Nonlinear Dyn (2010) 62: 867–874
DOI 10.1007/s11071-010-9769-4

ORIGINAL PAPER

Exponential synchronization of chaotic neural networks with time delays: a M-matrix approach

Z. Xing · J. Peng · K. Wang

Received: 30 November 2009 / Accepted: 23 June 2010 / Published online: 22 July 2010

Abstract Based on M-matrix theory, global exponential synchronization of a class of time-varying delayed chaotic neural networks is investigated. Without designing a Lyapunov function, some new criteria are established under less restrictive conditions using this approach. Finally, simulation examples are given to verify the effectiveness of the obtained conditions.

Keywords Synchronization · M-matrix · Time-varying delay · Chaotic neural networks

1 Introduction

The drive-response concept was proposed by Pecora and Corroll for constructing the synchronization of coupled chaotic systems [1]. Synchronization can be reached between the response system and its driving system via the controlling signal sent from the driving system when some conditions are satisfied. Since then, the study of chaotic synchronization has attracted increasing attention due to the its important application in secure communication [2–4], chemical and biological systems [5, 6], and so on. And a wide variety of extensions and alternative synchronization schemes have been proposed including state coupling control [7, 8], adaptive control [9–11], impulsive control [4, 12], and other control methods [13–17].

Artificial neural networks with or without time delays may exhibit chaotic behavior [18–20], and so there has been increasing interest in the study of the synchronization of chaotic neural networks [7, 8, 21, 22]. However, it should be noted that much of the previous work in this area is based on Lyapunov direct methods, where constructing a proper Lyapunov function is important for the synchronization analysis. How to construct a proper Lyapunov function for a given system is very difficult and there are no general rules to follow. Instead of constructing a Lyapunov function, He and Cao used a different method, matrix measure, to research the synchronization of chaotic neural networks [8]. However, we found that their results are quite limited. Here, we give another approach to discuss the exponential synchronization of coupled chaotic systems with time delays without constructing a Lyapunov function.

In this paper, we will consider a general class neural network with time-varying delays, which exhibit chaotic behavior (e.g., see [18–20]), described by

Z. Xing (✉) · J. Peng · K. Wang
School of Science, Xi'an Jiaotong University,
Xi'an 710049, China
e-mail: xzw.xing@gmail.com

J. Peng
e-mail: jgpeng@mail.xjtu.edu.cn

K. Wang
School of Science, Chang'an University, Xi'an 710064,
China
e-mail: kmwang@chd.edu.cn

the following differential equations:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(x_j\big(t-\tau_j(t)\big)\big) + I_i, \quad i = 1, 2, \ldots, n \tag{1}$$

or in vector form

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = -\mathbf{C}x(t) + \mathbf{A}f\big(x(t)\big) + \mathbf{B}g\big(x\big(t-\tau(t)\big)\big) + I, \tag{2}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in R^n$ is the state vector of the network at time t; $\mathbf{C} = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ is a positive diagonal matrix; $\mathbf{A} = (a_{ij})_{n\times n}$; $\mathbf{B} = (b_{ij})_{n\times n}$ are the output feedback and delayed output feedback weight matrices; f_i, g_i are output transfer functions, $\mathbf{f}(\mathbf{x}(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^{\mathrm{T}}$, $\mathbf{g}(\mathbf{x}(t-\tau(t))) = (g_1(x_1(t-\tau_1(t))), \ldots, g_n(x_n(t-\tau_n(t))))^{\mathrm{T}}$; $\mathbf{I} = (I_1, I_2, \ldots, I_n)^{\mathrm{T}} \in R^n$ is external input; $\tau_i(t)$ is a time-varying delay that satisfies $0 \le \tau_i(t) \le \tau$.

The initial conditions of system (1) are of the form: $x_i(\theta) = \phi_i(\theta) \in C([-\tau, 0], R)$, $\theta \in [-\tau, 0]$, $i = 1, 2, \ldots, n$.

