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On the Perturbations of Regular Linear Systems and Linear Systems with State and Output Delays

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Abstract. This paper is concerned with perturbation problems of regularity linear systems. Two types of perturbation results are proved: (i) the perturbed system $(A+P,B,C)$ generates a regular linear system provided both (A,B,C) and (A,B,P) generate regular linear systems; and (ii) the perturbed system $((A_{-1}+\Delta A)|_X,B,C_\Lambda^A)$ generates a regular linear system if both (A,B,C) and $(A,\Delta A,C)$ generate regular linear systems. These allow us to establish a new variation of constants formula of the control system $(A+P,B)$. Moreover, these results are applied to the linear systems with state and output delays. The regularity and the mild expressibility is deduced, and a necessary and sufficient condition for stabilizability of the delayed systems is proved.

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1. Introduction

The class of well-posed linear systems introduced by Salamon in [21] has become a well-understood class of systems (see, e.g., [24,28,30,31]). Many partial differential equations with boundary control and point observation can be formulated as well-posed linear systems [17,21,22]. In a well-posed linear system, the input and output functions are locally in L^p, and on any finite time-interval, the final state and output function depend continuously on the initial state and the input function. A regular linear system is among the well-posed systems whose output function corresponding to a step input function and zero initial state is not very discontinuous at zero (in detail,

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see the definition in Sect. 2). Regular linear systems constitute a large subclass of well-posed linear systems, whose basic properties are rich enough to develop a parallel of the theory of control systems with bounded control and observation operators, as presented in Curtain and Pritchard [4]. Weiss showed in [28] that regular linear systems with unbounded control and observation operators allow nice generalizations of finite dimensional systems by admitting the differential representation

$$x'(t) = Ax(t) + Bu(t), y(t) = C_\Lambda x(t) + Du(t),$$

where C_Λ is the Λ-extension of the observation operator C with respect to system operator A (see Sect. 2).

Admissibility of control as well as observation operators are necessary for a linear system to be regular (see, e.g., [21,22,28,30,31]). However, unfortunately, it is hard to test the admissibility, not to mention the well-posedness and regularity of an infinite-dimensional linear system with unbounded control and observation. There are many papers devoted to discussion on the admissibility of control and observation operators, most of which are interested in proving or disproving Weiss' conjecture (see, e.g., [6,12–15,25,29,34,35]). Here, we mention an important work due to Zwart [35]; he proved that the Weiss conjecture almost holds in Hilbert spaces and in the case that $p = 2$. Sometimes, if A generates a C_0-semigroup on the state space and the admissibility of the control operator (or observation operator) for A is easy to check, we can write the generator of semigroup of the system as a sum of two simple operators, $(A_{-1} + \Delta A)|_X$ (or $A + P$). Thus the perturbation method can be used to test the admissible for the considered generator of semigroup of the system (see [8,9]).

Inspired by such perturbations of admissibility, we consider to use perturbation methods to test whether an infinite dimensional linear system is regular or not. Clearly, output feedback can be regarded as an perturbation [31]. However, in practical applications, it is hard to check that whether an infinite dimensional system is the feedback of another one or not; in contrast, it is easier to decompose the generator of semigroup of the system into a sum of two operators as in [8,9]. Motivated by this, in this paper, we consider the later scheme. More specifically, we shall prove in Sect. 3 two types of perturbation results. In detail, if both (A, B, C) and (A, B, P) generate regular linear systems on appropriate Banach spaces, $(A + P, B, C)$ generates a regular linear system; if both (A, B, C) and $(A, \Delta A, C)$ generate regular linear system on appropriate Banach spaces, $((A_{-1} + \Delta A)|_X, B, C)$ generates a regular linear system. Furthermore, in Sect. 4, we will see that such perturbation results allow us to establish a new variation of constants formula for linear control system with perturbation in generator of the system semigroup.

An important application of our perturbation results is to deal with delay differential systems. It is well-known that such systems arise in the study of many problems with theoretical and practical importance. Semigroup theory was first systematically applied to delay systems in the book [11] for finite dimensional state spaces. B*á*tkai [1] generated such method to delay differential systems in general Banach space setting. Since then, there have many mathematicians followed the work (see e.g [2,7,9,10,18]). In Sect. 5, we apply our perturbation results to systems with state and output delays. Similar to [1], we establish in the sense of classical solution the equivalent relationship of delay system and the extended system without delay. Moreover, using the results developed in Sect. 4, we obtain a new variation of constants formula for functional differential equations in L^p-phase spaces in terms of the Λ-extension of the delay operator.

In the final section we consider by our perturbation results the stabilizability of delay differential systems. Such property is specially important for delay systems because it has been recognized that the delay presence in the state could induce bad performance and complicate controller design and system analysis (see [33]). We know that most authors focus on the stabilizability of state delay systems with a finite dimensional delay-free state space. It is shown by Olbrot [19] that the stabilizability of the state-input delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + Pu(t)$$

is equivalent to the condition

$$Rank[\Delta(\lambda), P] = n, \tag{1.1}$$

for $\lambda \in \mathbb{C}$ with $Re\lambda \geq 0$, where n is the dimension of the delay-free system and $\Delta(\lambda) := \lambda I - A_0 - A_1e^{-\lambda}$. It is not difficult to prove that (1.1) is equivalent to

$$Range[\Delta(\lambda), P] = \mathbb{C}^{n\times n}. \tag{1.2}$$

Combining our perturbation results and [32], we will develop a necessary condition extending (1.2) to system with state delay and unbounded control term. Moreover, if the input space is of finite dimensions and the semigroup is immediately compact, the condition becomes sufficient.

Throughout this paper, we assume $p \in (1,\infty)$ (see Definition 2.1). Let X be a Banach space, we denote by $M|_X = \{x \in D(M) : Mx \in X\}$ the part of M in X and I_X by the identity operator on X. Assume that A generates a C_0-semigroup $T := (T(t))_{t\geq 0}$ on X, without any additional statement. And X_1 denotes the domain $D(A)$ equipped with the graph norm. By definition in [5], the extrapolation space X_{-1} is the completion of X under the norm $\|R(\lambda_0, A)\cdot\|$ with $R(\lambda_0, A)$ the resolvent of A at λ_0. Denote by $\{T_{-1}(t)\}_{t\geq 0}$ the extrapolation semigroup of $\{T(t)\}_{t\geq 0}$ on X_{-1}, with A_{-1} denoting its generator. It follows from [5] that A_{-1} is the continuously extension from $D(A)$ to X. Define P_τ by $(P_\tau f)(t) = f(t)$ when $0 \leq t \leq \tau$ and $(P_\tau f)(t) = 0$ when $t > \tau$. Denote $R^+ = [0,\infty)$. For u, $v \in L^p(R^+, U)$ and $\tau \geq 0$, the τ-concatenation of u and v, denoted by $u\Diamond_\tau v$, is defined by

$$(u\Diamond_\tau v)(t) = \begin{cases} u(t), & t < \tau,\\ v(t-\tau), & t \geq \tau. \end{cases}$$

2. Preliminaries on Regular Linear Systems

This section is to recall in a very sketchy way some concepts and results related to regular linear system in the sense of Salamon [21] and Weiss [30], those are to be used as the main tool throughout our work. Throughout this section X, U and Y are Banach spaces.

Definition 2.1. Let $\Omega = L^p(R^+, U)$ and $\Gamma = L^p(R^+, Y)$. A well-posed linear system on Ω, X, and Γ is a quadruple $\Sigma = (T, \Phi, \Psi, F)$, where:

(i) $T = (T(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X.

(ii) $\Phi = (\Phi(t))_{t\geq 0}$ is a family of bounded linear operators from Ω to X such that

$$\Phi(t+\tau)(u\Diamond_\tau v) = T(t)\Phi(\tau)u + \Phi(t)v,$$

for any $u, v \in \Omega$ and any $\tau, t \geq 0$.

(iii) $\Psi = (\Psi(t))_{t\geq 0}$ is a family of bounded linear operators from X to Γ such that

$$\Psi(t+\tau)x = \Psi(\tau)x\Diamond_\tau \Psi(t)T(\tau)x, \tag{2.1}$$

for any $x \in X$ and any $\tau, t \geq 0$, and $\Psi(0) = 0$.

(iv) $F = (F(t))_{t\geq 0}$ is a family of bounded linear operators from Ω to Γ such that

$$F(t+\tau)(u\Diamond_\tau v) = F(\tau)u\Diamond_\tau(\Psi(t)\Phi(\tau)u + F(t)v), \tag{2.2}$$

for any $u, v \in \Omega$ and any $\tau, t \geq 0$, and $F(0) = 0$.

U is the input space of Σ, X is the state space of Σ, and Y is the output space of Σ. The operators $\Phi(\tau)$ are called input map. The operators $\Psi(\tau)$ are called output map. The operators $F(\tau)$ are called input/output map.

By the representation theorem due to Weiss [26], there exists a unique operator $B \in L(U, X_{-1})$, called *admissible control operator* for A, such that for any $t \geq 0$ and $u \in L^p_{loc}(R^+, U)$,

$$\Phi(t)u = \int_0^t T_{-1}(t-s)Bu(s)ds \in X$$

where the integral exists in X_{-1}. In this case, we say that (A, B) generates an *abstract linear control system* on (X, U) and denote $\Phi = \Phi_{A,B}$ for brief.

By [27,28], there are unique operators $\Psi(\infty) : X \to L^2_{loc}(R^+, Y)$ and $F(\infty) : L^p_{loc}(R^+, U) \to L^p_{loc}(R^+, Y)$ such that, for any $\tau \geq 0$, the operators $\Psi(\tau)$ and $F(\tau)$ are obtained by truncation:

$$\Psi(\tau) = P_\tau \Psi(\infty),\ F(\tau)(t) = F(\infty)(t),\ \text{for any } t \leq \tau.$$

We call $\Psi(\infty)$ the extended output map of Σ, and $F(\infty)$ the extended input/output map of Σ. It is not difficult to obtain the following two equations related to the extended output map and the extended input/output map:

$$\Psi(\infty)x = \Psi(\infty)x\Diamond_\tau \Psi(\infty)T(\tau)x, \forall x \in X, \forall \tau \geq 0, \tag{2.3}$$

and

$$F(\infty)(u\Diamond_\tau v) = F(\infty)u\Diamond_\tau(\Psi(\infty)\Phi(\tau)u + F(\infty)v), \tag{2.4}$$

for any u, $v \in L^p_{loc}(R^+, U)$ and $\tau \geq 0$. By the representation theorem in [27], there exists a unique operator $C \in L(X_1, Y)$, called *admissible observation operator* for A, such that for any $t \geq 0$ and $x \in D(A)$, $CT(t)x = (\Psi_\infty x)(t)$. In this case, we say that (A, C) generates an *abstract linear observation system* on (X, Y) and denote $\Psi = \Psi_{A,C}$ for brief.

