauchy problem $(\mathcal{ACP})$ on $\mathcal{X}$. Then, we consider in the section when $\mathcal{A}$ generates an exponentially bounded Lipschitzian semigroup on $\mathcal{X}$. For this, we begin with the following well-known result in the framework of Lipschitzian semigroups.

Lemma 3 (*[18]*). *Let A be the infinitesimal generator of linear strongly continuous semigroup $(T(t))_{t\ge 0}$ on X satisfying $\|T(t)\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$ and some $\omega \in \mathbb{R}$, $M \ge 1$. If $K \in Lip(X, X)$, then $A+K$ is the generator of an exponentially bounded Lipschitzian semigroup $(S(t))_{t\ge 0}$ on X satisfying $L(S(t)) \le M\mathrm{e}^{(\omega+M\cdot L(K))t}$ for all $t \ge 0$.*

Remark 3. It is obvious that $D(A+K) = D(A)$ is densely defined in X. If X has the Radon–Nikodym property, $A+K$ generates $(S(t))_{t\ge 0}$ from Remark 2.

We consider the sum form $\mathcal{A}_0 + \mathcal{K}$ of $\mathcal{A}$ where

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \tag{4.1}$$

with domain

$$D(\mathcal{A}_0) = D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1,0], X) : f(0) = x \right\} \tag{4.2}$$

and

$$\mathcal{K} = \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in Lip(\mathcal{X}, \mathcal{X}). \tag{4.3}$$

Lemma 4 (*[1, Proposition3.1]*). *Let A generate a linear strongly continuous semigroup $(T(t))_{t\ge 0}$ on X. Then $\mathcal{A}_0$ generates the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\ge 0}$ on $\mathcal{X}$ given by*

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ T_t & T_0(t) \end{pmatrix}, \tag{4.4}$$

where $(T_0(t))_{t\ge 0}$ is the nilpotent left translation semigroup on $L^p([-1,0], X)$ and $T_t : X \to L^p([-1,0], X)$ is defined by

$$(T_t x)(\tau) = \begin{cases} T(t+\tau)x, & -t < \tau \le 0, \\ 0, & -1 \le \tau \le -t. \end{cases}$$

Accordingly, if A is a generator of a linear strongly continuous semigroup on X, from Lemmas 3 and 4 and Remark 3, we can immediately derive that $\mathcal{A}_0 + \mathcal{K}$ generates an exponentially bounded Lipschitzian semigroup on $\mathcal{X}$.

Theorem 3. *Assume X to enjoy the Radon–Nikodym property. Let A be the generator of a linear strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. Then, $\mathcal{A}$ generates an exponentially Lipschitzian semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. Therefore, FDE is well posed in the sense of Definition 6.*

5. Principles of linearized stability for FDE

In this section, we plan to study the stability of FDE by a linearized method. For this aim, we firstly recall some basic concepts.

Let $(T(t))_{t\geq 0}$ be a linear strongly continuous semigroup on X with generator A. The *uniform growth bound* or *type* of the semigroup

$$\omega_0(A) = \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\| \leq M e^{\omega t}, \forall t \geq 0\}.$$

$(T(t))_{t\geq 0}$ is uniformly exponentially stable if $\omega_0(A) < 0$.

Definition 7. x^* is called an equilibrium point of FDE if $A(x^*) + \Phi(x^*) = 0$.

Without loss of generality, let $x^* = 0$. We assume that Φ is differentiable at $x^* = 0$ and $\Phi(0) = 0$. Then, the linearized equation of FDE at 0 is

$$\begin{cases} u'(t) = Au(t) + Lu_t, & t > 0, \\ u(0) = x, \\ u_0 = f, \end{cases} \tag{5.1}$$

where $L : L^p([-1, 0], X) \to X$ is a linear bounded operator and $L = \Phi'(0)$.

