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The Generating Theorem of Integrated Semigroups and the Wellposedness of Abstract Cauchy Problem

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1. Introduction

Let E be Banach space. A family of bounded linear operators $S(t)(t \geqslant 0)$ on E is called "integrated semigroup" iff S(t) is strongly continuous and satisfies:

$$S(0)=0, \quad S(t)S(s)x=\int_s^{t+s} S(r)x\,dr-\int_0^t S(r)x\,dr, \quad t,s \geqslant 0, x \in E$$

An unbounded linear operator A: D(A)→E is called the generator of S(t) iff for each x ∈ D(A) S(t)x is continuously differentiable in t and $S'(t)x = AS(t)x+x = S(t)Ax+x$(see [1]—[6])

The concept of integrated semigroup was introduced by Arendt[1] in 1987, since then the theory has been developed rapidly, and applied successfully to abstract Cauchy problem instead of C_0–semigroup. In order that integrated semigroup can be applied to study a given Cauchy problem, it is of key that A can generate an integrated semigroup, therefore, looking for the conditions that A generates integrated semigroup is a very important subject. Arendt[2] and Neubrander[3] obtained the following results:

Theorem[2][3]: *Let A be a densly defined linear operator, then the following assertions are equivalent:*

i) A generates integrated semigroup $S(t)$, $\| S(t) \| \leqslant Me^{\omega t}$.

ii) There exists $0 \leqslant \omega \in R$ *such that* $(\omega,\infty) \subseteq \rho(A)$ *and for all* $k \in \{0\} \cup N$ *Hille–yosida conditions are satisfied:*

$$\| (\lambda-\omega)^{k+1}[R(\lambda,A)/\lambda]^{(k)}/k! \| \leqslant M, \text{ for } \lambda>\omega.$$

Theorem[2][3]: *Let A be closed linear operator, then the following assertions are equiva-*

lent:

i) A generates local–Lipschitz continuous integrated semigroup $S(t)$.

ii) There exists $\omega \geqslant 0$ such that $(\omega,\infty) \subseteq \rho(A)$ and the following Hille–yosida conditions are satisfied:

$$\|(\lambda-\omega)^n(\lambda-A)^{-n}\| \leqslant M, \text{ for } n \in N, \lambda > \omega.$$

One main purpose of this paper is to obtain the conditions for A to generate integrated semigroup in case that A need not be densly defined or satisfy the Hille–yosida condition.

For the wellposedness of abstract Cauchy problem, the core problem is to give a proper definition, and under which to examine what class of operators A generates. The classical wellposedness was due to Hadamard[7], Cauchy problem is Hadmard′ s wellposed iff A generates C_0–semigroup[8]–[11]. Considering the situations that A is not densly defined, Neubrander[5] introduced (n,k)–wellposedness, but he only showed that Cauch problem is (n,n−1)–wellposed iff A genrates n−1–times integrated semigroup in cass that A is densly defined. The second part of this paper gives a new wellposedness in which A need not densly defined and the solutions of Cauchy problem need not exponentially bounded, meanwhile, it is showed that Cauchy problem is wellposed iff A generates integrated semigroup.

2. The generating theorem of integrated semigroup.

Lemma2.1[12]: *Let $\Phi(t): [0,\infty) \to E$ be continuous and exponentially bounded, then*

$$\lim_{k\to\infty}(k/t)^{k+1} \cdot 1/k! \cdot \int_0^\infty e^{-k\frac{u}{t}} u^k \Phi(u)du = \Phi(t)$$

Uniformly in $[0,\infty)$.

Lemma2.2: *Let A generates integrated semigroup $S(t)$, $\| S(t) \| \leqslant Me^{\omega t}$, then*

(2.1) $$S'(t)x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-(n+1)}x \qquad x \in D(A).$$

Uniformly in $[0,\infty)$.

Proof: Noting that:

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t} S'(t)x dt, \qquad \lambda > \omega.$$

then by Lemma2.1, the Lemma holds.

In fact, since for $x \in D(A)$ $\lambda R(\lambda,A)x \to x$ (as $\lambda \to \infty$), (2.1) can be improved so as: $S'(t)x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}x$ uniformly in compact subset of $(0,\infty)$.

Theorem2.1: *Let A be a closed linear operator, then the following assertions are equivalent:*

i) There exists $\omega \in R$ such that $[\omega,\infty) \subseteq \rho(A)$, for each $x \in D(A)$,

(2.2) $$\|\lambda^n R(\lambda+\omega,A)^n x\| \leqslant M(x), \quad \text{for all } \lambda > 0,\ n \in \{0\} \cup N.$$

and the limit $\alpha(t)=\lim_{n\to\infty}[I-\frac{t}{n}(A-\omega)]^{-n-1}x$ *exists uniformly in compact subset of* $[0,\infty)$.

ii) A generates a exponentially bounded integrated semigroup $S(t)$.