We say that the well-posed linear system Σ is *regular* if the limit

$$\lim_{t\to 0} \frac{1}{t} \int_0^t (F_\infty u_0)(s)ds = Dz \tag{2.5}$$

exists in Y for the constant input $u_0(t) = z, z \in U, t \geq 0$. In this case, we also say that the quadruple $\Sigma = (T, \Phi, \Psi, F)$ is a regular linear system on (X, U, Y) generated by (A, B, C, D), and we denote $\Sigma_{A,B,C,D} = \Sigma$ and $F_{A,B,C} = F$. Furthermore, we denote $(A, B, C) = (A, B, C, 0)$ and $\Sigma_{A,B,C} = \Sigma_{A,B,C,0}$ for brief.

In order to introduce the representation theorem of regular linear system, Weiss [30] introduced an extension of C, called Λ-*extension* with respect to A, which is defined by

$$C_\Lambda x = \lim_{\lambda\to\infty} C\lambda R(\lambda, A)x \tag{2.6}$$

with the domain $D(C_\Lambda) = \{x \in X : \text{this above limit exists in } Y\}$. It follows from [27, Theorem 4.5 and Proposition 4.7] that for any $x \in X$, $y(t) = C_\Lambda T(t)x$ a.e. in $t \geq 0$ whenever C is admissible for A.

With the above Λ-extension, for well-posed linear system $\Sigma_{A,B,C}$, the regularity condition (2.5) is equivalent to each of the following two conditions:

- *Range*$(R(\lambda, A_{-1})B) \subset D(C^A_\Lambda)$ holds for some (and hence for all) $\lambda \in \rho(A)$.
- For any $u \in U$, $G(\lambda)u$ has a limits when $\lambda \to \infty$, where G is the transfer function associated to $F(\infty)$.

In this case, the transfer function G is given explicitly by

$$G(\lambda) = C^A_\Lambda R(\lambda, A_{-1})B + D, \ Re(\lambda) > w_0(T), \tag{2.7}$$

where $w_0(T)$ is the growth bound of the semigroup T, and we denote $G_{A,B,C} = G$.

In order to state the following theorem, we define

$$\mathbb{D}^p(M) = \{f(\cdot) \in L^p_{loc}(R^+, X) : f \in D(M) \ for \text{ a.e. } t \geq 0, and \ Mf(\cdot) \in L^p_{loc}(R^+, X)\}.$$

Theorem 2.2. [28] *Let Σ be a regular linear system with generating operator A, B, C and D on (X, U, Y). Then, for given $(x_0, u) \in X \times L^p(R^+, U)$, the state trajectory $x(\cdot)$ of Σ, given by $x(t) := T(t)x_0 + \Phi_{A,B}u$, $t \geq 0$, is a.e. differential in X_{-1} and*

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \ x(0) = x_0 \ for \ a.e. \ t \geq 0. \tag{2.8}$$

Furthermore, $x(t) \in D(C^A_\Lambda)$ for a.e. $t \geq 0$ and the output function $y = \Psi_{A,C}(\infty)x + F_{A,B,C,D}(\infty)u$ of Σ is given by

$$y(t) = C^A_\Lambda x(t) + Du(t) \ for \ a.e. \ t \geq 0. \tag{2.9}$$

In particular, $\Phi_{A,B}(\cdot)u \in \mathbb{D}^p(C_\Lambda^A)$ *and the extended input–output map* $F(\infty)$ *is given by*

$$(F(\infty)u)(t) = C_\Lambda^A \int_0^t T_{-1}(t-s)Bu(s)ds + Du(t) \text{ for a.e. } t \geq 0. \quad (2.10)$$

Definition 2.3. [24,31] Let Σ be a regular linear system with input/output map $F(t)$. An operator $\Gamma \in L(Y,U)$ is called an admissible feedback for Σ if $I_Y - F(\cdot)\Gamma$ has uniformly bounded inverse.

Theorem 2.4. [24,31] *Assume that* (A,B,C) *generates a regular linear system* $\Sigma = (T,\Phi,\Psi,F)$ *on* (U,X,Y) *with admissible feedback operator* Γ*. Then feedback system* Σ^Γ *is also a regular system given by*

$$\begin{aligned}\Sigma^\Gamma &= \begin{pmatrix} T^\Gamma(\cdot) & \Phi^\Gamma(\cdot) \\ \Psi^\Gamma(\cdot) & F^\Gamma(\cdot) \end{pmatrix} \\ &= \begin{pmatrix} T(\cdot) + \Phi(\cdot)\Gamma(I_Y - F\Gamma)^{-1}\Psi & \Phi(\cdot)(I_U - \Gamma F(\cdot))^{-1} \\ (I_U - \Gamma F(\cdot))^{-1}\Psi & F(I_U - \Gamma F(\cdot))^{-1} \end{pmatrix}\end{aligned}$$

with the generating $(A^\Gamma, B^\Gamma, C^\Gamma)$:

$$\begin{aligned} A^\Gamma &= (A_{-1} + B\Gamma C_\Lambda)|_X \\ D(A^\Gamma) &:= \{z \in D((C)_\Lambda) : (A_{-1} + B\Gamma C_\Lambda)z \in X\} \\ B^\Gamma &= B, C^\Gamma = C_\Lambda^A. \end{aligned}$$

In addition, $D(C_\Lambda^A) = D(C_\Lambda^{A^\Gamma})$.

3. Perturbations of Regular Linear Systems

In this section, we develop two perturbation theorems of regular linear system based on the feedback theory [31]. Such results will be used throughout this paper. To do this, we first introduce some lemmas. Throughout this section X, U and Y are Banach spaces and we denote by I the identity operator on appropriate Banach space.

Lemma 3.1. *Let* (A,P) *generate an abstract observation system on* (X,X)*, then* (A,I_X,P) *generates a regular linear system with admissible feedback operator* I_X*.*

Proof. By [7, Proposition 3.3], for any $f \in L^p_{loc}(R^+,X)$, $\Phi_{A,I}(\cdot)f \in D^p(P_\Lambda)$ and $\|P_\Lambda\Phi_{A,I}(\cdot)f\|_{L^p([0,t],X)} \leq K(t)\|f\|_{L^p([0,t],X)}$, where $K(t) \to 0$ as $t \to 0$. Let $F(\tau) = P_\tau P_\Lambda \Phi(\cdot)$, it is not difficult to verify that F satisfies (2.2). It is not hard to show that (A,I_X,P) generates a regular linear system. Moreover, since $K(t) \to 0$ as $t \to 0$, $\|F(\tau)\| \to 0$ as $\tau \to 0$. Hence I_X is an admissible feedback operator. □

Lemma 3.2. *Assume that* (A,B,P) *generates a regular linear system on* (X,U,X)*. Then* $(A+P,B)$ *generates an abstract linear control system. Furthermore, we have* $\Phi_{A+P,B} = \Phi_{A+P,I}F_{A,B,P} + \Phi_{A,B}$*.*

Proof. Similar to the proof of [8, Theorem 3.3], consider the operators $\tilde{B} := (I, B) : X \times U \to X_{-1}$, $\tilde{C} = \begin{pmatrix} P \\ 0 \end{pmatrix} : X \to X \times U$.

By the definition and Lemma 3.1, it is easy to prove that $(A, \tilde{B}, \tilde{C})$ generates a regular linear system given by

$$\Sigma_{A,\tilde{B},\tilde{C}} := \begin{pmatrix} T & (\Phi_{A,I}, \Phi_{A,B}) \\ \begin{pmatrix} \Psi_{A,P} \\ 0 \end{pmatrix} & \begin{pmatrix} F_{A,I,P} & F_{A,B,P} \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

with admissible feedback operator $I_{X\times U}$. By Theorem 2.4, it follows that $A^{I_{X\times U}} = A + P$, $\tilde{B}^{I_{X\times U}} = \tilde{B}$, B is admissible for $A + P$ and

$$\begin{aligned} \Phi_{A+P,\tilde{B}^{I_{X\times U}}} &= (\Phi_{A+P,I}, \Phi_{A+P,B}) \\ &= (\Phi_{A,I}, \Phi_{A,B}) \begin{pmatrix} (I - F_{A,I,P})^{-1} & (I - F_{A,I,P})^{-1} F_{A,B,P} \\ 0 & I \end{pmatrix}. \end{aligned}$$

So we obtain

$$\Phi_{A+P,B} = \Phi_{A,I}(I - F_{A,I,P})^{-1} F_{A,B,P} + \Phi_{A,B} = \Phi_{A+P,I} F_{A,B,P} + \Phi_{A,B}. \quad (3.1)$$

□

The following lemma is due to Hadd and Idrissi [9].

Lemma 3.3. *Let (A, P) and (A, C) be abstract linear observation systems. Then $D(C_\Lambda^A) \cap D(P_\Lambda^A) = D(C_\Lambda^{A+P}) \cap D(P_\Lambda^A)$, and $C_\Lambda^{A+P} x = C_\Lambda^A x$, for any $x \in D(C_\Lambda^A) \cap D(P_\Lambda^A)$.*

Now we can prove the regularity invariance under some regularity perturbation.

Theorem 3.4. *Assume that (A, B, C) and (A, B, P) generate regular linear systems on (X, U, Y) and (X, U, X), respectively. Then $(A + P, B, C)$ generates a regular linear system with $F_{A+P,B,C} = F_{A+P,I,C} F_{A,B,P} + F_{A,B,C}$.*

Proof. By assumption, (A, B, P) and (A, B, C) generate regular linear systems, it follows from Theorem 2.2 that $Range(\Phi_{A,B}(t)) \subset D(C_\Lambda^A) \cap D(P_\Lambda^A)$ for a.e. $t \geq 0$. By lemma 3.3, we obtain $Range(\Phi_{A,B}(t)) \subset D(C_\Lambda^{A+P})$ and $C_\Lambda^{A+P} \Phi_{A,B}(t) = C_\Lambda^A \Phi_{A,B}(t)$, for a.e. $t \geq 0$. Moreover, we derive from Lemma 3.1 that $(A+P, I, C)$ generates a regular linear system, $Range(\Phi_{A+P,I}(t))) \subset D(C_\Lambda^{A+P})$ for a.e. $t \geq 0$. Thus, by (3.1), $\Phi_{A+P,B}(t)u = \Phi_{A+P,I}(t) F_{A,B,P}(t)u + \Phi_{A,B}(t)u \in D(C_\Lambda^{A+P})$ for any $u \in L^p_{loc}(R^+, U)$ and a.e. $t \geq 0$.

