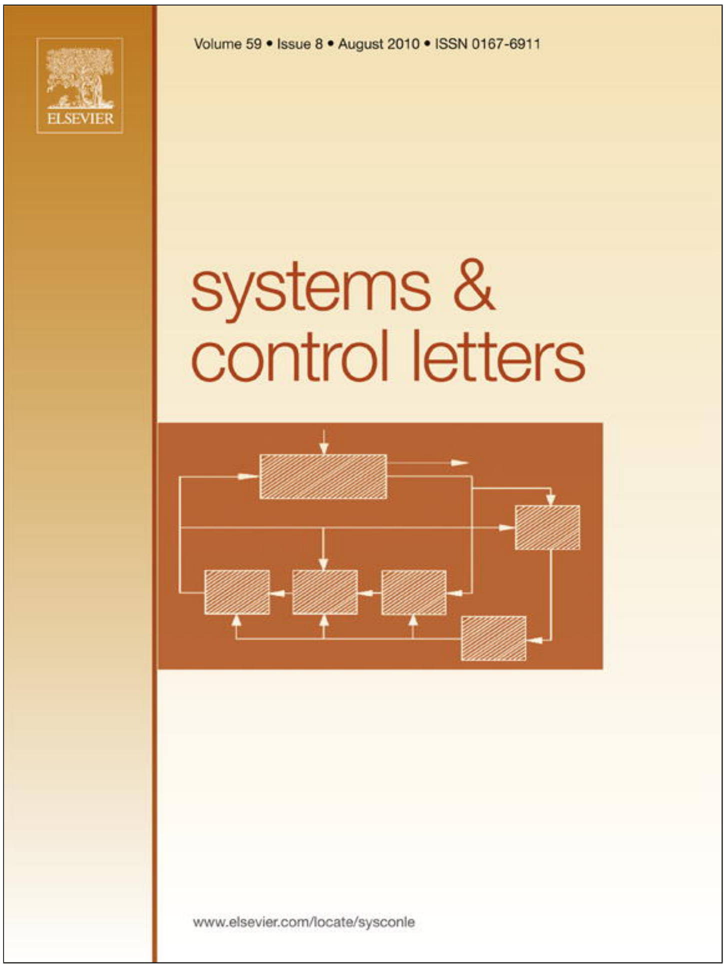
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On invariance of p-admissibility of control and observation operators to q-type of perturbations of generator of C_0-semigroup ☆

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ABSTRACT

In this paper, it is proved in general setting that p-admissibilities of control operators and observation operators are invariant to any q-type of perturbations of generator of C_0-semigroups on Banach space. Moreover, some relations between the Λ-extensions of observation operators with respect to the original generator and the perturbed generator are also characterized, so that the output can be expressed in the mild sense. As an application, the admissibility as well as the mild expressibility of output of a class of observation systems with time delay in state is deduced.

1. Introduction and definitions

Consider the infinite dimensional linear control systems described by the following differential equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \\ y(t) = Cx(t), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where the system state $x(t)$ takes values in Banach space E; the coefficient operator $A : D(A) \subset E \to E$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$; the input $u(\cdot)$ and output $y(\cdot)$ take values in Banach spaces U and Y, respectively; the control operator B is bounded from U to the extrapolation space E_{-1} of E; and the observation operator C is bounded from E_1 to Y, where E_1 denotes the domain $D(A)$ equipped with the graph norm. By definition [1], the extrapolation space E_{-1} is the completion of E under the norm $\|R(\lambda_0, A)\cdot\|$ with $R(\lambda_0, A)$ the resolvent of A at λ_0.

In the special case when the control operator B takes values in the state space E, the system Eq. (1.1) is mildly solved in the state space E by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\mathrm{d}s \tag{1.2}$$

and moreover, the solution $x(t)$ depends continuously on the input u in the space $L^p([0,\infty), U)$ for all $p \in [1,\infty)$; therefore, the control system (1.1) is well-posed. However, in our present setting, formula (1.2) may fail to make sense in E because $Bu(s)$ may be out of the state space E.

Let $\{T_{-1}(t)\}_{t\geq 0}$ be the extrapolation semigroup of $\{T(t)\}_{t\geq 0}$ on E_{-1}, with A_{-1} denoting its generator. Then, the control system (1.1) can be mildly solved in the extrapolation space E_{-1} by

$$x(t) = T_{-1}(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\mathrm{d}s. \tag{1.3}$$

Obviously, in order to make the control system (1.1) be well-posed, it is necessary that the integral term of formula (1.3) belongs to E for all $t \geq 0$ and depends continuously on the input $u \in L^p([0,\infty), U)$ for some $p \geq 1$. A notion, named p-admissible control operator, is thus introduced.

Definition 1.1. Let $p \geq 1$. The control operator B is said to be p-admissible for A if, for any $t \geq 0$, the mapping $\Phi_t : u \mapsto \int_0^t T_{-1}(t-s)Bu(s)\mathrm{d}s$ is continuous from $L^p([0,\infty), U)$ into E. The class of p-admissible control operators is denoted by $\mathscr{C}^p(A, U, E)$.

The p-admissibility notion was formally introduced by Weiss [2], although equivalent hypotheses had appeared several times before then (see, e.g., [3,4]). Just in the same paper, Weiss proved that B is p-admissible if $\Phi_T u \in E$ for a certain $T > 0$ and for all $u \in L^p([0,\infty), U)$.