For the purpose of synchronization, we consider the response system that is driven by (1) as follows:

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(y_j(t)\big) + I_i + u_i(t) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j\big(t-\tau_j(t)\big)\big), \quad i = 1, 2, \ldots, n \tag{3}$$

where $u_i(t)$ is the driving signals. And in this paper, we consider the state coupling and output coupling, i.e., for $i = 1, \ldots, n$,

$$u_i(t) = \sum_{j=1}^{n} \omega_{ij}\big(y_j(t) - x_j(t)\big) \tag{4}$$

and

$$u_i(t) = \sum_{j=1}^{n} \omega_{ij}\big(f_j\big(y_j(t)\big) - f_j\big(x_j(t)\big)\big), \tag{5}$$

respectively, where $\Omega = (\omega_{ij})_{n\times n}$ is the coupling matrix. The initial conditions of system (3) are of the form $y_i(\theta) = \varphi_i(\theta) \in C([-\tau, 0], R)$, $\theta \in [-\tau, 0]$, $i = 1, 2, \ldots, n$.

Throughout the paper, we do not make additional assumption on the system (1) and (3) except the following traditional requirement:

(H) There exist constants $L_i^f, L_i^g > 0$ such that

$$0 \le \big(f_i(\xi_1) - f_i(\xi_2)\big)/(\xi_1 - \xi_2) \le L_i^f,$$
$$0 \le \big(g_i(\xi_1) - g_i(\xi_2)\big)/(\xi_1 - \xi_2) \le L_i^g$$

hold for any $\xi_1, \xi_2 \in R$, $i = 1, 2, \ldots, n$.

In previous results, most authors always required that f_i, g_i satisfied (H), because the transfer functions f_i and g_i are frequently chosen as sigmoidal functions in neural network applications, i.e., f_i, g_i are monotonically increasing, odd and with the saturation properties that $\lim_{\xi\to\pm\infty} f_i(\xi) = \lim_{\xi\to\pm\infty} g_i(\xi) = \pm 1$. Furthermore, by the Lyapunov method, $0 \le \dot{\tau}_i(t) \le \theta < 1$ is also required besides (H), which is removed in this paper (see Example 1).

For convenience, we introduce the following notations: $\|\cdot\|_1$ is the vector 1-norm of R^n; $\mathbf{L}^f = \mathrm{diag}\{L_1^f, L_2^f, \ldots, L_n^f\}$; $\mathbf{L}^g = \mathrm{diag}\{L_1^g, L_2^g, \ldots, L_n^g\}$; and for $\mathbf{A} \in R^{n\times n}$, the notations $\mathbf{A}_d = \mathrm{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\}$, $|\mathbf{A}| = (|a_{ij}|)_{n\times n}$.

The organization of this paper is as follows. In Sect. 2, we state some definitions and lemmas needed in the later sections. In Sect. 3, we derive the sufficient conditions to ensure the systems synchronization and some simulations are given in Sect. 4. Finally, concluding remarks are given.

2 Preliminaries

In this section, some definitions and lemmas which will be used later are given.

Definition 1 Systems (1) and (3) are said to be exponentially synchronized if there exist constants $M > 0$, $\eta > 0$ such that

$$\big\|\mathbf{y}(t) - \mathbf{x}(t)\big\|_1 \le \mathbf{M}\|\varphi - \phi\|_\tau e^{-\eta(t-t_0)}, \quad t > t_0$$

where $\|\varphi - \phi\|_\tau = \sup_{s\in[\tau,0]} \|\varphi(s) - \phi(s)\|_1$.

Definition 2 ([23]) Let the matrix $\mathbf{Q} = (q_{ij})_{n\times n}$ have non-positive off-diagonal elements and all leading principal minors of $\mathbf{Q}$ are positive, then $\mathbf{Q}$ is said to be an M-matrix.

Lemma 1 ([24]) *Let* $\mathbf{x}(t) = (x_1, \ldots, x_n(t))^{\mathrm{T}}$ *be a solution of the differential equation*

$$\frac{d\mathbf{x}(t)}{dt} \le \mathbf{A}\mathbf{x}(t) + \mathbf{B}\bar{\mathbf{x}}(t), \quad t \le t_0,$$

where $\bar{\mathbf{x}}(t) = (\sup_{t-\tau\le s\le t} x(s), \ldots, \sup_{t-\tau\le s\le t} x(s))^{\mathrm{T}}$, $\mathbf{A}, \mathbf{B} \in R^{n\times n}$.