It is proved in [1] that (5.1) is equivalent to

$$\begin{cases} \mathcal{U}'(t) = \mathcal{B}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{cases} \tag{5.2}$$

where $\mathcal{B} : D(\mathcal{B}) \to \mathcal{X}$ is defined by

$$\mathcal{B} = \begin{pmatrix} A & L \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}$$

and with domain

$$D(\mathcal{B}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1, 0], X) : f(0) = x \right\}.$$

Moreover, it is showed in [1] that (5.1) is well defined in the sense of classical solutions if and only if $\mathcal{B}$ generates a strongly continuous semigroup $\mathcal{T}_{\mathcal{B}} = (\mathcal{T}_{\mathcal{B}}(t))_{t\geq 0}$. It is well known that if $\mathcal{T}_{\mathcal{B}}$ is uniformly exponentially stable, 0 is a locally exponentially stable equilibrium point of nonlinear Lipschitzian semigroup $(\mathcal{T}(t))_{t\geq 0}$, i.e., there exist positive constants N_1, N_2, N_3 such that

$$\left\| \mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{X}} \leq N_2 e^{-N_3 t} \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{X}} \quad \text{for } \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{X} \text{and } \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{X}} \leq N_1.$$

Subsequently, it is only needed to characterize uniform exponential stability of $\mathcal{T}_{\mathcal{B}}$. Let $F(J, Y)$ be the Banach space consisting of all Y-valued functions with domain $J \subseteq \mathbb{R}$. Let $T : X \to Y$ be bounded linear and $f : J \to \mathbb{C}$. If the map $f \otimes y : J \ni s \to f(s)y$ belongs to $F(J, Y)$, we define the linear operator $f \otimes T : X \to F(J, Y)$ by

$$((f \otimes T)x)(s) = (f \otimes Tx)(s) = f(s) \cdot Tx, \quad x \in X, s \in J.$$

The function $\epsilon_\lambda : \mathbb{R} \to \mathbb{R}^+$ is defined by $\epsilon_\lambda(s) = e^{\lambda s}$ for $s \in \mathbb{R}$.

Lemma 5 ([1]). *Take $p = 2$ and let X be a Hilbert space. If $\omega_0(A) < \alpha \leq 0$ and*

$$\sup_{\omega\in\mathbb{R}} \|L(\epsilon_{\alpha+i\omega} \otimes I)\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha + i\omega, A)\|},$$

then $\omega_0(\mathcal{B}) < \alpha \leq 0$, i.e., the semigroup $\mathcal{T}_{\mathcal{B}}$ is uniformly exponentially stable.

Consequently, we immediately derive the following local stability result.

Theorem 4. *Take $p = 2$ and let X be a Hilbert space. If $\omega_0(A) < \alpha \leq 0$ and*

$$\sup_{\omega\in\mathbb{R}} \|L(\epsilon_{\alpha+i\omega} \otimes I)\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha + i\omega, A)\|},$$

then, 0 is a locally exponentially stable equilibrium point of nonlinear Lipschitzian semigroup $\mathcal{T}$.

Lemma 6 ([1]). *Assume that A generates an immediately norm continuous semigroup, $\omega_0(A) < 0$ and let $\alpha \in (\omega_0(A), 0]$. If*

$$\sup_{\omega\in\mathbb{R}} \|L(\epsilon_{\alpha+\mathrm{i}\omega} \otimes I)\| < \frac{1}{\sup\limits_{\omega\in\mathbb{R}} \|R(\alpha + \mathrm{i}\omega, A)\|}, \tag{5.3}$$

then $\omega_0(\mathcal{B}) < \alpha \leq 0$.

Theorem 5. *Assume that A generates an immediately norm continuous semigroup, $\omega_0(A) < 0$ and let $\alpha \in (\omega_0(A), 0]$. If*

$$\sup_{\omega\in\mathbb{R}} \|L(\epsilon_{\alpha+\mathrm{i}\omega} \otimes I)\| < \frac{1}{\sup\limits_{\omega\in\mathbb{R}} \|R(\alpha + \mathrm{i}\omega, A)\|}, \tag{5.4}$$

then 0 is a local exponentially stable equilibrium point of nonlinear semigroup $\mathcal{T}$.

Lemma 7 ([19]). *Let $1 \leq p < \infty$. If the semigroup $(T(t))_{t\geq 0}$ generated by A is immediately compact, $\mathcal{T}_{\mathcal{B}}(t)$ is compact for $t \geq 1$, i.e., $\mathcal{T}_{\mathcal{B}}$ is eventually compact.*

It is well known that every eventually compact semigroup is eventually norm continuous. Consequently, we derive the following local stability criterion.