Hence we can define $F : (Fu)(t) = C_\Lambda^{A+P} \Phi_{A+P,B}(t)u$ for any $u \in L^p_{loc}(R^+, U)$ and a.e. $t \geq 0$. By simple calculation, we obtain that $F = F_{A+P,I,C}(\infty) F_{A,B,P}(\infty) + F_{A,B,C}(\infty)$. We can see from the definition of F that $F_0 = (P_t F)_{t\geq 0}$ satisfies (2.2). Taking Laplace transform on F, we obtain $G(\lambda) = G_{A+P,I,C}(\lambda) G_{A,B,P}(\lambda) + G_{A,B,F}(\lambda)$, thereby, for any $z \in U$, $G(\lambda)z \to 0$ as $\lambda \to 0$. The assertion follows immediately. □

In the above theorem, let $Y = X$ and $P = C$, we obtain the following corollary:

Corollary 3.5. *Assume that (A, B, C) generates a regular linear system on (X, U, X). Then, $(A + C, B, C)$ generates a regular linear system with the relation $F_{A+C,B,C} = F_{A+C,I,C}F_{A,B,C} + F_{A,B,C}$.*

Next, we will prove the second perturbation theorem. To do this, we begin with some important lemmas.

Lemma 3.6. *Assume that $(A, \Delta A, C)$ generates a regular linear system on (X, X, Y). Then, $((A_{-1} + \Delta A)|_X, C)$ generates an abstract linear observation system. Furthermore, $C_\Lambda^{(A_{-1}+\Delta A)|_X} = C_\Lambda^A$.*

Proof. Similar to the proof of [8, Theorem 3.3], consider the operators $\mathscr{B} := (\Delta A, 0) : X \times Y \to X_{-1}$, $\mathscr{C} = \begin{pmatrix} I \\ C \end{pmatrix} : X \to X \times Y$. Obversely, $D(\mathscr{C}_\Lambda^A) = D(C_\Lambda^A)$, $\mathscr{C}_\Lambda^A = \begin{pmatrix} I \\ C_\Lambda^A \end{pmatrix}$.

It is not hard to prove that $(A, \mathscr{B}, \mathscr{C})$ generates a regular linear system given by

$$\Sigma_{A,\mathscr{B},\mathscr{C}} := \begin{pmatrix} T & (\Phi_{A,\Delta A}, 0) \\ \begin{pmatrix} \Psi_{A,I} \\ \Psi_{A,C} \end{pmatrix} & \begin{pmatrix} F_{A,\Delta A,I} & 0 \\ F_{A,\Delta A,C} & 0 \end{pmatrix} \end{pmatrix}$$

with admissible feedback operator $I_{X\times Y}$. By Theorem 2.4, it follows that $A^{I_{X\times Y}} = (A_{-1} + \Delta A)|_X$, $\mathscr{C}_\Lambda^{(A_{-1}+\Delta A)|_X} = \begin{pmatrix} I \\ C_\Lambda^A \end{pmatrix}$ with $D(\mathscr{C}_\Lambda^{(A_{-1}+\Delta A)|_X}) = D(C_\Lambda^A)$ and C is admissible for $(A + \Delta A)|_X$. Furthermore, $C_\Lambda^{(A_{-1}+\Delta A)|_X} = C_\Lambda^A$. □

Lemma 3.7. *Let $(A, \Delta A)$ and (A, B) generate abstract linear control systems on (X, X) and (X, U), respectively. Then $((A_{-1} + \Delta A)|_X, B)$ generates an abstract linear control system. Moreover,*

$$\Phi_{(A_{-1}+\Delta A)|_X,B} = \Phi_{(A_{-1}+\Delta A)|_X,\Delta A}F_{A,B,I} + \Phi_{A,B}.$$

Proof. By [8, Theorem 3.3], it follows that $((A_{-1} + \Delta A)|_X, B)$ generates an abstract linear control system and

$$\begin{aligned}(\Phi_{(A_{-1}+\Delta A)|_X,\Delta A}, &\Phi_{(A_{-1}+\Delta A)|_X,B}) \\ &= (\Phi_{A,\Delta A}, \Phi_{A,B}) \begin{pmatrix} (I - F_{A,\Delta A,I})^{-1} & (I - F_{A,\Delta A,I})^{-1}F_{A,B,I} \\ 0 & I \end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}\Phi_{(A_{-1}+\Delta A)|_X,B} &= \Phi_{A,\Delta A}(I - F_{A,\Delta A,I})^{-1}F_{A,B,I} + \Phi_{A,B} \\ &= \Phi_{(A_{-1}+\Delta A)|_X,\Delta A}F_{A,B,I} + \Phi_{A,B}.\end{aligned}$$

□

Theorem 3.8. *Suppose that (A, B, C) and $(A, \Delta A, C)$ generate regular linear systems on (X, U, Y) and (X, X, Y), respectively. Then $((A_{-1} + \Delta A)|_X, B, C_\Lambda^A)$ generates a regular linear system with the relation*

$$F_{(A_{-1}+\Delta A)|_X,B,C_\Lambda^A} = F_{A,\Delta A,C}(I - F_{A,\Delta A,I})^{-1}F_{A,B,I} + F_{A,B,C}.$$

Proof. By Lemma 3.6 and Lemma 3.7, it follows that $C_\Lambda^{(A_{-1}+\Delta A)|_X} = C_\Lambda^A$ and $\Phi_{(A_{-1}+\Delta A)|_X,B} = \Phi_{A,\Delta A}(I - F_{A,\Delta A,I})^{-1}F_{A,B,I} + \Phi_{A,B}$. The combination of this and the assumption that (A, B, C) generates a regular linear system implies that $Range(\Phi_{(A_{-1}+\Delta A)|_X,B}(t)) \subset D(C_\Lambda^{(A_{-1}+\Delta A)|_X})$ for a.e. $t \geq 0$. Hence, we can define $F : (Fu)(t) = C_\Lambda^{(A_{-1}+\Delta A)|_X}\Phi_{(A_{-1}+\Delta A)|_X,B}(t)u$, for a.e. $t \geq 0$ and any $u \in L^p_{loc}(R^+, U)$. Through a simple calculation, $F = F_{A,\Delta A,C}(\infty)(I-F_{A,\Delta A,I}(\infty))^{-1}F_{A,B,I}(\infty)+F_{A,B,C}(\infty)$. We see from the definition of F that $F_1 = (P_tF)_{t\geq 0}$ satisfies (2.2). Moreover, the transfer function G of F satisfies $G(\lambda) = G_{A,\Delta A,C}(\lambda)(I-G_{A,\Delta A,I}(\lambda))^{-1}G_{A,B,I}(\lambda)+G_{A,B,C}(\lambda)$, thereby, for any $z \in U$, $G(\lambda)z \to 0$ as $\lambda \to 0$. By the definition of regular linear systems, our assertion follows immediately. □

In the above theorem, let $U = X$ and $\Delta A = B$, we can obtain the following corollary:

Corollary 3.9. *Let (A, B, C) generate a regular linear system on (X, X, Y), then $((A_{-1} + B)|_X, B, C_\Lambda^A)$ generates a regular linear system.*

4. Variation of Constants Formula of Admissible Observation Perturbed Control System

In this section, we will see that our perturbation results allow us to establish a new variation of constants formula for the mild solutions of (4.1) in terms of the initial semigroup $T(t)$ and the Λ-extension P_Λ^A of P with respect to A. To obtain the main theorem in this section, we first introduce the following lemma due to Hadd [7]:

Lemma 4.1. *Assume X to be a Banach space. Let $P \in L(X_1, X)$ be an admissible observation operator for A. Then, we have*

$$T_{A+P}(t)x = T(t)x + \int_0^t T(t-s)P_\Lambda^A T_{A+P}(s)x ds,$$

for any $x \in X$ and $t \geq 0$. Here $T = (T(t))_{t\geq 0}$ and $T_{A+P} = (T_{A+P}(t))_{t\geq 0}$ are the C_0-semigroups generated by A and $A + P$, respectively.

Now we state our variation of constants formula as follows.

Theorem 4.2. *Let (A, B, P) generate a regular linear system on the triple of Banach spaces (X, U, X). Then the mild solution of the following equation*

$$\dot{x}(t) = (A + P)x(t) + Bu(t), \ t \geq 0 \tag{4.1}$$

with $x(0) = x$ satisfies $x(\cdot) \in \mathbb{D}^p(P_\Lambda^A)$ and is given by

$$x(t) = T(t)x + \int_0^t T(t-s)P_\Lambda x(s)ds + \Phi_{A,B}(t)u, \tag{4.2}$$

for any $x \in X$ *and* $u \in L^p(R^+, U)$. *Moreover* $x(\cdot)$ *is a.e. differential in* X_{-1}^A.

Proof. In (3.1), taking $B = I$, we obtain

$$\begin{aligned}\Phi_{A+P,I}(t) &= \Phi_{A,I}(t) + \Phi_{A,I}(t)(I - F_{A,I,P}(t))^{-1} F_{A,I,P}(t) \\ &= \Phi_{A,I}(t)(F_{A+P,I,P}(t) + I).\end{aligned}$$

Multiplying the above equation by $F_{A,B,P}(t)$ from the right, we obtain

$$\Phi_{A+P,I}(t) F_{A,B,P}(t) = \Phi_{A,I}(t)(F_{A+P,I,P}(t) F_{A,B,P}(t) + F_{A,B,P}(t)). \quad (4.3)$$

From (2.10), Corollary 3.3 and 3.5, we obtain that for $u \in L^p(R^+, U)$,

$$\begin{aligned}\Phi_{A+P,I}(t) F_{A,B,P}(t) u &= \Phi_{A,I}(t) F_{A+P,B,P}(t) u \\ &= \int_0^t T(t-s) P_\Lambda^{A+P} \Phi_{A+P,B}(s) u ds. \quad (4.4)\end{aligned}$$

On the other hand, it follows from [7] that $T_{A+P}(\cdot)x \in \mathbb{D}^p(P_\Lambda^A)$ for any $x \in X$. By the definition of state trajectory and (3.1), we have

$$\begin{aligned}x(\cdot) &= T_{A+P}(\cdot)x + \Phi_{A+P,B}(t)u \\ &= T_{A+P}(\cdot)x + \Phi_{A+P,I}(\cdot) F_{A,B,P}(\cdot) u + \Phi_{A,B}(\cdot) u \in \mathbb{D}^p(P_\Lambda^A). \quad (4.5)\end{aligned}$$

Substituting (4.5) into (4.4), we obtain

$$\Phi_{A+P,I}(t) F_{A,B,P}(t) u = \int_0^t T(t-s) P_\Lambda^{A+P} (x(s) - T_{A+P}(s)x) ds.$$