If

(a) $a_{ij} \ge 0 (i \ne j), b_{ij} \ge 0, \quad i, j = 1, 2, \ldots, n$; $\sum_{j=1}^n \bar{x}_j(t_0) > 0$
(b) *the matrix* $-(A + B)$ *is an* M*-matrix*

then there exist constants λ, $r_i > 0$ *such that*

$$x_i(t) \le r_i \sum_{j=1}^n \bar{x}_j(t_0) e^{\lambda(t-t_0)}.$$

Define the error $e_i(t) = y_i(t) - x_i(t)$ between the systems (1) and (3), and for any diagonal positive-definite matrix $\mathbf{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$, $\|\mathbf{e}\|_1 \to 0$ if and only if $\|\mathbf{P}^{-1}\mathbf{e}\|_1 \to 0$. From the networks (1) and (3), we have that

$$\begin{aligned}\frac{\mathrm{d}\tilde{e}_i(t)}{\mathrm{d}t} &= -c_i\tilde{e}_i(t) + p_i^{-1}\sum_{i=1}^n a_{ij}F_j\big(p_j\tilde{e}_j(t)\big) \\ &\quad + p_i^{-1}\sum_{i=1}^n b_{ij}G_j\big(p_j\tilde{e}_j\big(t-\tau_j(t)\big)\big) \\ &\quad + p_i^{-1}\sum_{i=1}^n \omega_{ij}p_j\tilde{e}_j(t) \qquad (6)\end{aligned}$$

for $u_i(t) = \sum_{j=1}^n \omega_{ij}(y_j(t) - x_j(t))$ and

$$\begin{aligned}\frac{\mathrm{d}\tilde{e}_i(t)}{\mathrm{d}t} &= -c_i\tilde{e}_i(t) + p_i^{-1}\sum_{i=1}^n a_{ij}F_j\big(p_j\tilde{e}_j(t)\big) \\ &\quad + p_i^{-1}\sum_{i=1}^n b_{ij}G_j\big(p_j\tilde{e}_j\big(t-\tau_j(t)\big)\big) \\ &\quad + p_i^{-1}\sum_{i=1}^n \omega_{ij}\big(f_j\big(p_jy_j(t)\big) - f_j\big(p_jx_j(t)\big)\big). \qquad (7)\end{aligned}$$

for $u_i(t) = \sum_{j=1}^n \omega_{ij}(f_j(p_j(y_j(t)) - f_j(p_jx_j(t)))$, where

$$\begin{aligned}&F_j\big(p_j\tilde{e}_j(t)\big) = f_j\big(p_jy_j(t)\big) - f_j\big(p_jx_j(t)\big), \\ &G_j\big(p_j\tilde{e}_j\big(t-\tau(t)\big)\big) \\ &\quad = g_j\big(p_jy_j\big(t-\tau_j(t)\big)\big) \\ &\qquad - g_j\big(p_jx_j\big(t-\tau_j(t)\big)\big), \quad i = 1, 2, \ldots, n.\end{aligned}$$

3 Main results

In this section, we will give some sufficient conditions for synchronizing the response system (3) with the drive system (1) for different coupling conditions.

Theorem 1 *Assume* (*H*) *holds, and* $u_i(t)$ *is defined as in* (4). *If there exists a diagonal matrix* $\mathbf{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$, $p_i > 0$ *such that* $-(\Delta + \mathbf{P}^{-1}|\mathbf{B}|\mathbf{L}^g\mathbf{P})$ *is an* M*-matrix, then the system* (3) *will globally exponentially synchronize with the system* (1), *where* $\Delta = \mathbf{K} + \mathbf{P}^{-1}(|\Omega - \Omega_d| + |\mathbf{A} - \mathbf{A}_d|\mathbf{L}^f)\mathbf{P}$ *and* $\mathbf{K}$ *is a diagonal matrix with* $k_i = -c_i + \omega_{ii} + a_{ii}L_i^f$.