It follows that if B is p-admissible, then for any $t \geq 0$, there exists a positive K_t such that

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$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\mathrm{d}s \right\|_E \le K_t \|u\|_{L^p([0,\infty),U)},$$
$$\forall u \in L^p([0,\infty),U). \quad (1.4)$$

In general, the bound K_t, or more precisely the operator norm $\|\Phi_t\|$ of Φ_t, depends on the time t.

As a dual pattern of p-admissibility for control operator, the notion, p-admissible observation operator, was formally defined by Weiss [5] as follows.

Definition 1.2. Let $p \ge 1$. The observation operator C is said to be p-admissible for A if for some $T > 0$ there exists a constant $K > 0$ such that

$$\int_0^T \|CT(t)x\|_Y^p \mathrm{d}t \le K\|x\|_E^p, \quad \forall x \in D(A). \quad (1.5)$$

The class of p-admissible observation operators is denoted by $\mathscr{O}^p(A,E,Y)$. In the special case when $Y = E$, $p = 1$ and the constant K in (1.5) is less than 1, $\mathscr{O}^p(A,E,Y)$ is briefly denoted by $\mathscr{S}^{MV}(A)$.

By the semigroup property of $\{T(t)\}_{t\ge 0}$ it is easy to show that the inequalities really make sense for all $T \ge 0$ if C is p-admissible. Moreover, it follows from the inequalities (1.5) that if C is p-admissible for A, then for any fixed $T > 0$ the mapping $\Psi : D(A) \to L^p([0,T),Y)$ defined by $\Psi x = CT(\cdot)x$ is continuous and so can be extended to the whole space E. However, $CT(t)x$ may fail to make sense for some t when x is out of $D(A)$, since C is just defined in $D(A)$. In order to express the output function $y(t)$ in the mild sense, Weiss [6] introduced an extension of C, called Λ-*extension* with respect to A, which is defined by

$$C_\Lambda x = \lim_{\lambda\to\infty} C\lambda R(\lambda, A)x \quad (1.6)$$

with the domain $D(C_\Lambda) = \{x \in E : \text{this above limit exists in } Y\}$. It follows from [5, Theorem 4.5 and Proposition 4.7] that for any $x \in E$, $y(t) = C_\Lambda T(t)x$ a.e. in $t \ge 0$ if C is p-admissible for A.

From the above definitions, it is clear to see that admissibility is an important property that is closely related to well-posedness of system (1.1). The topic of "admissible" control and observation in Hilbert spaces is more than 30 years old and was originally motivated by boundary control and point observation. It was at that time when it became clear that "unbounded" inputs can be incorporated into classical semigroup theory by considering it on the extended spaces $[D(A^*)]'$ (see [7]). The precursor of this approach was A.V. Balakrishnan when he formulated (1975) boundary control problem via variation of parameter formula with the semigroup acting on an extended space (i.e. domains of dual to the generator). This development gave an impetus to the field where both parabolic and hyperbolic control systems with point and boundary controls were shown "admissible" (though this terminology was not used at that time since the emphasis was put on proving inequalities rather than "labeling" or axiomatizing some properties (see [8,9,7,10,11])).

Since then, the theory of control systems with unbounded inputs became fully developed and the framework $x(t) = Ax + Bu$, $y = Cx$ with B and C^* are "admissible" so that the variation of parameters formula provides well defined element $x(t) \in E$ in the original state space E, became a standard tool in control theory. Questions such as regularity of PDE control systems, optimal control theory, Riccati theory, controllability, stabilization of boundary-point control systems (see, e.g., [12,13]) are resolved by using the above framework (see [8,9,7,14,10,11] and references therein). At the same time, it became obviously clear that the main issue in this field is to be able to verify the abstract postulates of admissibility. And in fact, this was and till is the main trend in the developing theory. The existing investigations on admissibility problems can be mainly casted into two aspects: *necessary and/or sufficient conditions for admissibility*, and *admissibility invariance to a certain type of perturbations of coefficient operator A*. There have been established a lot of criteria for admissible control operators as well as for admissible observation operators [15–25], among which the semigroup criterion has to be mentioned here because it is helpful for us to obtain the main results of the present paper. The original semigroup criterion was established by Grabowski and Callier [26] for infinite-time 2-admissibility of observation operators in Hilbert space, under the assumption that the C_0-semigroup generated by the coefficient operator A is exponentially stable. Later, it was proved by [27] that the assumption is not necessary. Applying the method developed in [28], [29] gave a semigroup criterion for p-admissibility of control and observation operators in Banach space.

Admissibility invariance for control operator (for observation operator) to perturbation ΔA of A refers to that $B \in \mathscr{C}^p(A + \Delta A, U, E)$ ($C \in \mathscr{O}^p(A + \Delta A, E, Y)$) whenever $B \in \mathscr{C}^p(A, U, E)$ ($C \in \mathscr{O}^p(A, E, Y)$). Admissibility invariance problems have two main causes. The first one is that in most cases a real control system cannot be precisely described by such system equations as (1.1), the coefficient operator A is just an approximation to the real one. And, the second one is that the verification of most of admissibility criteria is not an easy task and for many important operators A is even impossible to be performed in a direct way. Sometimes, it is helpful to write the coefficient operator as a sum of two operators, one of which generates C_0-semigroup and for it the p-admissibility is easy to be verified.