Proof: i)⇒ii). Let $\alpha(n,t)x=[I-\frac{t}{n}(A-\omega]^{-n-1}x=[\frac{n}{t}R(\frac{n}{t},A-\omega)]^{n+1}x$. Obviously, for each n, $\alpha(n,t)x$ is continuous in $(0,\infty)$ and $\alpha(n,0)x=x$. Since for each $x\in D(A)$ $\frac{n}{t}R(\omega+\frac{n}{t},A)x-x$ $=\frac{n}{t}R(\omega+\frac{n}{t},A)\frac{t}{n}(Ax-\omega x)$, then by (2.2),

$\lim_{t\to 0}\frac{n}{t}R(\frac{n}{t}+\omega,A)x=x$, furthermore, $\lim_{t\to 0}\alpha(n,t)x=x$.

For $x\in D(A)$, $\lambda>0$, $n\in N$, the following formula

$$I(\lambda,n)=:\int_0^\infty e^{-\lambda t}\alpha(n,t)x\,dt=\int_0^\infty e^{-\lambda t}[I-\frac{t}{n}(A-\omega)]^{-n-1}x\,dt$$

$$=(-1)^n\frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}[R(t+\omega,A)]^{(n)}x\,dt.$$

$$=(-1)^{n-1}\frac{1}{(n-1)!}\int_0^\infty [R(t+\omega,A)]^{(n-1)}x\frac{\partial}{\partial t}[exp(-\frac{n}{t}\lambda)t^{n-1}]dt$$

holds by condition (2.2). On the analogy of this one can get

$$I(\lambda,n)=\frac{1}{(n-1)!}\int_0^\infty [R(t+\omega,A)]x\frac{\partial^n}{\partial t^n}[exp(-\frac{n}{t}\lambda)t^{n-1}]dt$$

Following Euler formula[12], $\frac{\partial^n}{\partial t^n}[exp(-\frac{n}{t}\lambda)t^{n-1}]=exp(-\frac{n}{t}\lambda)(n\lambda)^n t^{-n-1}$, we have

$$I(\lambda,n)=\frac{1}{(n-1)!}(n\lambda)^n\int_0^\infty e^{-n\lambda u}u^{n-1}R(\frac{1}{u}+\omega,A)x\,du$$

therefore, $I(\lambda,n)\to R(\lambda+\omega,A)x$ (as $n\to\infty$) by Lemma2.1, that is

$$\lim_{n\to\infty}\int_0^\infty e^{-\lambda t}\alpha(n,t)x\,dt=R(\lambda+\omega,A)x,\ \lambda>0,\ x\in D(A).$$

if denote $\lim_{n\to\infty}\alpha(n,t)x$ by $\alpha(t)x$ ($t\geqslant 0$) for each $x\in D(A)$, then $(*)--\int_0^\infty e^{-\lambda t}\alpha(t)x\,dt$ $=R(\lambda+\omega,A)x$ holds by (2.2) and Lebesgue's control convergence theorem.

Define a class of operators $S(t)$: $E\to E$ ($t\geqslant 0$) as following: $S(t)x=R(\omega,A)x-\alpha(t)R(\omega,A)x$ for $x\in E$. Obviously $S(t):E\to E$ is linear for each $t\geqslant 0$ and $S(0)x=0$ for all $x\in E$. Since for each $x\in D(A)$ $\alpha(t)x$ is continuous in $[0,\infty)$, then $S(t)$ is strongly continuous in $[0,\infty)$. For $\lambda>0$, $x\in E$, the equality

$$R(\lambda+\omega,A)x=\lambda\int_0^\infty e^{-\lambda t}S(t)x\,dt$$

follows from (*) and resolvent equation, since then S(t) is a integrated semigroup generated by $A-\omega$, therefore, the assertion ii) holds by the perturbation theorem.

ii)$\Rightarrow$i) It follows from Lemma2.2. ∎

Theorem2.2: *Let E be Hilbert space and A be closed linear operator, if there exists $\omega \in R$ such that $\{\lambda: Re\lambda > \omega\} \subseteq \rho(A)$ and*

$$M^2 =: \sup_{r>\omega} \int_{-\infty}^{\infty} \|R(r+iy,A)\|^2 dy < \infty. \tag{2.3}$$

then A generates integrated semigroup S(t) ($t \geqslant 0$) satisfying $\| S(t) \| \leqslant M e^{at} \sqrt[2]{t}$ and $\| S(t+h)-S(t) \| \leqslant M \sqrt[2]{h}\, e^{a(t+h)}$ ($t \geqslant 0$, $h \geqslant 0$) (in which $a = \max\{0,\omega\}$).