Theorem 6. *Assume that A generates an immediately compact semigroup, $\omega_0(A) < 0$ and let $\alpha \in (\omega_0(A), 0]$. If*

$$\sup_{\omega\in\mathbb{R}} \|L(\epsilon_{\alpha+\mathrm{i}\omega} \otimes I)\| < \frac{1}{\sup\limits_{\omega\in\mathbb{R}} \|R(\alpha + \mathrm{i}\omega, A)\|}, \tag{5.5}$$

then 0 is a local exponentially stable equilibrium point of nonlinear semigroup $\mathcal{T}$.

Example 1. Consider the following reaction–diffusion equation with delays

$$\begin{aligned}
&\frac{\partial}{\partial t} y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) + \sum_{k=0}^{n} B_k y(x, t+h_k), \quad x \in \Omega, t \geq 0, h_k \in [-1, 0],\\
&y(x,t) = 0, \quad x \in \partial\Omega, t \geq 0,\\
&y(x,t) = f(x,t), \quad (x,t) \in \Omega \times [-1, 0],
\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $B_k \in Lip(L^2(\Omega), L^2(\Omega))$ and $B_k(0) = 0$ for $k = 0, 1, \ldots, n$, $f(\cdot, t) \in L^2(\Omega)$ for all $t \geq 0$, $f(\cdot, 0) \in W^{1,2}(\Omega)$ and the map $[-1, 0] \ni t \mapsto f(\cdot, t) \in L^2(\Omega)$ belongs to $W^{1,2}([-1, 0], L^2(\Omega))$.

Take $X := L^2(\Omega)$, $\Phi y_t := \sum_{k=0}^{n} B_k y(x, t + h_k)$ and $A := \Delta$ where Δ is the Laplacian operator with usual domain. It is well known that Δ generates a compact semigroup on X and satisfies $(0, +\infty) \subset \rho(\Delta)$ and $\|R(\lambda, \Delta)\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. Lipschitz continuity of B_k implies that Φ is Lipschitz continuous form $L^2([-1, 0], X)$ to X. Then, the reaction–diffusion equation can be reformulated in X as follows:

$$\begin{cases} u'(t) = \Delta u(t) + \Phi u_t, & t \geq 0,\\ u(0) = x,\\ u_0 = f. \end{cases} \tag{5.6}$$

Hence, the well posedness follows from Theorem 3, i.e., $\mathcal{A}$ generates a nonlinear Lipschitzian semigroup on $\mathcal{X} = L^2(\Omega) \times L^2([-1, 0], L^2(\Omega))$, where $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} \Delta & \Phi \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \tag{5.7}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Delta) \times W^{1,2}([-1, 0], X) : f(0) = x \right\}. \tag{5.8}$$

It is obvious that 0 is an equilibrium point of the reaction–diffusion equation. Assume that B_k is differentiable at 0 for $k = 0, 1, \ldots, n$. The linearized equation of (5.6) at 0 is

$$\begin{cases} u'(t) = \Delta u(t) + L u_t, & t \geq 0,\\ u(0) = x,\\ u_0 = f, \end{cases} \tag{5.9}$$

where $L = \sum_{k=0}^{n} B'_k(0)$. We should note that L satisfies

$$\|L(\epsilon_\lambda \otimes I)x\| = \left\| \sum_{k=0}^{n} B'_k(0)\mathrm{e}^{\lambda s} x \right\| \leq \sum_{k=0}^{n} \|B'_k(0)\| \|x\|, \quad s \in [-1, 0], x \in \Omega. \tag{5.10}$$

Since A is a normal operator on a Hilbert space (see, [20, Sect. V.3.8]), we have

$$\sup_{\omega \in \mathbb{R}} \|R(\mathrm{i}\omega, A)\| = \sup_{\omega \in \mathbb{R}} \frac{1}{\mathrm{d}(\mathrm{i}\omega, \sigma(A))} = \frac{1}{\mathrm{d}(0, \sigma(A))} = \frac{1}{|\lambda_1|} \tag{5.11}$$

where λ_1 is the first eigenvalue of the Laplacian operator. Consequently, from Lemma 6, we conclude that the solutions of (5.9) are uniformly exponentially stable if

$$\sum_{k=0}^{n} \|B'_k(0)\| \leq \frac{1}{|\lambda_1|}. \tag{5.12}$$

From Theorem 6, we know that 0 is locally exponentially stable equilibrium point of nonlinear semigroup generated by $\mathcal{A}$ defined by (5.7) and (5.8).

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