Obviously, the premise for admissibility invariance is that the perturbed operator $A + \Delta A$ or the part of $A_{-1} + \Delta A$ on E generates a C_0-semigroup on E. A sufficient condition for $A + \Delta A$ to generate a C_0-semigroup on E is that ΔA is bounded from E into itself. In fact, it is easy to prove the admissibility invariance to bounded perturbation of the coefficient operator A, which was really pointed out by Weiss [5, Remark 5.4]. Several years later, Weiss [30] studied the regularity invariance to admissible feedback in the setting of Hilbert space, that can be viewed as a generalization of the previous result. As for unbounded ΔA, two important corollaries from [1] are worth mentioning. First, according to [1, Corollary 3.4, page 188], if ΔA is bounded from E into E_{-1}, and for some $r > 0$ and $q \in [1,\infty)$,

$$\int_0^r T_{-1}(r-t)\Delta A f(t)\mathrm{d}t \in E, \quad \forall f \in L^q([0,r],E), \quad (1.7)$$

that is, if $\Delta A \in \mathscr{C}^q(A, E, E)$, then the part of $A_{-1} + \Delta A$ on E generates a C_0-semigroup on E. Secondly, by [1, Corollary 3.16, page 199], if ΔA is bounded from E_1 into E, and for some constants $T > 0$ and $0 \le \delta < 1$,

$$\int_0^T \|\Delta A T(t)x\|\mathrm{d}t \le \delta\|x\|, \quad \forall x \in D(A) \quad (1.8)$$

that is, if $\Delta A \in \mathscr{S}^{MV}(A)$, then $A + \Delta A$ generates a C_0-semigroup on E. Accordingly, a natural question is whether or not the p-admissibility of control operator (or observation operator) is invariant to either perturbation of the class $\mathscr{C}^q(A, E, E)$ or perturbation of the class $\mathscr{S}^{MV}(A)$. Recently, Said Hadd considered the question in the special case when $p = q$ [31,32]. He proved in [31] the p-admissibility invariance to perturbation of the class $\mathscr{C}^p(A, E, E)$ for control operators, and proved in [32] the p-admissibility invariance to perturbation of the class $\mathscr{O}^p(A, E, E) \bigcap \mathscr{S}^{MV}(A)$ for observation operators. The discussions both in [31] and [32] were based on the perturbation results of [33] and the well-posed linear system theory [21].

In this paper, we consider the above question in general cases. By using the semigroup criterion for p-admissibilities of control operators and observation operators, we prove that, for any

$q \in [1, \infty)$, the p-admissibilities of control operator and observation operator are invariant to perturbation of the class of $\mathscr{C}^q(A, E, E)$ and to perturbation of the class of $\mathscr{O}^q(A, E, E) \bigcap \mathscr{S}^{MV}(A)$, respectively. As a result, we give some relationships between the Λ-extensions (see Section 2) of admissible observation operators with respect to the original generator and the perturbed generator, so that the output can be expressed in the mild sense. As an application, we consider a class of observation systems with time delays in states, for which the admissibility as well as the expressibility of output in the mild sense is deduced.

Throughout this paper, the operators A, B and C and the C_0-semigroup $T(t)$ are all specified as in the control system (1.1), without any additional statement.

2. Main results

In the following, we use the notations $W^{1,p}([0, t_0], U) = \{f \in L^p([0, t_0], U) : f' \in L^p([0, t_0], U)\}$ and $W_r^{1,p}([0, t_0], U) = \{g \in W^{1,p}([0, t_0], U) : g(t_0) = 0\}$, and we denote by δ_0 the point evaluation at $s = 0$, and by $\mathscr{L}(X, Z)$ the Banach space of bounded linear operators from Banach space X to Banach space Z.

Lemma 2.1 (*[29]*). *Let $p \in [1, \infty)$. Then,*

(a) *the control operator $B \in \mathscr{L}(U, E_{-1})$ is p-admissible for A if and only if there exists $t_0 > 0$ such that the operator matrix*

$$\mathscr{C}_B(A) = \begin{pmatrix} A_{-1} & B\delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix},$$

$$D(\mathscr{C}_B(A)) = \left\{ \begin{pmatrix} x \\ g \end{pmatrix} \in E \times W_r^{1,p}([0, t_0], U) : A_{-1}x + Bg(0) \in E \right\}$$

generates a C_0-semigroup $\mathscr{T}(t)$ on $E \times L^p([0, t_0], U)$, and

(b) *the observation operator $C \in \mathscr{L}(E_1, Y)$ is p-admissible for A if and only if there exists $t_0 > 0$ such that the operator matrix*

$$\mathscr{O}_C(A) = \begin{pmatrix} A & 0 \\ 0 & -\dfrac{d}{ds} \end{pmatrix},$$

$$D(\mathscr{O}_C(A)) = \left\{ \begin{pmatrix} x \\ g \end{pmatrix} \in D(A) \times W^{1,p}([0, t_0], Y) : g(0) = Cx \right\}$$

generates a C_0-semigroup $\mathscr{S}(t)$ on $E \times L^p([0, t_0], Y)$.