Proof: Without loss of generality, we assume $\omega = 0$. For each $x \in E$, by inequality (2.3) and Paley−winner theorem[13], there exists $g(t)$: $(0,\infty) \to E$ such that $\int_0^{\infty} \|g(t)\|^2 dt \leqslant M^2 \|x\|^2$ and

$$R(\lambda,A)x = \int_0^{\infty} e^{-\lambda t} g(t)dt, \text{ for all } Re\lambda > 0. \tag{2.4}$$

Define $\alpha(\cdot,x)$: $[0,\infty) \to E$ by $\alpha(t,x) = \int_0^t g(s)ds$ ($x \in E$), since $\alpha(\cdot,x)$ is continuous, then, by the uniqueness theorem for Laplace tranform, $\alpha(t,x)$ is linear in $x \in E$, furthermore, S(t): E $\to$ E denoted by $S(t)x = \alpha(t,x)$ ($x \in E$) is a class of strongly continuous linear bouned operators, and $S(0)x = \alpha(0,x) = 0$. From (2.4) $R(\lambda,A)x = \lambda \int_0^{\infty} e^{-\lambda t} S(t)x dt$ follows for all $Re\lambda > 0$, then by [2, Th3.1] S(t) ($t \geqslant 0$) is an integraced semigroup generaced by A, which satisfies that $\| S(t) \|^2 \leqslant \| \int_0^t g(s)ds \|^2 \leqslant t \int_0^t \|g(s)\|^2 ds \leqslant M^2 t$ and $\| S(t+h)-S(t) \| \leqslant M \sqrt[2]{t}$. Therefore, the proof is complete. ∎

Collary2.1: *If there exists $\omega \in R$ such that $\{\lambda: Re\lambda > \omega\} \subseteq \rho(A)$ and $\| R(\lambda,A) \| \leqslant \frac{M}{|\lambda|}$ (for a certain constant $M > 0$), then A generates integrated semigroup S(t), $\| S(t) \| \leqslant M \cdot \sqrt[2]{t}\, e^{at}$, $\| S(t+h)-S(t) \| \leqslant M \sqrt[2]{h}\, e^{a(t+h)}$ ($t \geqslant 0$, $h \geqslant 0$) where $a = \max\{0,\omega\}$.*

Remark: In a general space, since the Paley−winner theorem may be invalid, theorem2.2 need not hold. But we have the following result.

Theorem2.3: *Let E be a Banach space, if there is $\omega \in R$ such that $\{\lambda \in C: Re\lambda > \omega\} \subseteq \rho(A)$ and $\| R(\lambda,A) \| \leqslant \frac{M}{|\lambda|}$ ($Re\lambda > \omega$) (where M is a constant), then A generates integrated semigroup S(t) ($t \geqslant 0$), $\| S(t) \| \leqslant M e^{at}$ (in which $a = \{0,\omega\}$).*

Proof: Let $p > a$ be fixed. For $x \in D(A)$, $Re\lambda \geqslant p$, since $R(\lambda,A)x = 1/\lambda[x+R(\lambda,A)Ax]$, then $v(\cdot,x)$: $[0,\infty) \to E$ defined by

$$v(t,x)=\frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty}e^{\lambda t}R(\lambda,A)xd\lambda.$$

is continuous by the condition that $\|R(\lambda,A)\|\leqslant\frac{M}{|\lambda|}$, and there exists a constant $M(x)$ such that $\|v(t,x)\|\leqslant M(x)e^{pt}$. By the closedness of A, it is easy to show that

(2.5) $$A\int_0^t v(s,x)ds=v(t,x)-x.$$

therefore $v(0,x)=x$. Define $S(t)\cdot:E\to E$ (for $t\geqslant 0$) by $S(t)x=R(p,A)x-e^{pt}v(t,R(p,A)x)$ ($x\in E$), then $S(t)$ is a class of strongly continuous liear bouned operators satisfying that $S(0)x=R(p,A)x-v(0,R(p,A)x)=0$. Noting the equation (2.5), one can show that $S(t)x=(A-p)\int_0^t S(r)xdr+tx$, then by, $S(t)$ is an integrated semigroup generated by A, therefore, the theorem holds by the perturbation theorem. ∎

3. The wellposedness of abstract Cauchy problem

In Banach space E, we conside the abstract Cauchy problem:

(ACP) $$u'(t)=Au(t),\ u(0)=x.$$

Definition3.1: *If for each $x\in D(A)$ (ACP) has a unique mild solution and for each $x\in D(A^2)$ (ACP) has a solution $u(\cdot,x)$ depending continuously on the initial data in the sense that $\|x_n\|_1=:\|x_n\|+\|Ax_n\|\to 0$ (as $n\to\infty$) implies $u(t,x_n)\to 0$ (as $n\to\infty$) uniformly on compact subset of $[0,\infty)$, then we say that (ACP) is wellposed.*

If, in addition, each mild solution is exponentially bounded, we say that (ACP) is exponentialy wellposed.