Proof By (H), we have

$$\begin{aligned}&F_i(e_i)\,\mathrm{sign}(e_i) = \big|F_i(e_i)\big| \le L_i^f|e_i|, \\ &G_i(e_i)\,\mathrm{sign}(e_i) = \big|G_i(e_i)\big| \le L_i^g|e_i|,\end{aligned} \qquad (8)$$

for any $e_i \in R$. Let $z_i(t) = |\tilde{e}_i(t)|$, then the upper right derivative $\mathrm{D}^+z_i(t)$ along the solutions of (6) is as follows:

$$\begin{aligned}\frac{\mathrm{D}^+z_i(t)}{\mathrm{d}t} &= \Bigg[-c_i\tilde{e}_i(t) + p_i^{-1}\sum_{j=1}^n a_{ij}F_j\big(p_j\tilde{e}_j(t)\big) \\ &\quad + p_i^{-1}\sum_{j=1}^n b_{ij}G_j\big(p_j\tilde{e}_j\big(t-\tau(t)\big)\big) \\ &\quad + p_i^{-1}\sum_{j=1}^n \omega_{ij}p_j\tilde{e}_i(t)\Bigg]\mathrm{sign}\big(\tilde{e}_i(t)\big) \\ &\le -c_iz_i(t) + \omega_{ii}z_i(t) + p_i^{-1}a_{ii}\big|F_i\big(p_i\tilde{e}_i(t)\big)\big| \\ &\quad + p_i^{-1}\sum_{j=1,i\ne j}^n |\omega_{ij}|p_j\big|\tilde{e}_j(t)\big| \\ &\quad + p_i^{-1}\sum_{j=1,i\ne j}^n |a_{ij}|\big|F_j\big(p_j\tilde{e}_j(t)\big)\big|\end{aligned}$$

$$
\begin{aligned}
&+ p_i^{-1} \sum_{j=1}^{n} |b_{ij}| \big| G_j\big(p_j \tilde{e}_j\big(t-\tau(t)\big)\big)\big| \\
&\le \big(-c_i + \omega_{ii} + a_{ii} L_i^f\big) z_i(t) \\
&+ p_i^{-1} \sum_{j=1, j\neq i}^{n} |\omega_{ij}| p_j \big|\tilde{e}_j(t)\big| \\
&+ p_i^{-1} \sum_{j=1, j\neq i}^{n} |a_{ij}| L_j^f p_j \big|\tilde{e}_j(t)\big| \\
&+ p_i^{-1} \sum_{j=1}^{n} |b_{ij}| L_j^g p_j \big|\tilde{e}_j\big(t-\tau(t)\big)\big|.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\mathrm{D}^+ z_i(t)}{\mathrm{d}t} &\le \big(-c_i + \omega_{ii} + a_{ii} L_i^f\big) z_i(t) \\
&+ p_i^{-1} \sum_{j=1, j\neq i}^{n} \big(|\omega_{ij}| + |a_{ij}| L_j^f\big) p_j z_j(t) \\
&+ p_i^{-1} \sum_{j=1}^{n} |b_{ij}| L^g p_j \bar{z}_j(t).
\end{aligned}
$$

That is,

$$
\begin{aligned}
\frac{\mathrm{D}^+ \mathbf{z}(t)}{\mathrm{d}t} &\le \big(\mathbf{K} + \mathbf{P}^{-1}\big(|\Omega - \Omega_d| + |\mathbf{A} - \mathbf{A}_d| \mathbf{L}^f\big)\mathbf{P}\big) z(t) \\
&+ \mathbf{P}^{-1} |\mathbf{B}| \mathbf{L}_j^g \mathbf{P} \bar{\mathbf{z}}(t),
\end{aligned}
$$

where $\mathbf{K}$ is a diagonal matrix with $k_i = -c_i + \omega_{ii} + a_{ii} L_i^f$. Let $\Delta = \mathbf{K} + \mathbf{P}^{-1}(|\Omega - \Omega_d| + |\mathbf{A} - \mathbf{A}_d| \mathbf{L}^f)\mathbf{P}$. Obviously, $\Delta_{ij} \ge 0$, $p_i^{-1} |b_{ij}| p_j \ge 0$, $i \neq j$. By Lemma 1, if $-(\Delta + \mathbf{P}^{-1}|\mathbf{B}|\mathbf{L}^g\mathbf{P})$ is an M-matrix, there must exist constants $\mu \ge 0, r_i \ge 0$ such that

$$
z_i(t) = \big|e_i(t)\big| \le r_i \sum_{j=1}^{n} \bar{z}_i(0) e^{-\mu t}, \quad i = 1, 2, \ldots, n.
$$