From the lemma's proof given in [29] (or [28]), it is seen that the semigroup $\mathscr{T}(t)$ in the part (a) of Lemma 2.1 is given by

$$\mathscr{T}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S_l(t) \end{pmatrix},$$

where $T(t)$ is the semigroup generated by A, $R(t) \in \mathscr{L}(L^p([0, t_0], U), E)$ $(\forall t \geq 0)$, and $S_l(t)$ is the left-shift semigroup on $L^p([0, t_0], U)$. Dually, the semigroup $\mathscr{S}(t)$ in the part (b) of Lemma 2.1 takes on the form

$$\mathscr{S}(t) = \begin{pmatrix} T(t) & 0 \\ Q(t) & S_r(t) \end{pmatrix},$$

where $T(t)$ is the semigroup generated by A, $Q(t) \in \mathscr{L}(E, L^p([0, t_0], U))$ $(\forall t \geq 0)$, and $S_r(t)$ is the right-shift semigroup on $L^p([0, t_0], U)$.

Lemma 2.2 (*Desch, Schappacher, 1985, [1]*). *Let $\tilde{A}$ be the generator of C_0-semigroup $\tilde{T}(t)$ on Banach space X and let $\tilde{C} \in \mathscr{L}(X)$. Assume that there exist constants $t_0 > 0$ and $1 \leq p < \infty$ such that*

$$\int_0^{t_0} \tilde{T}(t_0 - r)\tilde{C}f(r)\mathrm{d}r \in D(\tilde{A}), \quad \forall f \in L^p([0, t_0], X). \tag{2.1}$$

Then, both $(Id + \tilde{C})\tilde{A}$ and $\tilde{A}(Id + \tilde{C})$ are generators of C_0-semigroups on E.

Theorem 2.3. *Let $p, q \in [1, \infty)$. If $B \in \mathscr{C}^p(A, U, E)$, then $B \in \mathscr{C}^p((A_{-1} + \Delta A)_{|E}, U, E)$ for any $\Delta A \in \mathscr{C}^q(A, E, E)$, where $(A_{-1} + \Delta A)_{|E}$ stands for the part of $A_{-1} + \Delta A$ on E* (i.e., $(A_{-1} + \Delta A)_{|E}x =$ *$(A_{-1} + \Delta A)x$, the domain $D((A_{-1} + \Delta A)_{|E}) = \{x \in E : (A_{-1} + \Delta A)x \in E\}$).*

Proof. Let $B \in \mathscr{C}^p(A, U, E)$, and any given $\Delta A \in \mathscr{C}^q(A, E, E)$. By [1, Corollary 3.4, page 188], it follows that $(A_{-1} + \Delta A)_{|E}$ generates a C_0 semigroup on E. Hence, by Lemma 2.1, it suffices to prove that the operator matrix

$$\mathscr{C}_B((A_{-1} + \Delta A)_{|E}) = \begin{pmatrix} ((A_{-1} + \Delta A)_{|E})_{-1} & B\delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix}$$

generates a C_0 semigroup on the product space $E \times L^p([0, t_0], U)$ for some $t_0 > 0$.

Without loss of generality, we assume that A, hence A_{-1}, is invertible (indeed, otherwise we can replace A with $A - \lambda$ for a certain $\lambda \in \rho(A)$). Then, noticing that $((A + \Delta A)_{|E})_{-1} = A_{-1} + \Delta A$, we have that

$$\begin{aligned} \mathscr{C}_B((A_{-1} + \Delta A)_{|E}) &= \begin{pmatrix} A_{-1} & B\delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix} \begin{pmatrix} Id + A_{-1}^{-1}\Delta A & 0 \\ 0 & Id \end{pmatrix} \\ &=: \mathscr{C}_B(A)(Id + \tilde{C}) \end{aligned} \tag{2.2}$$

on the set

$$\mathscr{D} = \left\{ \begin{pmatrix} x \\ g \end{pmatrix} \in E \times L^p([0, t_0], U) : \begin{pmatrix} Id + A_{-1}^{-1}\Delta A & 0 \\ 0 & Id \end{pmatrix} \times \begin{pmatrix} x \\ g \end{pmatrix} \in D(\mathscr{C}_B(A)) \right\}.$$

It is easy to check that the set $\mathscr{D}$ is consistent with the following set

$$\left\{ \begin{pmatrix} x \\ g \end{pmatrix} \in E \times W_r^{1,p}([0, t_0], U) : (A_{-1} + \Delta A)x + Bg(0) \in E \right\},$$

which is just the domain of the operator $\mathscr{C}_B((A_{-1} + \Delta A)_{|E})$.