Theorem3.1: *Suppose A is closed operator with nonempty resolvent set, then (ACP) is wellposed ifand only ifA generates integrated semigroup.*

Proof: Necessity: By the perturbation principle, without loss of generality, we assume that $0\in\rho(A)$. For $x\in D(A)$, let $u(t,x)$ be the mild solution of (ACP), and for each $x\in E$, let $S(t)x=:R(0,A)x-u(t,R(0,A)x)$, then $S(0)x=0$, $S(t)x$ is continuous in t and is linear in x.

From the definition of $u(t,R(0,A)x)$, the equation $A\int_0^t S(r)xdr=S(t)x-tx$ ($t\geqslant 0$) follows. For all $t,s>0$, $x\in E$, we have

(3.1) $S(s)S(t)x=S(s)R(0,A)x-R(0,A)u(t,R(0,A)x)+u(s,R(0,A)u(t,R(0,A)x))$

Let $u(t)=:R(0,A)u(t,R(0,A)x)$, since $u(t,R(0,A)x)=A\int_0^t u(r,R(0,A)x)dr+R(0,A)x$, then $R(0,A)u(t,R(0,A)x)=-\int_0^t u(r,R(0,A)x)dr+R(0,A)^2x$, thus $u(t)$ is a solution of (ACP) with initial data $u(0)=R(0,A)^2x$, and by the uniqueness of solution, $u(t)\equiv u(t,R(0,A)^2x)$, therefore, (3.1) can be improved as:

$$(3.2)\qquad S(s)S(t)x=-\int_0^s S(r)xdr-\int_0^t S(r)xdr+(t+s)R(0,A)x$$
$$+u(s,u(t,R(0,A)^2x))-R(0,A)^2x.$$

If define $v(\cdot)=:u(s,u(\cdot,R(0,A)^2x))$, it can be showed that $v(t)$ is a solution of (ACP) with initial data $u(s,R(0,A)^2x)$, then, by the uniqueness of solution, $v(t)\equiv u(t+s,R(0,A)^2x)$ and

$$(3.3)\qquad S(s)S(t)x=-\int_0^s S(r)xdr-\int_0^t S(r)xdr+\int_0^{t+s} S(r)xdr$$

For $t>0$ fixed, let $x_n\to 0$, $S(t)x_n\to y$ (as $n\to\infty$), then $\| R(0,A)^2x_n \|_1 =: \| R(0,A)^2x_n \| + \| AR(0,A)^2x_n\to 0$ (as $n\to\infty$), thus, by the wellposedness of (ACP), $S(t)R(0,A)x_n\to 0$. Since $AS(t)R(0,A)x_n=-S(t)x_n\to -y$ (as $n\to\infty$), then by the closedness of A $y=0$, and S(t) is a closed oprator on E, thus S(t) is bounded (for each $t\geqslant 0$). By now, we have proved that S(t) is an integrated semigroup on E.

Denote the generator of S(t) by A_0. For $x\in D(A_0)$, since $S'(t)x=S(t)A_0x+x$ and $S'(t)x=-u'(t,R(0,A)x)=-Au(t,R(0,A)x)=AS(t)x+x$, then $AS(t)x\equiv S(t)A_0x$, this implies that $x=R(0,A)A_0x$, i.e., $x\in D(A)$. On the other hand, $D(A)\subseteq D(A_0)$ is obvious. Therefore, S(t) is generated by A.

The sufficiency follows from [5] [6]. ∎

Using the same methods as the proof of theorem3.1, we have.

Theorem3.2: *Let A be closed operator with nonempty resolvent set, then (ACP) is exponentially bounded wellposed if and only if A generates exponentially bounded integrated semigroup.*

Remark: 1) The wellposedness of (ACP) can be generalized as: *if for each $x\in D(A^n)$ (ACP) has a unique mild solution and for each $x\in D(A^{n+1})$ (ACP) has a solution $u(t,x)$ depending continuously on the initial data in the sense that $\| x_k \|_n =: \sum_{i=0}^{n} \| A^i x_k \| \to 0$ (as $k\to\infty$) implies $u(t,x_k)\to 0$ uniformly in compact subset of $[0,\infty)$, then we say that (ACP) is n−times wellposed.* It can be showed that (ACP) is n−times wellposed iff A generates n−times integrated semigroup.

2) In the previous theorems A need not be densely defined, and the integrated semigroup generated by A need not be exponentially bounded in theorem3.1.

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