Then

$$
\big|y_i(t) - x_i(t)\big| \le r_i e^{-\mu t} \sum_{j=1}^{n} \sup_{-\tau \le s \le 0} \big|y_i(s) - x_i(s)\big|,
$$

which implies that the system (3) is globally exponentially synchronous with the system (1). The proof is completed. □

Theorem 2 *Assume* (H) *holds, and* $u_i(t)$ *is defined as in* (5). *For some given diagonal matrix* $\mathbf{P} = \{p_1, p_2, \ldots, p_n\}$ *with* $p_i > 0$, *if* $-(\Delta + \mathbf{P}^{-1}|\mathbf{B}|\mathbf{L}^g\mathbf{P})$ *is an* M*-matrix, where* $\Delta = \mathbf{K} + \mathbf{P}^{-1}(|\Omega - \Omega_d + \mathbf{A} - \mathbf{A}_d|)\mathbf{L}^f\mathbf{P}$ *and* $\mathbf{K}$ *is a diagonal matrix with* $k_i = -c_i + (\omega_{ii} + a_{ii})L_i^f$, *then the system* (3) *is globally exponentially synchronous with the system* (1).

Proof Equation (8) holds by (H). Let $z_i(t) = |\tilde{e}_i(t)|$. Analogously to the proof of Theorem 1, we then calculate the upper right derivative $\mathrm{D}^+ z_i(t)$ along the trajectories of system (7).

$$
\begin{aligned}
\frac{\mathrm{D}^+ z_i(t)}{\mathrm{d}t} &= \Bigg[-c_i \tilde{e}_i(t) + p_i^{-1} \sum_{j=1}^{n} (a_{ij} + \omega_{ij}) \\
&\quad \times F_j\big(p_j \tilde{e}_j(t)\big) \\
&\quad + p_i^{-1} \sum_{j=1}^{n} b_{ij} G_j\big(p_j \tilde{e}_j\big(t-\tau(t)\big)\big)\Bigg] \\
&\quad \times \operatorname{sign}\big(\tilde{e}_i(t)\big) \\
&\le \big(-c_i + (\omega_{ii} + a_{ii}) L_i^f\big) z_i(t) \\
&\quad + p_i^{-1} \sum_{j=1, j\neq i}^{n} \big(|a_{ij} + \omega_{ij}|\big) L_j^f p_j z_j(t) \\
&\quad + p_i^{-1} \sum_{j=1}^{n} |b_{ij}| L_j^g p_j z_j\big(t - \tau_j(t)\big) \\
&\le \big(-c_i + (\omega_{ii} + a_{ii}) L_i^f\big) z_i(t) \\
&\quad + p_i^{-1} \sum_{j=1, j\neq i}^{n} \big(|a_{ij} + \omega_{ij}|\big) L_j^f p_j z_j(t) \\
&\quad + p_i^{-1} \sum_{j=1}^{n} |b_{ij}| L_j^g p_j \bar{z}_j(t), \\
i &= 1, 2, \ldots, n.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\frac{\mathrm{D}^+ \mathbf{z}(t)}{\mathrm{d}t} &\le \big(\mathbf{K} + \mathbf{P}^{-1}\big(|\Omega - \Omega_d + \mathbf{A} - \mathbf{A}_d| \mathbf{L}^f\big)\mathbf{P}\big) \mathbf{z}(t) \\
&+ \mathbf{P}^{-1} |\mathbf{B}| \mathbf{L}^g \mathbf{P} \bar{\mathbf{z}}(t),
\end{aligned}
$$

where K is a diagonal matrix with $k_i = -c_i + (\omega_{ii} + a_{ii}) L_i^f$. The rest of the proof is similar to that of Theorem 1, and omitted here. The proof is completed. □

Taking $\mathbf{P} = \mathbf{I}$ in Theorem 1, one can easily obtain the following corollary.