Since B is p-admissible for A, by Lemma 2.1 it follows that the operator $\mathscr{C}_B(A)$ generates a C_0-semigroup $\mathscr{T}(t)$ on $E \times L^p([0, t_0], U)$ for some $t_0 > 0$. And, since $\Delta A \in \mathscr{C}^q(A, E, E)$, there exists a $t_1 > 0$ such that $\int_0^{t_1} T_{-1}(t_1 - r)\Delta Af(r)\mathrm{d}r \in E$ for all $f \in L^q([0, t_1], E)$. Let $X = E \times L^p([0, t_0], U)$. Then, for any $f = (f_1, f_2)^T \in L^q([0, t_1], X)$, we have that

$$\int_0^{t_1} \mathscr{T}(t_1 - r)\tilde{C}f(r)\mathrm{d}r = \begin{pmatrix} \displaystyle\int_0^{t_1} T(t_1 - r)A_{-1}^{-1}\Delta Af_1(r)\mathrm{d}r \\ 0 \end{pmatrix}, \tag{2.3}$$

where

$$\int_0^{t_1} T(t_1 - r)A_{-1}^{-1}\Delta Af_1(r)\mathrm{d}r = A_{-1}^{-1}\int_0^{t_1} T_{-1}(t_1 - r)\Delta Af_1(r)\mathrm{d}r \in D(A),$$

which implies that

$$\int_0^{t_1} \mathscr{T}(t_1 - r)\tilde{C}f(r)\mathrm{d}r \in D(\mathscr{C}_B(A)).$$

Therefore, by Lemma 2.2, $\mathscr{C}_B((A_{-1} + \Delta A)_{|E})$ generates a C_0-semigroup on $X = E \times L^p([0, t_0], U)$. $\square$

Remark 2.4. It should be pointed out that Theorem 2.3 can be proved by the standard fixed point argument, similar to [9, Theorem 2.1]. Here, we adopt the pure semigroup method to prove the theorem so as to offer a way to unify the proofs of admissible control and admissible observation, particularly for the non-reflexive case.

Remark 2.5. As mentioned in the introduction section, the theorem for the special case when $q = p$ is attributed to Said Hadd [31]. In fact, when $q < p$, the theorem can also be concluded from Said Hadd's result, if the inclusion relation obtained by Weiss [2] that $\mathscr{C}^q(A, E, E) \subset \mathscr{C}^p(A, E, E)$ is used. However, when $q > p$, the theorem cannot be obtained directly along the proof of the paper [31], because in this case the implication that $\Delta A \in \mathscr{C}^p(A, E, E)$ and $B \in \mathscr{C}^q(A, U, E) \Rightarrow [\Delta A, B] \in \mathscr{C}^p(A, E \times U, E)$ is no longer available.

Remark 2.6. In general, it is not an easy task to verify that $\Delta A \in \mathscr{C}^q(A, E, E)$. In [1], a sufficient condition for $\Delta A \in \mathscr{C}^1(A, E, E)$ was given that the range $\mathbb{R}(\Delta A)$ is contained in the set $F_0 = \{x \in E_{-1} : \sup_{t>0} \|\frac{1}{t}(e^{-\omega t}T_{-1}(t) - Id)x\| < \infty\}$, where ω is a constant which is larger than the growth bound of $(T(t))_{t\geq 0}$.

The following corollary is a joint result of the above theorem and Theorem 3.3 of [31].

Corollary 2.7. *For any* $\Delta A \in \mathscr{C}^q(A, E, E)$, $\mathscr{C}^p(A, U, E) = \mathscr{C}^p((A_{-1} + \Delta A)_{|E}, U, E)$.

According to [5], in the case that the state space E is a reflexive Banach space, the following theorem may be viewed as a dual pattern of Theorem 2.3, that is, it can be obtained immediately by taking the dual form of Theorem 2.3. But when E is not reflexive, we cannot even guarantee that $T(t)^*$, the dual of $T(t)$, generates a C_0-semigroup. Therefore, the proof after the following theorem becomes necessary.

Theorem 2.8. *Let* $1 \leq p, q < \infty$. *If* $C \in \mathscr{O}^p(A, E, Y)$, *then* $C \in \mathscr{O}^p(A + \Delta A, E, Y)$ *for any* $\Delta A \in \mathscr{O}^q(A, E, E) \cap \mathscr{S}^{MV}(A)$.

Proof. In the case that E is reflexive, the result is followed, by duality, from Theorem 2.3. Below we prove the case that E is a general Banach space.

Suppose that $\Delta A \in \mathscr{O}^q(A, E, E) \cap \mathscr{S}^{MV}(A)$ and $C \in \mathscr{O}^p(A, E, Y)$. By [1, Corollary 3.16, p. 199], $A + \Delta A$ generates a C_0-semigroup on E. Let the operator $\mathscr{O}_C(A)$ be defined as in the part (b) of Lemma 2.1. Then, $\mathscr{O}_C(A)$ generates a C_0-semigroup $\mathscr{S}(t)$ on $E \times L^p([0, t_0], Y)$. It is clear to see that $D(\mathscr{O}_C(A + \Delta A)) = D(\mathscr{O}_C(A))$ and

$$\mathscr{O}_C(A + \Delta A) = \mathscr{O}_C(A) + \begin{pmatrix} \Delta A & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, $\widetilde{\Delta A} =: \begin{pmatrix} \Delta A & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{L}(D(\mathscr{O}_C(A)), E \times L^p([0, t_0], Y))$, with $D(\mathscr{O}_C(A))$ being endowed with the graph norm.