Observe that by Theorem 4.1, $T_{A+P}(t)x = T(t)x + \int_0^t T(t-s) P_\Lambda^A T_{A+P}(s) x ds$, so we obtain

$$\begin{aligned}x(t) &= T(t)x + \int_0^t T(t-s) P_\Lambda^A T_{A+P}(s) x ds \\ &\quad + \int_0^t T(t-s) P_\Lambda^{A+P} (x(s) - T_{A+P}(s)x) ds + \Phi_{A,B}(t) u \\ &= T(t)x + \int_0^t T(t-s) P_\Lambda x(s) ds + \Phi_{A,B}(t) u,\end{aligned}$$

which is just (4.2). □

5. The Classical and Mild Solutions of Linear Systems with State and Output Delays

As an application example of the results obtained in the last two sections, we consider in this section the following linear system with time delays in state and output:

$$\begin{cases} \frac{d}{dt}x(t) = A_{-1}x(t) + Lx_t + Bu(t), t \geq 0, \\ x(0) = z, \\ x_0 = f, \\ y(t) = Cx(t) + Dx_t, \end{cases} \tag{5.1}$$

where the state space X, input space U and output space Y are Banach spaces, $A : D(A) \subset X \to X$ be a generator of C_0-semigroup on X, $L \in \mathscr{L}(W^{1,p}([-1,0],X),X)$, $B \in \mathscr{L}(U, X_{-1}^A)$, $z \in X$, $f \in L^p([-1,0],X)$, $1 < p < \infty$, $C \in \mathscr{L}(D(A),Y)$ where $D(A)$ is equipped with the graph norm, and $D \in \mathscr{L}(W^{1,p}([-1,0],X),Y)$.

Similar to [1,10], we give the definition of the classical solution of system (5.1) as follows.

Definition 5.1. Let $u \in W^{1,p}_{loc}(R^+,U)$, we call $x(\cdot)$ to be the classical solution of system (5.1) if

(i) $x(\cdot) \in C^1([0,\infty),X) \cap C([-1,\infty),X)$,
(ii) $A_{-1}x(t) + Bu(t) \in X$ and $x_t \in W^{1,p}([-1,0],X)$ for any $t \geq 0$,
(iii) $x(\cdot)$ satisfies $\frac{d}{dt}x(t) = A_{-1}x(t) + Lx_t + Bu(t)$, for any $t \geq 0$.

In order to investigate system (5.1), similar to [1], we convert it into a system without delay. To do this, we consider in this section the linear system with the state space $\mathscr{E} = X \times L^p([-1,0],X)$, input space U and the output space Y:

$$\begin{cases} \frac{d}{dt}\mathcal{X}(t) = (\mathcal{A}_L)_{-1}\mathcal{X}(t) + \mathcal{B}u(t), \\ \mathcal{X}(0) = \begin{pmatrix} z \\ f \end{pmatrix}, \\ y(t) = \mathfrak{C}\mathcal{X}(t), \end{cases} \tag{5.2}$$

where $\mathcal{A}_L = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, whose domain is given by

$$D(\mathcal{A}_L) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1,0],X) : f(0) = x \right\},$$

$\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$ and $\mathfrak{C} = (C, D)$, whose domain is $D(C) \times L^p([-1,0],X)$.

Definition 5.2. For $u \in W^{1,p}_{loc}(R^+,U)$, we call $\mathscr{X} : R^+ \to \mathscr{E}$ to be a classical solution of system (5.2), if $(\mathcal{A}_L)_{-1}\mathscr{X}(t) + \mathcal{B}u(t) \in \mathscr{E}$ and $\mathscr{X}(t) \in C^1(R^+,\mathscr{E})$ for $t \geq 0$.

Write $\mathcal{A}_L$ as $\mathcal{A}_L = \mathcal{A} + \Delta\mathcal{A}$, with

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \Delta\mathcal{A} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}. \tag{5.3}$$

According to [1], $\mathcal{A}$, with the domain $D(\mathcal{A}) = D(\mathcal{A}_L)$, generates a C_0-semigroup on $\mathscr{E}$ given by

$$\mathscr{T}(t) = \begin{pmatrix} T(t) & 0 \\ T_t & S_X(t) \end{pmatrix}, \tag{5.4}$$

where $T(t)$ is the C_0-semigroup generated by A, $T_t : X \to L^r([-1,0],X)$ are operators defined by

$$(T_t x)(\theta) = \begin{cases} T(t+\theta)x, & t+\theta \geq 0 \\ 0, & if\ not \end{cases}$$

and $(S_X(t))_{t\geq 0}$ is the left shift semigroup on $L^p([-1,0],X)$ with generator Q_X defined by

$$Q_X = \frac{d}{d\sigma} \text{ on } D(Q_X) = \{f \in W^{1,p}([-1,0],X) : f(0) = 0\}.$$

Moreover, it is easy to show that $\Delta\mathcal{A} \in \mathscr{L}(D(\mathcal{A}),\mathscr{E})$, where $D(\mathcal{A})$ is equipped with the graph norm.

Assume that $\phi = \phi(t)_{t\geq 0}$ is a family of bounded linear operators from $L^p(R^+,X)$ to $L^p([-1,0],X)$ defined by

$$(\phi(t)f)(\theta) = \begin{cases} f(t+\theta), & t+\theta \geq 0 \\ 0, & if\ not. \end{cases}$$

It follows from [30] that (S_X,ϕ) generates an abstract linear control system. We denote such control operator by β_X. In order to derive our main theorem in this section, we introduce in X the mass operator $\overline{L}$ (see [10]) as follows:

$$\begin{aligned} \overline{L}x &= \lim_{\lambda\to+\infty} Le_\lambda x, \\ D(\overline{L}) &= \{x \in X : \lim_{\lambda\to+\infty} Le_\lambda x \text{ exists}\}, \end{aligned} \tag{5.5}$$

where $e_\lambda : X \to L^r([-1,0],X)$, $e_\lambda x = e^{\lambda\cdot}x$. For mass operator, there exists an important proposition due to Said Hadd et al. [10] as follows.

Lemma 5.3. *Assume E to be a Banach space. Let $K \in L(W^{1,p}([-1,0],X),E)$, then (Q_X,β_X,K) is a regular triple if and only if $D(\overline{K}) = X$.*

Here we say (A,B,C) to be a regular triple on (X,U,Y) if, (i) (A,B) generates an abstract linear control system; (ii) (A,C) generates an abstract linear observation system; and (iii) $Range(R(\lambda,A_{-1})B) \subset D(C_\Lambda^A)$ holds for some (and hence for all) $\lambda \in \rho(A)$. Obversely, if (A,B,C) generates a regular linear system, (A,B,C) is a regular triple. Using Lemma 5.3, we can obtain the following lemma.

Lemma 5.4. *Let $K \in L(W^{1,p}([-1,0],X),E)$, where E is a Banach space. Let (Q_X,β_X,K) generate a regular linear system. Then*

$$K_\Lambda^{Q_X} = K - \overline{K}\delta_0 \tag{5.6}$$

on $W^{1,p}([-1,0],X)$. Here δ_0 denotes the point evaluation at zero.

Proof. It follows from the proof of [10, Proposition 3] that (5.6) holds on $D(K_\Lambda^{Q_X}) \cap W^{1,p}([-1,0],X)$. So it is sufficient to prove that $W^{1,p}([-1,0],X) \subset D(K_\Lambda^{Q_X})$. Let $f \in W^{1,p}([-1,0],X)$, $\lambda > 0$, we have

$$K\lambda R(\lambda, Q_X)f = K\lambda R(\lambda, Q_X)(f - e_0 f(0)) + K\lambda R(\lambda, Q_X)e_0 f(0). \quad (5.7)$$

The first term of the right hand of (5.7) has a limit $K(f - e_0 f(0))$. Since (Q_X, β_X, K) generates a regular linear system, by the combination of [10, (3.4)] and Theorem 2.2, $e_0 f(0) \in D(K_\Lambda^{Q_X})$. Our assertion follows immediately. □

In order to make our computation convenient, we define a diagonal operator by $\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, with the domain $D(\mathcal{A}_0) = D(A) \times D(Q_X)$. Obversely, the semigroup generated by $\mathcal{A}_0$ is $\mathbb{T} = \begin{pmatrix} T & 0 \\ 0 & S_X \end{pmatrix}$ with the extrapolation semigroup $\mathbb{T}_{-1} = \begin{pmatrix} T_{-1} & 0 \\ 0 & (S_X)_{-1} \end{pmatrix}$. Now we can obtain the following lemma, part of which can be found in [5, Exercise 3.8 (4)] without proof.

Lemma 5.5. *Let* $\mathcal{K} := -(\mathcal{A}_0)_{-1} \begin{pmatrix} 0 & 0 \\ (1 \otimes Id) & 0 \end{pmatrix}$, *where* $1 \otimes Id : X \to L^p([-1,0],X)$ *is defined by* $((1 \otimes Id)x)(\theta) = x$, *for any* $x \in X$ *and* $\theta \in [-1,0]$. *Then,* $\mathcal{K}$ *is admissible for* $\mathcal{A}_0$ *and* $\Phi_{\mathcal{A}_0,\mathcal{K}} = \begin{pmatrix} 0 & 0 \\ \phi(\cdot) & 0 \end{pmatrix}$.

Proof. For any $(u,v)^T \in L^p(R^+, \mathscr{E}) = L^p(R^+, X) \times L^p(R^+, L^p([-1,0],X))$ and $t > 0$,

$$\int_0^t \mathbb{T}_{-1}(t-s)\mathcal{K} \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} ds = - \begin{pmatrix} 0 \\ (Q_X)_{-1} \int_0^t S_X(t-s)(1 \otimes u(s))ds \end{pmatrix}. \quad (5.8)$$

It is easy to check that $\int_0^t S_X(t-s)(1 \otimes u(s))ds = \int_{max\{t+\cdot,0\}}^t u(s)ds \in D(Q_X)$. By definition, $\mathcal{K}$ is admissible for $\mathcal{A}_0$ and $\Phi_{\mathcal{A}_0,\mathcal{K}}(t) \begin{pmatrix} u \\ v \end{pmatrix}$ is equal to (5.8). Moreover, we can compute

$$\left(Q_X \int_0^t S_X(t-s)(1 \otimes u(s))ds \right)(\theta) = \begin{cases} -u(t+\theta), & t+\theta \geq 0, \\ 0, & t+\theta < 0, \end{cases}$$

combining which with the definition of ϕ we obtain $\Phi_{\mathcal{A}_0,\mathcal{K}}(t) = \begin{pmatrix} 0 & 0 \\ \phi(t) & 0 \end{pmatrix}$ for any $t \geq 0$. □

By the perturbation method, we can obtain the following theorem.