Corollary 1 *Assume* (H) *holds, and* $u_i(t)$ *is defined as in* (4). *If* $-(\Delta + |\mathbf{B}|\mathbf{L}^g)$ *is an* M*-matrix, then the system* (3) *will globally exponentially synchronize with the system* (1), *where* $\Delta = \mathbf{K} + |\Omega - \Omega_d| + |\mathbf{A} - \mathbf{A}_d|\mathbf{L}^f$ *and* $\mathbf{K}$ *is a diagonal matrix with* $k_i = -c_i + \omega_{ii} + a_{ii}L_i^f$.

Similarly, taking $\mathbf{P} = \mathbf{I}$ in Theorem 2, we have the following corollary.

Corollary 2 *Assume* (H) *holds, and* $u_i(t)$ *is defined as in* (5). *If* $-(\Delta + |\mathbf{B}|\mathbf{L}^g)$ *is an* M*-matrix, where* $\Delta = \mathbf{K} + (|\Omega - \Omega_d + \mathbf{A} - \mathbf{A}_d|)\mathbf{L}^f$ *and* $\mathbf{K}$ *is a diagonal matrix with* $k_i = -c_i + (\omega_{ii} + a_{ii})L_i^f$, *then the system* (3) *will globally exponentially synchronize with the system* (1).

4 Numerical simulations

In this section, we will give some examples to illustrate the effectiveness of the controlling laws presented in the last section. In the simulations, the programs dde23 and ddesd in Matlab are used to solve delay differential equations for constant delays and general delays, respectively.

Example 1 Consider the following two chaotic neural networks with time delays as the following:

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{2} a_{ij} f\big(x_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(x_j(t-\tau(t))\big), \quad i = 1, 2, \tag{9}$$

where

$$\mathbf{A} = \begin{pmatrix} 2.0 & -0.12 \\ -5.1 & 3.2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1.6 & -0.1 \\ -0.2 & -2.4 \end{pmatrix}$$

and $f(x) = \tanh(x)$, $\tau(t) = 1 + e^{-t}$. Obviously, $0 < \tau(t) \le 2$, $t \ge 0$ and assumption (H) holds with $L_i^f = L_i^g = 1$. For the parameters chosen above, the system exhibits chaotic behavior with the initial condition $x_1(s) = 0.08$, $x_2(s) = 0.15$, $-2 \le s < 0$. Figure 1 shows a chaotic attractor of this system.

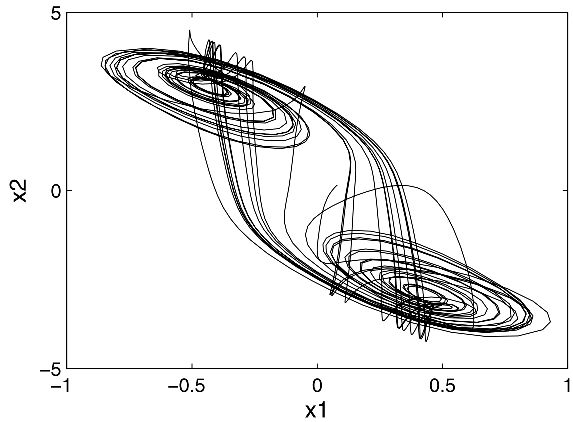

Fig. 1 The chaotic attractor of the system (9)