Since $\Delta A \in \mathscr{O}^q(A, E, E)$, by definition there holds on $D(A)$ that $\int_0^{t_1} \|\Delta A T(t)x\|^q dt \leq \alpha(t_1)^q \|x\|^q$ for some constants $t_1 > 0$ and $\alpha(t_1) > 0$ (if $q = 1$, we can take $\alpha(t_1) \in (0, 1)$). Hence, it follows that for any $f = (x, g)^T \in D(\mathscr{O}_C(A))$,

$$\begin{aligned} &\int_0^{t_1} \|\widetilde{\Delta A}\mathscr{S}(t)f\|^q dt \\ &= \int_0^{t_1} \|\Delta A T(t)x\|^q dt \\ &\leq \alpha(t_1)^q \|x\|^q \leq \alpha(t_1)^q \|f\|^q_{E\times L^p([0,t_0],Y)}, \end{aligned}$$

that is, $\widetilde{\Delta A} \in \mathscr{O}^q_{E\times L^p([0,t_0],Y)} \bigcap \mathscr{S}^{MV}(\mathscr{O}_C(A))$. Therefore, by [1, Corollary 3.16, page 199], $\mathscr{O}_C(A + \Delta A)$ is a generator of C_0-semigroup on $E \times L^p([0, t_0], Y)$, and so by Lemma 2.1, $C \in \mathscr{O}^p(A + \Delta A, E, Y)$. □

Remark 2.9. The theorem was proved early in [32] in the case when $q = p$. As pointed out in [5], there always holds the inclusion relationship that $\mathscr{O}^q(A, E, E) \subset \mathscr{O}^p(A, E, E)$ whenever $q > p$. This means that Said Hadd [32] virtually proved the theorem in the more general cases when $q \geq p$. However, when $q < p$, we cannot prove the theorem directly along the proof of Said Hadd's [32].

Similar to Corollary 2.7, we have

Corollary 2.10. *For any* $\Delta A \in \mathscr{O}^q(A, E, E) \bigcap \mathscr{S}^{MV}(A)$, $\mathscr{O}^p(A, E, Y) = \mathscr{O}^p(A + \Delta A, E, Y)$.

With the above theorem, we can obtain the relationships between the Λ-extensions of the observation operators with respect to the original generator A and the perturbed generator $A + \Delta A$.

Corollary 2.11. *Assume that* $\Delta A \in \mathscr{O}^q(A, E, E)$ *and* $C \in \mathscr{O}^p(A, E, Y)$. *Let* C_Λ, C'_Λ *be the* Λ*-extensions of* C *w.r.t* A *and* $A + \Delta A$, *respectively. Then, we have that*

$$D(C_\Lambda) \cap D(\Delta A_\Lambda) = D(C'_\Lambda) \cap D(\Delta A_\Lambda)$$

and

$$C'_\Lambda x = C_\Lambda x, \quad \forall x \in D(C_\Lambda) \cap D(\Delta A_\Lambda)$$

where ΔA_Λ *stands for the* Λ*-extension of* ΔA *with respect to* A.

Proof. The proof is similar to that of Theorem 3.2 of [32]. □

Remark 2.12. The significance of the above corollary lies in that if $C \in \mathscr{O}^p(A, E, Y)$, then the output function $y(t)$ can be expressed almost everywhere with the mild state trajectory (see, for example, [5,6]). That will be seen in the next section.

3. Application to observation systems with time delays in states

As an application example of the results obtained in the last section, we consider in this section the admissibility of observation operator of the following observation system with time delays in states:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + Lx_t, \quad t \geq 0, \\ x(0) = x, \\ x_0 = f, \\ y(t) = Cx(t), \end{cases} \tag{3.1}$$

where both state space X and output space Y are Banach spaces, $A : D(A) \subset X \to X$ is a generator of C_0-semigroup on X, $L \in \mathscr{L}(W^{1,r}([-1, 0], X), X)$, $f \in L^r([-1, 0], X)$, $1 \leq r < \infty$, x_t is the history function on $[-1, 0]$ at t defined by $x_t(\sigma) = x(t + \sigma)$, and $C \in \mathscr{L}(D(A), Y)$ where $D(A)$ is equipped with the graph norm.

According to [34], the above linear observation system (3.1) is equivalent to the following un-delayed linear system with the state space $\mathscr{E} = X \times L^r([-1, 0], X)$ and the output space Y:

$$\begin{cases} \dfrac{d}{dt}\mathfrak{X}(t) = \mathcal{A}_L \mathfrak{X}(t), \\ \mathfrak{X}(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \\ y(t) = \mathfrak{C}\mathfrak{X}(t), \end{cases} \tag{3.2}$$

where $\mathcal{A}_L = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, whose domain is given by

$$D(\mathcal{A}_L) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,r}([-1, 0], X) : f(0) = x \right\},$$

and $\mathfrak{C} = (C, 0)$, whose domain is $D(C) \times L^r([-1, 0], X)$.

Write $\mathcal{A}_L$ as $\mathcal{A}_L = \mathcal{A} + \Delta\mathcal{A}$, with

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \qquad \Delta\mathcal{A} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}. \tag{3.3}$$

According to [34], $\mathcal{A}$, with the domain $D(\mathcal{A}) = D(\mathcal{A}_L)$, generates a C_0-semigroup on $\mathscr{E}$ given by

$$\mathscr{T}(t) = \begin{pmatrix} T(t) & 0 \\ T_t & T_0(t) \end{pmatrix}, \tag{3.4}$$

where $T(t)$ is the C_0-semigroup generated by A, $T_t : X \to L^r([-1,0],X)$ are operators defined by

$$(T_t x)(\theta) = \begin{cases} T(t+\theta)x, & t+\theta \geq 0 \\ 0, & \text{if not} \end{cases}$$

and $T_0(t)$ is the left-shift semigroup on $L^r([-1,0],X)$. Moreover, it is easy to show that $\Delta\mathcal{A} \in \mathscr{L}(D(\mathcal{A}),\mathscr{E})$, where $D(\mathcal{A})$ is equipped with the graph norm.