Theorem 5.6. *Assume that* (A,B,C) *generates a regular linear system, both* (Q_X, β_X, L) *and* (Q_X, β_X, D) *generate regular linear systems. Then,*

$(\mathcal{A}+\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0},\mathcal{B},\mathcal{C}_{\Lambda}^{\mathcal{A}_0})$ *generates a regular linear system. If in addition,* $\overline{L}=0$ *and* $\overline{D}=0$*, then the system* (5.2) *is a regular linear system.*

Proof. By Lemma 5.5, $\mathcal{K}$ is admissible for $\mathcal{A}_0$. For any $u\in L^p(R^+,U)$ and $t\geq 0$, we have

$$\int_0^t \mathbb{T}_{-1}(t-s)\mathcal{B}u(s)ds=\begin{pmatrix}\Phi_{A,B}(t)u\\ 0\end{pmatrix}\in\mathscr{E}.$$

So $\mathcal{B}$ is admissible for $\mathcal{A}_0$ and $\Phi_{\mathcal{A}_0,\mathcal{B}}(t)=\begin{pmatrix}\Phi_{A,B}(t)\\ 0\end{pmatrix}$.

For any $(x,f)^T\in D(\mathcal{A}_0)$,

$$\int_0^{t_0}\left\|\mathcal{C}\mathbb{T}(s)\begin{pmatrix}x\\ f\end{pmatrix}\right\|^p ds\leq 2^p\int_0^{t_0}\|CT(s)x\|^p ds+2^p\int_0^{t_0}\|DS_X(s)f\|^p ds.$$

Obversely, (A,C) and (Q_X,D) generates abstract linear observation systems. So $\mathcal{C}$ is admissible for $\mathcal{A}_0$, $\Psi_{\mathcal{A}_0,\mathcal{C}}=(\Psi_{A,C},\Psi_{Q_X,D})$ and $\mathcal{C}_{\Lambda}^{\mathcal{A}_0}=(C_{\Lambda}^{A},D_{\Lambda}^{Q_X})$ with $D(\mathcal{C}_{\Lambda}^{\mathcal{A}_0})=D(C_{\Lambda}^{A})\times D(D_{\Lambda}^{Q_X})$. By the assumption that (Q_X,β_X,L) generates a regular linear system, it follows that $\Delta\mathcal{A}$ is admissible for $\mathcal{A}_0$, $\Psi_{\mathcal{A}_0,\Delta\mathcal{A}}=\begin{pmatrix}0 & \Psi_{Q_X,L}\\ 0 & 0\end{pmatrix}$ and $\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0}=\begin{pmatrix}0 & L_{\Lambda}^{Q_X}\\ 0 & 0\end{pmatrix}$ with $D(\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0})=X\times D(L_{\Lambda}^{Q_X})$.

By the assumption that (A,B,C) generates a regular linear system and Theorem 2.2, we obtain that $Range(\Phi_{\mathcal{A}_0,\mathcal{B}}(t))\subset D(\mathcal{C}_{\Lambda}^{\mathcal{A}_0})$ for a.e. $t\geq 0$. Let $F_1=\mathcal{C}_{\Lambda}^{\mathcal{A}_0}\Phi_{\mathcal{A}_0,\mathcal{B}}(\cdot)$ and $F(\tau)=P_\tau F_1$ for $\tau\geq 0$. Then it is not difficult to verify that $F=F_{A,B,C}$, thus F satisfies (2.2). By the definition of $F_{A,B,C}$, the Laplace transform of F_1 is $G_{A,B,C}$. Hence $(\mathcal{A}_0,\mathcal{B},\mathcal{C})$ generates a regular linear system and $F_{\mathcal{A}_0,\mathcal{B},\mathcal{C}}=F_{A,B,C}$.

Similarly, we can show that $(\mathcal{A}_0,\mathcal{K},\mathcal{C})$, $(\mathcal{A}_0,\mathcal{B},\Delta\mathcal{A})$ and $(\mathcal{A}_0,\mathcal{K},\Delta\mathcal{A})$ generate regular linear systems with the relations $F_{\mathcal{A}_0,\mathcal{K},\mathcal{C}}=(F_{Q_X,\beta_X,D},0)$, $F_{\mathcal{A}_0,\mathcal{B},\Delta\mathcal{A}}=\begin{pmatrix}F_{Q_X,\beta_X,L} & 0\\ 0 & 0\end{pmatrix}$ and $F_{\mathcal{A}_0,\mathcal{K},\Delta\mathcal{A}}=0$, respectively.

By Theorem 3.8, both $(\mathcal{A},\mathcal{B},\mathcal{C}_{\Lambda}^{\mathcal{A}_0})$ and $(\mathcal{A},\mathcal{B},\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0})$ generate regular linear systems. It follows from Theorem 3.4 that $(\mathcal{A}+\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0},\mathcal{B},\mathcal{C}_{\Lambda}^{\mathcal{A}_0})$ generates a regular linear system.

We can obtain by Lemma 5.4 that $L_{\Lambda}^{Q_X}=L-\overline{L}\delta_0$ and $D_{\Lambda}^{Q_X}=D-\overline{D}\delta_0$ on $W^{1,p}([-1,0],X)$. Thus under the assumption $\overline{L}=0$ and $\overline{D}=0$, we obtain $L_{\Lambda}^{Q_X}=L$ and $D_{\Lambda}^{Q_X}=D$ on $W^{1,p}([-1,0],X)$. Hence $\mathcal{A}+\Delta\mathcal{A}_{\Lambda}^{\mathcal{A}_0}=\mathcal{A}_L$ and $\mathcal{C}_{\Lambda}^{\mathcal{A}_0}=\mathcal{C}$ on $D(\mathcal{A})$. This implies that (5.2) is a regular linear system. □

Remark 5.7. It is reasonable to consider the conditions $\overline{L} = 0$ and $\overline{D} = 0$. In fact, as is described in [10], the frequently-used delay operators given by

$$Rz := \int_{-1}^{0} d\mu(\sigma)z(\sigma), z \in C([-1,0], X),$$

where $\mu : [-1, 0] \to L(X, Y)$ is an operator valued function of bounded variation with no mass at 0, i.e.,

$$\lim_{\epsilon \searrow 0} |\mu|([-\epsilon, 0]) = 0, \tag{5.9}$$

satisfies $\overline{R} = 0$. In particular, if $Y = X$ and $\mu := \mathcal{X}_{[a,0]}(\cdot)I_X$ for some $a \in [-1, 0)$, then $Rz = \delta_a z := z(a)$. In this case μ satisfies (5.9), so we have $\overline{R} = 0$.

Now we prove the equivalence of system (5.1) and (5.2) under some assumptions. To do this, we first introduce the following lemma:

Lemma 5.8. [1] *For $\lambda \in \mathbb{C}$ we have $\lambda \in \rho(\mathcal{A}_L)$ if and only if $A + Le_\lambda$. Moreover, for $\lambda \in \rho(\mathcal{A}_L)$ the resolvent $R(\lambda, \mathcal{A}_L)$ is given by*

$$R(\lambda, \mathcal{A}_L) = \begin{pmatrix} R(\lambda, A + L_\lambda) & R(\lambda, A + L_\lambda)LR(\lambda, Q_X) \\ e_\lambda R(\lambda, A + L_\lambda) & [e_\lambda R(\lambda, A + L_\lambda)L + I]R(\lambda, Q_X) \end{pmatrix},$$

where $L_\lambda = Le_\lambda$.

Theorem 5.9. *Assume that (A, B) generates an abstract linear control system, (Q_X, β_X, L) generates a regular linear system, and $\overline{L} = 0$. Let $u \in W^{1,p}_{loc}(R^+, U)$. Denote by $\mathscr{X}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$ the classical solution of system (5.2). Let*

$$m(t) = \begin{cases} x(t), & t \geq 0; \\ f(t), & -1 \leq t < 0. \end{cases} \tag{5.10}$$

Then $m(\cdot)$ is classical solution of system (5.1) and $z(t) = x_t$.

Conversely, if $x(t)$ is the classical solution of system (5.1), $(x(t), x_t)^T$ is the classical solution of (5.2).

Proof. Assume $\mathscr{X}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$ to be the classical solution of system (5.2). By Theorem 3.4, Theorem 3.8, Lemma 5.5 and Theorem 5.6, we obtain

$$\begin{aligned} F_{\mathcal{A}_L,\mathcal{B},I} &= F_{\mathcal{A}+\Delta\mathcal{A},\mathcal{B},I} \\ &= F_{\mathcal{A}+\Delta\mathcal{A},I,I}F_{\mathcal{A},\mathcal{B},\Delta\mathcal{A}} + F_{\mathcal{A},\mathcal{B},I} \\ &= F_{\mathcal{A}+\Delta\mathcal{A},I,I}F_{((\mathcal{A}_0)_{-1}+\mathcal{K})|_{\mathscr{E}},\mathcal{B},\Delta\mathcal{A}} + F_{((\mathcal{A}_0)_{-1}+\mathcal{K})|_{\mathscr{E}},\mathcal{B},I} \\ &= F_{\mathcal{A}+\Delta\mathcal{A},I,I}[F_{\mathcal{A}_0,\mathcal{K},\Delta\mathcal{A}}(I - F_{\mathcal{A}_0,\mathcal{K},I})^{-1}F_{\mathcal{A}_0,\mathcal{B},I} + F_{\mathcal{A}_0,\mathcal{B},\Delta\mathcal{A}}] \\ &\quad + F_{\mathcal{A}_0,\mathcal{K},I}(I - F_{\mathcal{A}_0,\mathcal{K},I})^{-1}F_{\mathcal{A}_0,\mathcal{B},I} + F_{\mathcal{A}_0,\mathcal{B},I}. \end{aligned}$$

For λ big enough, taking Laplace transform on both sides of the above equation, we obtain the following equation of the corresponding transfer function

$$\begin{aligned} & R(\lambda,(\mathcal{A}_L)_{-1})\begin{pmatrix} B \\ 0 \end{pmatrix} \\ & \quad = R(\lambda,\mathcal{A}_L)\begin{pmatrix} L_\Lambda^{Q_X}e_\lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ e_\lambda & I \end{pmatrix}\begin{pmatrix} R(\lambda,A_{-1})B \\ 0 \end{pmatrix} \\ & \qquad + \begin{pmatrix} 0 & 0 \\ e_\lambda & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ e_\lambda & I \end{pmatrix}\begin{pmatrix} R(\lambda,A_{-1})B \\ 0 \end{pmatrix} + \begin{pmatrix} R(\lambda,A_{-1})B \\ 0 \end{pmatrix} \\ & \quad = \begin{pmatrix} I \\ e_\lambda \end{pmatrix}[R(\lambda,A+L_\lambda)L_\Lambda^{Q_X}e_\lambda + I]R(\lambda,A_{-1})B. \end{aligned} \tag{5.11}$$

Here the result $R(\lambda,(Q_X)_{-1})\beta_X = e_\lambda$ in [10] and Lemma 5.8 are applied.