It is not hard to see that $\Delta\mathcal{A} \in \mathscr{O}^q(\mathcal{A},\mathscr{E},\mathscr{E}) \bigcap \mathscr{S}^{MV}(\mathcal{A})$ is equivalent to

(H_1) for some α_1, $\gamma > 0$ ($\gamma < 1$ when $q = 1$),

$$\int_0^{\alpha_1} \|L(T_t x + T_0(t)f)\|^q \mathrm{d}t \leq \gamma \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|^q, \quad \forall \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}), \tag{3.5}$$

and that for $1 \leq p < \infty$, $\mathfrak{C}$ is p-admissible for $\mathcal{A}$ is equivalent to

(H_2) $C \in \mathscr{O}^p(A,X,Y)$.

Then, by Theorem 2.8, under conditions (H_1) and (H_2) the observation operator $\mathfrak{C}$ is p-admissible for $\mathcal{A}_L$.

Below we further discuss the expression problem of the output function $y(t)$ with system state $x(t)$. For doing this, denote by C_Λ and L_Λ^0 the Λ-extensions of C and L with respect to A and A_0, respectively, where A_0 is the generator of the left-shift semigroup $T_0(t)$, and define in X a operator $\bar{L}$ as follows:

$$\bar{L}x = \lim_{\lambda\to+\infty} Le_\lambda x, \tag{3.6}$$

$$D(\bar{L}) = \left\{ x \in X : \lim_{\lambda\to+\infty} Le_\lambda x \text{ exists} \right\},$$

where $e_\lambda : X \to L^r([-1,0],X)$, $e_\lambda x = \mathrm{e}^{\lambda\cdot}x$. Then, we can prove the following theorem.

Theorem 3.1. *Assume that the conditions* (H_1) *and* (H_2) *are satisfied. If $D(\bar{L}) = X$, then $x(t) \in D(C_\Lambda)$, $x_t \in D(L_\Lambda^0)$, and $y(t) = C_\Lambda x(t)$ for a.e. $t \geq 0$, where $x(t)$ is the mild state trajectory of system* (3.1)*, which satisfies*

$$x(t) = T(t)x + \int_0^t T(t-r)\left[\bar{L}x(r) + L_\Lambda^0 x_r\right]\mathrm{d}r, \quad t \geq 0. \tag{3.7}$$

Proof. Denote by $\mathfrak{C}_\Lambda$ and $\mathfrak{C}'_\Lambda$ the Λ-extensions of $\mathfrak{C}$ with respect to $\mathcal{A}$ and $\mathcal{A}_L$, respectively. By definitions, it is a routine matter to show that

$$D(\mathfrak{C}_\Lambda) = D(C_\Lambda) \times L^P([-1,0],X), \quad \mathfrak{C}_\Lambda \begin{pmatrix} x \\ f \end{pmatrix} = C_\Lambda x. \tag{3.8}$$

Further, denote by $\Delta\mathcal{A}_\Lambda$ the Λ-extensions of $\Delta\mathcal{A}$ with respect to $\mathcal{A}$. Then, since $D(\bar{L}) = X$, by [35] it follows that

$$D(\Delta\mathcal{A}_\Lambda) = X \times D(L_\Lambda^0), \quad \Delta\mathcal{A}_\Lambda = \begin{pmatrix} \bar{L} & L_\Lambda^0 \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

which, associated with (3.8), leads to that

$$\begin{aligned} D(\mathfrak{C}_\Lambda) \cap D(\Delta\mathcal{A}_\Lambda) &= (D(C_\Lambda) \times L^r([-1,0],X)) \cap (X \times D(L_\Lambda^0)) \\ &= D(C_\Lambda) \times D(L_\Lambda^0) \subset D(\mathfrak{C}'_\Lambda). \end{aligned} \tag{3.10}$$

So, by Corollary 2.11,

$$\mathfrak{C}'_\Lambda \begin{pmatrix} y \\ g \end{pmatrix} = C_\Lambda x, \quad \forall \begin{pmatrix} y \\ g \end{pmatrix} \in D(C_\Lambda) \times D(L_\Lambda^0). \tag{3.11}$$

Let $x(t)$ be a mild state trajectory of system (3.1). Using Hölder's inequality, it is easy to show that the conditions (H_1) and (H_2) are satisfied for all $s \leq q$ and for all $l \leq p$, respectively. So, by Hadd and Idrissi [32, theorem 3.1], it follows that, for almost every $t \geq 0$,

$$\begin{pmatrix} x(t) \\ x_t \end{pmatrix} = \mathscr{T}_L(t) \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Delta\mathcal{A}_\Lambda) \cap D(\mathfrak{C}_\Lambda), \tag{3.12}$$

where $\mathscr{T}_L(t)$ is the C_0-semigroup generated by $\mathcal{A}_L$. Hence, by (3.8) and (3.9), $x(t) \in D(C_\Lambda)$ and $x_t \in D(L_\Lambda^0)$ for a.e. $t \geq 0$. Therefore, by (3.10) and (3.11) it follows that $y(t) = \mathfrak{C}'_\Lambda \begin{pmatrix} x(t) \\ x_t \end{pmatrix} = C_\Lambda x(t)$ make sense for a.e. $t \geq 0$. And, (3.7) is directly from [35, Theorem 4 and Proposition 2]. □