Since $\mathscr{X}$ is the classical solution, we obtain

$$(\mathcal{A}_L)_{-1}\mathscr{X}(t) + \begin{pmatrix} B \\ 0 \end{pmatrix}u(t) \in \mathscr{E}, \text{ for any } t \geq 0. \tag{5.12}$$

Let $\lambda \in \rho(\mathcal{A}_L)$ be fixed. From (5.12), we obtain

$$\mathscr{X}(t) - R(\lambda,(\mathcal{A}_L)_{-1})\begin{pmatrix} B \\ 0 \end{pmatrix}u(t) \in D(\mathcal{A}_L),$$

that is

$$\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} - \begin{pmatrix} I \\ e_\lambda \end{pmatrix}[R(\lambda,A+L_\lambda)L_\Lambda^{Q_X}e_\lambda + I]R(\lambda,A_{-1})Bu(t) \in D(\mathcal{A}_L).$$

This means

$$z(t)(0) = x(t) \tag{5.13}$$

and

$$z(t) \in W^{1,p}([-r,0],X). \tag{5.14}$$

In particular,

$$f = z(0) \in W^{1,p}([-r,0],X). \tag{5.15}$$

By Lemma 5.8, we obtain

$$\begin{aligned} & R(\lambda,\mathcal{A}_L)\mathscr{X}(t) \\ & \quad = \begin{pmatrix} R(\lambda,A+L_\lambda)x(t) + R(\lambda,A+L_\lambda)LR(\lambda,Q_X)z(t) \\ e_\lambda R(\lambda,A+L_\lambda)x(t) + [e_\lambda R(\lambda,A+L_\lambda)L + Id]R(\lambda,Q_X)z(t) \end{pmatrix}. \end{aligned} \tag{5.16}$$

On the other hand, multiplying (5.2) by $R(\lambda,\mathcal{A}_L)$ from the left, it follow that

$$\begin{aligned} \frac{d}{dt}R(\lambda,\mathcal{A}_L)\mathscr{X}(t) &= R(\lambda,(\mathcal{A}_L)_{-1})(\mathcal{A}_L)_{-1}\mathscr{X}(t) + R(\lambda,(\mathcal{A}_L)_{-1})\mathcal{B}u(t) \\ &= -\mathscr{X}(t) + \lambda R(\lambda,\mathcal{A}_L)\mathscr{X}(t) + R(\lambda,(\mathcal{A}_L)_{-1})\mathcal{B}u(t). \end{aligned} \tag{5.17}$$

Combining (5.16) and (5.17), we obtain

$$\begin{aligned} &R(\lambda, A+L_\lambda)\frac{d}{dt}x(t) + R(\lambda, A+L_\lambda)LR(\lambda, Q_X)\frac{d}{dt}z(t) \\ &\quad = -x(t) + \lambda R(\lambda, A+L_\lambda)x(t) + \lambda R(\lambda, A+L_\lambda)LR(\lambda, Q_X)z(t) \\ &\qquad + [R(\lambda, A+L_\lambda)L_\Lambda^{Q_X} e_\lambda + I]R(\lambda, A_{-1})Bu(t) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &e_\lambda R(\lambda, A+L_\lambda)\frac{d}{dt}x(t) + [e_\lambda R(\lambda, A+L_\lambda)L + Id]R(\lambda, Q_X)\frac{d}{dt}z(t) \\ &\quad = -z(t) + \lambda e_\lambda R(\lambda, A+L_\lambda)x(t) + \lambda[e_\lambda R(\lambda, A+L_\lambda)L + Id]R(\lambda, Q_X)z(t) \\ &\qquad + e_\lambda[R(\lambda, A+L_\lambda)L_\Lambda^{Q_X} e_\lambda + I]R(\lambda, A_{-1})Bu(t). \end{aligned} \tag{5.19}$$

Subtracting e_λ(5.18) from (5.19), it follows that

$$R(\lambda, Q_X)\frac{d}{dt}z(t) = e_\lambda x(t) - z(t) + \lambda R(\lambda, Q_X)z(t). \tag{5.20}$$

Multiplying (5.20) by $\lambda - Q_X$ from the left, we obtain $\frac{d}{dt}z(t) = \frac{d}{d\sigma}z(t)$, for any $t \geq 0$. Observe (5.13) and (5.14), we rewrite the relations of $z(t)$ and $x(t)$ as follows:

$$\begin{cases} \frac{d}{dt}z(t) = \frac{d}{d\sigma}z(t), \quad t \geq 0 \\ z(t)(0) = x(t), \\ z(0) = f. \end{cases} \tag{5.21}$$

Moreover, the combining of (5.15) and the assumption that $\mathscr{X}(\cdot)$ is classical solution implies that $x \in W^{1,p}([-1,0], X) \cap C^1(R^+, X)$. It follows from [10, Lemma 1] that $x \in W^{1,p}_{loc}([-r,\infty), X)$. Hence from [1], we obtain

$$\frac{d}{dt}x_t = \frac{d}{d\sigma}x_t. \tag{5.22}$$

As in [1], we set $y(t) = z(t) - x_t$. Then, the combining (5.21) and (5.22) implies that $y(t)$ is the solution of the abstract Cauchy problem associated to Q_X on $L^p([-1,0], X)$ with initial value $y(0) = 0$. Hence $y(t) = 0$ for any $t \geq 0$. This implies that $z(t) = x_t$.

Multiplying (5.18) by $\lambda - (A + L_\lambda)$ from the left, and using (5.21), we obtain

$$\frac{d}{dt}x(t) = A_{-1}x(t) + Lx_t + Bu(t) - \overline{L}R(\lambda, A_{-1})Bu(t).$$

By the assumption that the mass operator $\overline{L} = 0$, the above equation becomes

$$\frac{d}{dt}x(t) = A_{-1}x(t) + Lx_t + Bu(t).$$

Thus, $x(t)$ is the classical solution of system (5.1).

Now we consider the converse case. Assume $x(t)$ to be the classical solution of system (5.1). Then, $x(\cdot) \in W^{1,p}_{loc}([-r,\infty), X)$. It follows from [10, Lemma 1] that $x_\cdot \in C^1(R^+, L^p([-r,0], X))$. Define $\mathscr{X}(t) = \begin{pmatrix} x(t) \\ x_t \end{pmatrix}$, using the converse procedure of the last part, it is not difficult to verify that $\mathscr{X}(t)$ is the classical solution of system (5.2). □

Lemma 5.10. [23, Theorem 3.1] *If $(\mathscr{A}, \mathscr{B})$ generates an abstract linear control system on Banach spaces E and F, then for any $u \in W^{1,p}_{loc}(R^+, F)$ with $\mathscr{A}_{-1}x(0) + \mathscr{B}u(0) \in E$, the system $\dot{x}(t) = (\mathscr{A})_{-1}x(t) + \mathscr{B}u(t)$ has a classical solution.*

By Theorem 5.9 and Lemma 5.10, $(\mathcal{A}_L)_{-1}\mathscr{X}(0) + \mathfrak{B}u(0) \in \mathscr{E}$ implies that system (5.1) has a classical solution under the same assumption as in Theorem 5.9. Furthermore, from the proof of Theorem 5.9, it follows that the combining of $A_{-1}z + Bu(0) \in X$, $f \in W^{1,p}([-1, 0], X)$ and $f(0) = z$ implies the existence of mild solution of system (5.1). This can be specifically described by the following corollary.

Corollary 5.11. *Assume that (A, B) generates an abstract linear control system, (Q_X, β_X, L) generates a regular linear system, and $\overline{L} = 0$. Let $u \in W^{1,p}_{loc}(R^+, U)$, $A_{-1}x + Bu(0) \in X$, $f \in W^{1,p}([-1, 0], X)$ and $f(0) = z$. Then system (5.1) has a classical solution.*

With the equivalence of system (5.1) and system (5.2), it is reasonable to define the mild solution of (5.1) by the mild solution of (5.2) as in [1]. Moreover, from the proof of Lemma 5.10 (see [23]), we obtain that system (5.2) has a mild solution for any $(z, f, u)^T \in \mathscr{E} \times L^p_{loc}(R^+, U)$. Hence we are able to introduce the following concept.

Definition 5.12. Assume that (A, B) generates an abstract linear control system, (Q_X, β_X, L) generates a regular linear system, and $\overline{L} = 0$. For all $(z, f, u)^T \in \mathscr{E} \times L^p_{loc}(R^+, U)$, denote by $\mathscr{X}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$ the mild solution of system (5.2), we call m defined by (5.10) the mild solution of system (5.1).

Theorem 5.13. *Assume that (A, B, C), (Q_X, β_X, L) and (Q_X, β_X, D) generate regular linear systems. Let $\overline{L} = 0$ and $\overline{D} = 0$. Then the mild solution $\mathscr{X}(t)$ of the system (5.2) is of the form $\begin{pmatrix} x(t) \\ x_t \end{pmatrix}$ and satisfies $x_t \in D(L_\Lambda)$ for a.e. $t \geq 0$ and*

$$x(t) = T(t)z + \int_0^t T(t-s)L_\Lambda x_s ds + \Phi_{A,B}u. \tag{5.23}$$

Moreover, the output is given by

$$y(t) = C^A_\Lambda x(t) + D^{Q_X}_\Lambda x_t, \tag{5.24}$$

for a.e. $t \geq 0$.

Proof. Similar to the proof of [16, Proposition 2.5], the family $(\mathscr{T}_{L,U}(t))_{t\geq 0}$ given by

$$\mathscr{T}_{L,U}(t)\begin{pmatrix} x \\ f \\ g \end{pmatrix} = \begin{pmatrix} (\mathscr{T}_L)_{-1}(t)\begin{pmatrix} x \\ f \end{pmatrix} + \int_0^t \mathscr{T}_L(t-s)\begin{pmatrix} B \\ 0 \end{pmatrix} g(s)ds \\ S(t)g \end{pmatrix}$$

with $\mathscr{T}_L$ being the C_0-semigroup generated by $\mathcal{A}_L$ generates a C_0-semigroup on $\mathscr{E} \times L^p(R^+, U)$. The generator of $(\mathscr{T}_{L,U}(t))_{t\geq 0}$ is given by

$$\mathcal{A}_{L,U}\begin{pmatrix} x \\ f \\ g \end{pmatrix} = \begin{pmatrix} (\mathcal{A}_L)_{-1}\begin{pmatrix} x \\ f \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} g(0) \\ Qg \end{pmatrix}$$

with the domain

$$D(\mathcal{A}_{L,U}) = \left\{ \begin{pmatrix} x \\ f \\ g \end{pmatrix} : \begin{pmatrix} x \\ f \end{pmatrix} \in \mathscr{E}, g \in W^{1,p}(R^+, U), \right.$$
$$\left. (\mathcal{A}_L)_{-1}\begin{pmatrix} x \\ f \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} g(0) \in \mathscr{E} \right\}.$$

Hence, for $\begin{pmatrix} z \\ f \\ u \end{pmatrix} \in D(\mathcal{A}_{L,U})$, $\mathscr{X}(t)$ is the classical solution of system (5.2). By the density of $D(\mathcal{A}_{L,U})$ in $\mathscr{E} \times L^p(R^+, X)$ and the fact that $\mathscr{X}(t)$ is depend continuously on the initial values, the mild solution is of the form $\begin{pmatrix} x(t) \\ x_t \end{pmatrix}$.