Example 3.2. Consider the following diffusion equations:

$$\begin{cases} \dfrac{\partial}{\partial t} z(x,t) = \dfrac{\partial^2}{\partial x^2} z(x,t) - az(x,t) - bz(x,t-1), \\ \quad x \in \mathbb{R},\ t \geq 0 \\ z(x,t) = \varphi(x,t), \quad x \in \mathbb{R},\ t \in [-1,0] \\ y(t) = z(0,t). \end{cases} \tag{3.13}$$

In order to write the system (3.13) as the abstract form of system (3.1), we take

- the state space $X := L^r(\mathbb{R})$ ($1 \leq r \leq 2$ and when $r = 1$ we need $b < 1$; moreover, when $r = 1$, X is non-reflexive);
- the operator $A := \Delta - aId$, with $D(A) := \{f \in L^r(\mathbb{R}) : f, f'$ absolutely continuous, $f'' \in L^r(\mathbb{R})\}$;
- the function $\mathbb{R}_+ \ni t \mapsto z(t) = z(\cdot,t) \in L^r(\mathbb{R})$, and the history function $z_t : [-1,0] \to L^r(\mathbb{R})$, $z_t(s) := z(t+s)$;
- the state delay operator $L : W^{1,r}([-1,0], L^2(R)) \to L^r(R)$, as $L := b\delta_{-1}$;
- the output space $Y = \mathbb{R}$;
- the observation operator $C = \delta_0$.

It is well known that $A + aId$ generates a C_0-semigroup on X, denoted by $T(t)$ (see [36, page 578]). Moreover, $T(t)$ is expressed as follows

$$(T(t)g)(s) := \begin{cases} (4\pi t)^{-1/2} \displaystyle\int_R \mathrm{e}^{-(s-\theta)^2/4t} g(\theta)\mathrm{d}\theta & t > 0, \\ g(s) & t = 0, \end{cases}$$

where $s \in R$. For $x \in D(A)$ and $f \in W^{1,r}([-1,0],X)$ satisfying $f(0) = x$, it is easy to show that

$$\int_0^1 \|L(T_t x + T_0(t)f)\|^r \mathrm{d}t = b^r \int_0^1 \|f(t-1)\|^r \mathrm{d}t = b^r \|f\|^r,$$

which implies that the condition (H_1) is satisfied for $q = r$, or equivalently, $\Delta\mathcal{A} \in \mathscr{O}^r(\mathcal{A},\mathscr{E},\mathscr{E}) \bigcap \mathscr{S}^{MV}(\mathcal{A})$. Now we check the admissibility of C for A, which is equivalent to the admissibility of C for $A + aId$.

$$\begin{aligned} \int_0^\alpha |C(T(t)g)|^p \mathrm{d}t &= \int_0^\alpha \left| (4\pi t)^{-1/2} \int_R \mathrm{e}^{-\theta^2/4t} g(\theta)\mathrm{d}\theta \right|^p \mathrm{d}t \\ &\leq \int_0^\alpha (4\pi t)^{-p/2} \left(\int_R \mathrm{e}^{-\theta^2 r'/4t} \mathrm{d}\theta \right)^{p/r'} \|g\|^p \mathrm{d}t \\ &= \int_0^\alpha (4\pi t)^{-p/2} \left(\frac{4\pi t}{r'} \right)^{p/2r'} \mathrm{d}t \|g\|^p \\ &= \int_0^\alpha (4\pi t)^{-p/2r} (r')^{-p/2r'} \mathrm{d}t \|g\|^p, \end{aligned}$$

where r' satisfies $1/r + 1/r' = 1$ and $\alpha > 0$.

Obversely, $\int_0^\alpha (4\pi t)^{-p/2r} (r')^{-p/2r'} \mathrm{d}t$ is convergent whenever $p/2r < 1$. Thus, for any $1 \leq p < 2r$, C is p-admissibility for A, which means that the condition (H_2) is satisfied for $1 \leq p < 2r$, or equivalently, that $\mathfrak{C} =: (C,0) \in \mathscr{O}^p(\mathcal{A},\mathscr{E},Y)$ whenever $1 \leq p < 2r$. Therefore, by Theorem 2.8 it follows that $(C,0)$ is also p-admissible for $\mathcal{A}_L$ for any $p \in [1,2r)$, and hence the output $y(t)$ can be expressed in the mild sense $y(t) = (\delta_0)_\Lambda z(\cdot,t)$ for a.e. $t \geq 0$.

However, it is not hard to check that (H_1) does not hold for any $q > r$, that is, $\Delta\mathcal{A} \notin \mathcal{O}^q(\mathcal{A}, \mathcal{E}, \mathcal{E})$ whenever $q > r$. So, for $p > r$ the p-admissibility of $(C, 0)$ for $\mathcal{A}_L$ cannot be concluded directly from the admissibility invariance theorem developed by Said Hadd [32].

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