It follows from Theorem 4.2 that

$$\mathscr{X}(t) = \mathscr{T}(t)\begin{pmatrix} z \\ f \end{pmatrix} + \int_0^t \mathscr{T}(t-s)\Delta A_\Lambda^{\mathcal{A}}\mathscr{X}(s)ds + \Phi_{\mathcal{A},\mathcal{B}}(t)u. \tag{5.25}$$

By Lemma 3.7, it follows that

$$\begin{aligned} \Phi_{\mathcal{A},\mathcal{B}}(t)u &= \Phi_{\mathcal{A}_0+K,\mathcal{B}}(t)u \\ &= \Phi_{\mathcal{A}_0,\mathcal{K}}(t)(I - F_{\mathcal{A}_0,\mathcal{K},I})^{-1}F_{\mathcal{A}_0,\mathcal{B},I}u + \Phi_{\mathcal{A}_0,\mathcal{B}}(t)u \\ &= \begin{pmatrix} 0 & 0 \\ \phi(t) & 0 \end{pmatrix}\begin{pmatrix} \Phi_{A,B}(\cdot)u \\ 0 \end{pmatrix} + \begin{pmatrix} \Phi_{A,B}(t)u \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \Phi_{A,B}(t)u \\ \phi(t)\Phi_{A,B}(\cdot)u \end{pmatrix}. \end{aligned} \tag{5.26}$$

Substituting (5.26) into (5.25) and, taking the first component, we obtain (5.23). The formula (5.24) follows immediately from (2.9). □

6. Stabilizability of Linear System with State and Output Delays

This section discusses the stabilizability of system (5.1). Assume (M, N) to generate a control linear system on the Banach spaces $\mathbb{E}$ and $\mathbb{F}$. We say that the pair (M, N) is *stabilizable* if there exists $F \in L(D(M), \mathbb{F})$, such that (i) (M, N, F) generates a regular linear system Σ; (ii) the identity operator $I_{\mathbb{F}} : \mathbb{F} \to \mathbb{F}$ is an admissible feedback operator for Σ; (iii) the semigroup generated by $M_{-1} + NF_\Lambda^M$ is exponentially stable. In this case, we say that

F stabilizes (M, N). For this concept and for further comments we refer to [20,32]. Now we introduce a necessary condition for stabilizability.

Lemma 6.1. [32] *Assume (M, N) to be stabilizable on the Banach spaces $\mathbb{E}$ and $\mathbb{F}$, then there exists a constant $\delta > 0$ such that*

$$Ran[\lambda - M_{-1}, N] \supset \mathbb{E}$$

for all complex number λ with $Re\lambda > -\delta$.

Combining the above lemma and our perturbation results, we derive a necessary condition for stabilizability for delay equation (5.1).

Theorem 6.2. *Assume that (A, B) generates an abstract linear control system, (Q_X, β_X, L) generates a regular linear system, and $\overline{L} = 0$. If system (5.1) is stabilizable,*

$$X \subset (\lambda - A_{-1} - L_\lambda)X + BU \tag{6.1}$$

holds for all $\lambda \in \sigma(\mathcal{A}_L)$ with $Re\lambda \geq 0$.

Proof. Assume system (5.1) to be stabilizable. This means by Theorem 5.9 that system (5.2) is stabilizable. It follows from Lemma 6.1 that there exists a constant $\alpha > 0$ such that

$$Ran[\lambda - (\mathcal{A}_L)_{-1}, \mathcal{B}] \supset \mathscr{E}$$

for any $\lambda \in C$ with $Re\lambda > -\alpha$. This implies, for any $x \in X$ and $f \in L^p([-1, 0], X)$ there exist $(x_0, f_0) \in \mathscr{E}$ and $u \in U$ such that, for $Re\lambda > -\alpha$,

$$(x, f)^T = (\lambda - (\mathcal{A}_L)_{-1})(x_0, f_0)^T + \mathcal{B}u.$$

Let $\mu \in \rho(\mathcal{A}_L) = \rho((\mathcal{A}_L)_{-1})$ be fixed. By (5.11), it follows that

$$\begin{aligned}\begin{pmatrix} x \\ f \end{pmatrix} &= (\lambda - (\mathcal{A}_L)_{-1}) \left(\begin{pmatrix} x_0 \\ f_0 \end{pmatrix} + \begin{pmatrix} I \\ e_\mu \end{pmatrix} [R(\mu, A + L_\mu)L_\Lambda e_\mu + I]R(\mu, A_{-1})Bu \right) \\ &\quad + (\mu - \lambda) \begin{pmatrix} I \\ e_\mu \end{pmatrix} [R(\mu, A + L_\mu)L_\Lambda e_\mu + I]R(\mu, A_{-1})Bu \in \mathscr{E}.\end{aligned}$$

This indicates that

$$\begin{pmatrix} x_0 \\ f_0 \end{pmatrix} + \begin{pmatrix} I \\ e_\mu \end{pmatrix} [R(\mu, A + L_\mu)L_\Lambda e_\mu + I]R(\mu, A_{-1})Bu \in D(\mathcal{A}_L)$$

and

$$\begin{pmatrix} x \\ f \end{pmatrix} = (\lambda - \mathcal{A}_L) \left(\begin{pmatrix} x_0 \\ f_0 \end{pmatrix} + \begin{pmatrix} I \\ e_\mu \end{pmatrix} [R(\mu, A + L_\mu)L_\Lambda e_\mu + I]R(\mu, A_{-1})Bu \right) \tag{6.2}$$

$$+ (\mu - \lambda) \begin{pmatrix} I \\ e_\mu \end{pmatrix} [R(\mu, A + L_\mu)L_\Lambda e_\mu + I]R(\mu, A_{-1})Bu$$

$$= \begin{pmatrix} \lambda x_0 - A_{-1}X_0 - Lf_0 + Bu - \overline{L}R(\mu, A_{-1})Bu \\ (\lambda - \frac{d}{d\sigma})f_0 \end{pmatrix}. \tag{6.3}$$

Substituting the assumption $\overline{L} = 0$ into (6.2), we can derive

$$x = \lambda x_0 - A_{-1}X_0 - Lf_0 + Bu \tag{6.4}$$

and

$$f = \left(\lambda - \frac{d}{d\sigma}\right) f_0. \tag{6.5}$$

From (6.5), we obtain $f_0 = e_\lambda x_0 + R(\lambda, Q_X)f$. Substituting f_0 into (6.4) and putting $f = 0$, we obtain

$$x = \lambda x_0 - A_{-1}x_0 - L_\lambda x_0 + Bu.$$

Hence the condition (6.1) holds. □

In the rest of this section, we will give a necessary and sufficient condition of stabilizability of system with finite-dimensional input space and impact original semigroup by combining our perturbation theorem and [3]. To do this, we first introduce a lemma.

Lemma 6.3. *Assume that (A, B) generates an abstract linear control system and the semigroup $(T(t))_{t\geq 0}$ generated by A is immediately compact, that is, $T(t)$ is compact for $t > 0$. Then $t \to (F(t) = \int_0^t T_{-1}(s)Bds : U \to X)$ is compact for $t > 0$.*

Proof. Let ϵ be small enough. we can compute

$$(F(t) - F(\epsilon))u = \int_\epsilon^t T_{-1}(s)Buds = T(\epsilon)F(t-\epsilon)u.$$

By the assumption, $T(\epsilon)$ is compact, so $F(t) - F(\epsilon)$ is compact. From the admissibility of B for A, we obtain

$$\|(F(t) - (F(t) - F(\epsilon)))u\| = \|F(\epsilon)u\| \leq \epsilon^{1/p}\|\Phi_{A,B}(\epsilon)\|\|u\|, \text{ for any } u \in U.$$

This implies that $F(t) - (F(t) - F(\epsilon)) \to 0$, as $\epsilon \to 0$ in the uniform operator topology. Hence $F(t)$ is compact for $t > 0$. □

Theorem 6.4. *Assume that (A, B) generates an abstract linear control system, (Q_X, β_X, L) generates a regular linear system, and $\overline{L} = 0$. In addition, let $T(t)$ be compact for $t > 0$ and the input space be finite dimensional. Then, system (5.1) is stabilizable if and only if*

$$X = \{(\lambda - A_{-1} - L_\lambda)x + Bu : \ x \in X, u \in U, A_{-1}x - Bu \in X\}$$

holds for all $\lambda \in \sigma(\mathcal{A}_L)$ with $Re\lambda \geq 0$.

Proof. The necessity has been proved in Theorem 6.2. Now we prove the sufficiency. It is easy to see that the operator $\begin{pmatrix} A_{-1} & B \\ 0 & 0 \end{pmatrix}$ with domain $D\left(\begin{pmatrix} A_{-1} & B \\ 0 & 0 \end{pmatrix}\right) = X \times U$ generates a C_0-semigroup on $X_{-1} \times U$ given by

$$\mathscr{S}(t) = \begin{pmatrix} T_{-1}(t) & \int_0^t T_{-1}(\sigma)Bd\sigma \\ 0 & I_U \end{pmatrix}.$$

Because of the admissibility of B for A, the restriction of $\mathscr{S}(t)$ to $X \times U$ is also a C_0-semigroup given by

$$\mathscr{S}_0(t) = \begin{pmatrix} T(t) & \int_0^t T_{-1}(\sigma) B d\sigma \\ 0 & I_U \end{pmatrix},$$

which is generated by

$$\mathfrak{A} = \begin{pmatrix} A_{-1} & B \\ 0 & 0 \end{pmatrix}$$

with the domain

$$D(\mathfrak{A}) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in X \times U : \; A_{-1}x + Bu \in X \right\}.$$

By Lemma 6.3, the operator $t \to (\int_0^t T_{-1}(t-\sigma) B ds : U \to X)$ is immediately compact. Combining this with the compactness of I_U and the the immediate compactness of $T(\cdot)$, we obtain that $\mathscr{S}_0(t)$ is immediately compact.

Define

$$\mathfrak{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} : \; W^{1,p}([-r,0], X \times U)$$

and $\xi = \begin{pmatrix} x \\ u \end{pmatrix} : [-r, \infty) \to X \times U$ and $v = \dot{u}$ where u is assumed to be smooth with $u = 0$ on $[-1, 0]$. Then the system (5.1) is converted into

$$\begin{cases} \dot{\xi}(t) = \mathfrak{A}\xi(t) + \mathfrak{L}\xi_t + \mathfrak{B}v(t), \; t \geq 0 \\ \xi(t) = (f(t), 0), \; \text{a.e. } t \in [-1, 0], \end{cases} \tag{6.6}$$

where $\mathfrak{B} : U \to X \times U$ is defined by $\mathfrak{B}v = (0, v)^T$. Obversely, $\mathfrak{B}$ is bounded. By the analysis in Sect. 5 and [7], system (6.6) is equivalent to

$$\begin{cases} \dot{\tau}(t) = \mathbb{M}_{\mathfrak{L}}\tau(t) + \mathbb{B}v(t), \; t \geq 0, \\ \tau(0) = (z, u(0), \phi, 0)^T, \end{cases} \tag{6.7}$$

where $\tau(t) = (\xi(t), \xi_t)^T$, $\mathbb{M}_{\mathfrak{L}} = \begin{pmatrix} \mathfrak{A} & \mathfrak{L} \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$ with $D(\mathbb{M}_{\mathfrak{L}}) = \left\{ \begin{pmatrix} \xi \\ g \end{pmatrix} \in D(\mathfrak{A}) \times W^{1,p}([-1,0], X \times U) : \; g(0) = \xi \right\}$ and $\mathbb{B} = (\mathfrak{B}, 0)^T$. By Lemma 5.8,

$$\lambda \in \sigma(\mathbb{M}_{\mathfrak{L}}) \Leftrightarrow \lambda \in \sigma(\mathfrak{A} + \mathfrak{L}E_\lambda)$$

with $E_\lambda = \begin{pmatrix} e_\lambda & 0 \\ 0 & e_\lambda \end{pmatrix}$ and $\mathfrak{A} + \mathfrak{L}E_\lambda = \begin{pmatrix} A_{-1} + Le_\lambda & B \\ 0 & 0 \end{pmatrix}$.

Since $Le_\lambda \in L(X)$, $\sigma(\mathfrak{A} + \mathfrak{L}E_\lambda) = \sigma(A + Le_\lambda) \cup \{0\} = \sigma(\mathcal{A}_L) \cup \{0\}$. By [18], $\mathscr{T}$ is eventually compact, specifically, $\mathscr{T}(t)$ is compact for $t > 1$. Hence the set

$$\sigma^+ = \{\lambda \in \sigma(\mathbb{M}_{\mathfrak{L}}) : \Re\lambda \geq 0\}$$

is finite.

Take any $(x,u) \in X \times U$ and let $\lambda \in \sigma^+$. By assumption, there exist $x_0 \in X$ and $u_0 \in U$ such that $A_{-1}x_0 - Bu_0 \in X$ and $(\lambda - A_{-1} - Le_\lambda)x_0 + Bu_0 = x$, which implies that

$$(\lambda - \mathfrak{A}) \begin{pmatrix} x_0 \\ -u_0 \end{pmatrix} - \mathfrak{L} \begin{pmatrix} e_\lambda x_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u + \lambda u_0 \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}.$$

Take any ϕ and ψ such that

$$(x, u, \phi, \psi)^T \in X \times U \times L^p([-1,0], X) \times L^p([-1,0], U),$$

and define

$$\phi_0 = e_\lambda x_0 + R(\lambda, Q_X).$$

Then,

$$\begin{aligned} (\lambda - \mathfrak{A}) \begin{pmatrix} x_0 \\ -u_0 \end{pmatrix} - \mathfrak{L} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u + \lambda u_0 \end{pmatrix} \\ = \begin{pmatrix} x \\ u \end{pmatrix} - \mathfrak{L} R(\lambda, Q_{X\times U}) \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \end{aligned}$$

The combination of the last two identities implies

$$(\lambda - \mathbb{M}_{\mathfrak{L}})(x_0, -u_0, \phi_0, 0) + \mathbb{B} v_0 = \begin{pmatrix} I & -\mathfrak{L} R(\lambda, Q_{X\times U}) \\ 0 & I \end{pmatrix} (x, u, \phi, \psi)^T,$$

where $v_0 = u + \lambda u_0$. This is equivalent to

$$Range(\lambda - \mathbb{M}_{\mathfrak{L}}) + Range(\mathbb{B}) = X \times U \times L^p([-1,0], X) \times L^p([-1,0], U)$$

for any $\lambda \in \sigma^+$. By [3, Theorem 1], system (6.7), or system (5.1) is stabilizable. □

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References

[1] Bátkai, A., Piazzera, S.: Semigroup and linear partial differential equations with delay. J. Math. Anal. Appl. **264**, 1–20 (2001)

[2] Bátkai, A., Piazzera, S.: Semigroups for Delay Equations, vol. 10. A K Peters, Ltd, Wellesley (2005)

[3] Bhat, K.P.M., Wonham, W.M.: Stabilizability and detectability for evolution systems on Banach spaces. In: Proceedings of IEEE Conference Decision Control 15th Symposium Adaptive Processes, pp. 1240–1243 (1976)

[4] Curtain, R.F., Pritchard, A.J.: Infinite dimensional linear systems theory. In: Lecture Notes in Information Sciences, vol. 8. Springer-Verlag, Berlin (1978)

[5] Engel, K.J., Nagel, R.: One Parameter Semigroups for Linear Evolutional Equations. Springer-Verlag, New York (2000)

[6] Haak, B., Kunstmann, P.C.: Admissibility of unbounded operators and well-posedness of linear systems in Banach spaces. Integr. Equ. Oper. Theory **55**(4), 497–533 (2006)

[7] Hadd, S.: Unbounded perturbations of C_0-semigroups on Banach spaces and applications. Semigroup Forum **70**, 451–465 (2005)

[8] Hadd, S.: Exact controllability of infinite dimensional systems persists under small perturbations. J. Evol. Equ. **5**, 545–555 (2005)

[9] Hadd, S., Idrissi, A.: On the admissibility of observation for perturbed C_0-semigroups on Banach spaces. Syst. Control Lett. **55**, 1–7 (2006)

[10] Hadd, S., Idrissi, A., Rhandi, A.: The regular linear systems associated to the shift semigroups and application to control delay systems. Math. Control Signals Syst. **18**, 272–291 (2006)

[11] Hale, J.K.: Functional Differential Equations, Applied Mathematical Sciences, vol. 3. Springer-Verlag, Berlin (1971)

[12] Jacob, B., Partington, J.R.: The Weiss conjecture on admissibility of observation operators for contraction semigroups. Integr. Equ. Oper. Theory **40**(2), 231–241 (2001)

[13] Jacob, B., Partington, J.R., Pott, S.: Conditions for admissibility of observation operators and boundedness of Hankel operators. Integr. Equ. Oper. Theory **47**(3), 315–338 (2003)

[14] Jacob, B., Zwart, H.: Counterexamples concerning observation operators for C_0 semigroups. SIAM J. Control Optim. **43**(1), 137–153 (2004)

[15] Maciá, F., Zuazua, E.: On the lack of observability for wave equations: Gaussian beam approach. Asymptot. Anal. **32**(1), 1–26 (2002)

[16] Malinen, J., Staffans, O.J., Weiss, G.: When is a linear system conservative?. Quart. Appl. Math. **64**, 61–91 (2006)

[17] Malinen, J., Staffans, O.J.: Conservative boundary control systems. J. Differ. Equ. **231**, 290–312 (2006)

[18] Mátrai, T.: On perturbations of eventually compact semigroups preserving eventual compactness. Semigroup Forum **69**, 317–340 (2004)

[19] Olbrot, A.W.: Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays. IEEE Trans. Automat. Control **23**(5), 887–890 (1978)

[20] Rebarber, R.: Conditions for the equivalence of internal and external stability for distributed parameter systems. IEEE Trans. Automat. Control **38**, 994–998 (1993)

[21] Salamon, D.: Infinite-dimensional linear system with unbounded control and observation: a functional analytic approach. Trans. Am. Math. Soc. **300**, 383–431 (1987)

[22] Salamon, D.: Realization theory in Hilbert space. Math. Syst. Theory **21**, 147–164 (1989)

[23] Staffans, O.J., Weiss, G.: Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup. Trans. Am. Math. Soc. **354**, 3229–3262 (2002)

[24] Staffans, O.J.: Well-Posed Linear Systems. Cambridge University Press, Cambridge (2005)

[25] Tucsnak, M., Weiss, G.: Observation and Control for Operators Semigroups. Birkhäuser Verlag, Basel (2009)

[26] Weiss, G.: Admissibility of unbounded control operators. SIAM J. Control Optim. **27**, 527–545 (1989)

[27] Weiss, G.: Admissible observation operators for linear semigroups. Isr. J. Math. **65**, 17–43 (1989)

[28] Weiss, G.: The representation of regular linear systems on Hilbert spaces. In: Kappel, F., Kunisch, K., Schappacher, W. (eds.) Control and Estimation of Distributed Parameter Systems (Proceedings Vorau 1988), pp. 401–416. Birkhäuser, Basel

[29] Weiss, G.: Two conjectures on the admissibility of control operators. In: Desch, W., Kappel, F. (eds.) Estimation and Control of Distributed Parameter Systems, pp. 367–378. Birkhäuser, Basel (1991)

[30] Weiss, G.: Transfer functions of regular linear systems. Part I: characterizations of regularity. Trans. Am. Math. Soc. **342**(2), 827–854 (1994)

[31] Weiss, G.: Regular linear systems with feedback. Math. Control Signals Syst. **7**, 23–57 (1994)

[32] Weiss, G., Rebarber, R.: Optimizability and estimatability for infinite-dimensional linear systems. SIAM J. Control Optim. **39**, 1204–1232 (2000)

[33] Zhong, Q.C.: Robust Control of Time-Delay Systems. Springer-Verlag, London (2006)

[34] Zwart, H., Jacob, B., Staffans, O.: Weak admissibility does not imply admissibility for analytic semigroups. Syst. Control Lett. **48**(3), 341–350 (2003)

[35] Zwart, H.: Sufficient conditions for admissibility. Syst. Control Lett. **54**, 973–979 (2005)

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