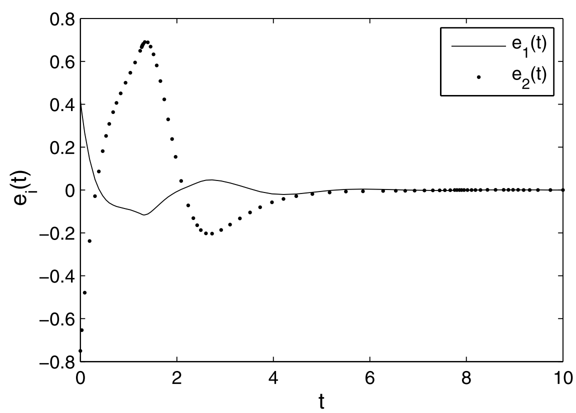

Fig. 2 The error states $e_i(t)$ between systems with $\omega = -12$

We consider the following coupled response system

$$\dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{2} a_{ij} f\big(y_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(y_j(t-\tau(t))\big) + u_i(t), \quad i = 1, 2, \tag{10}$$

where $u_i(t)$ is defined as in (5). Take $\Omega = \begin{pmatrix} -3 & 1 \\ 4.3 & -5 \end{pmatrix}$. By Corollary 2 and Definition 2, one can readily obtain that $-(\Delta + |\mathbf{B}|\mathbf{L}^g) = \begin{pmatrix} 1.4 & -0.98 \\ -1 & 0.8 \end{pmatrix}$ is an M-matrix. Then the networks (9) and (10) can reach synchronization. Figure 2 shows the error states $e_i(t)$ of (9) and (10) with initial condition $(x_1(s), x_2(s), y_1(s), y_2(s))^{\mathrm{T}} = (0.08, 0.15, 0.5, -0.6)^{\mathrm{T}}$ for $-2 \le s < 0$.

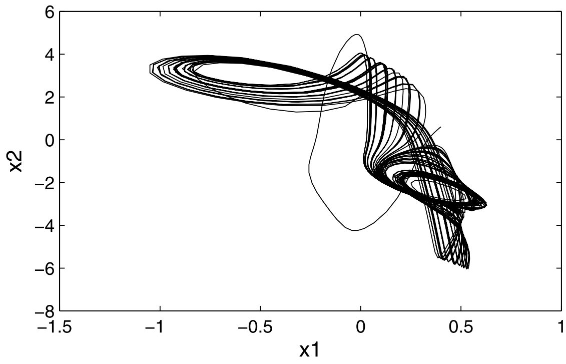

Fig. 3 The chaotic attractor of the system (11)

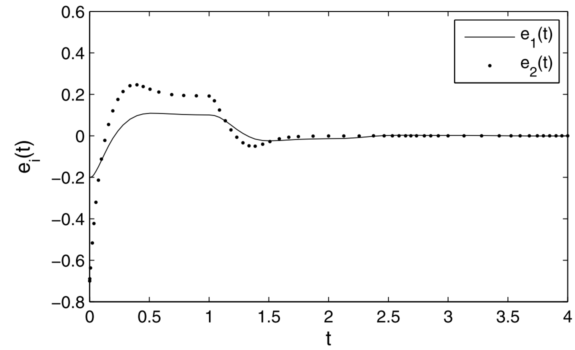

Fig. 4 The error states $e_i(t)$ between systems with $\omega = -12$

For constant time delays, our results are better than that in previous works because our methods give sufficient conditions for drive-response synchronization over a larger portion of the parameter space. To show this, we give another two examples in the following.

Example 2 Consider the following two chaotic neural networks with time delays as the following (see [18]):

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{2} a_{ij} f\big(x_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(x_j(t-1)\big), \quad i = 1, 2, \tag{11}$$

where

$$\mathbf{A} = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & 4.5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{pmatrix}$$

and $f(x) = \tanh(x)$. Then assumption (H) holds with $L_i^f = L_i^g = 1$. For the parameters chosen above, the system exhibits chaotic behavior with the initial condition $x_1(s) = 0.4$, $x_2(s) = 0.6$, $-1 \le s \le 0$. Figure 3 shows a chaotic attractor of this system.

We consider the following coupled response system:

$$\dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{2} a_{ij} f\big(y_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(y_j(t-1)\big) + u_i(t), \quad i = 1, 2, \tag{12}$$

where $u_i(t) = \sum_{j=1}^{2} \omega_{ij}(y_j(t) - x_j(t))$. For convenience of comparison, we choose $\Omega = \begin{pmatrix} -12 & 4 \\ 4 & \omega \end{pmatrix}$, which is the same as that in [7, 8] except one variable ω. One can readily obtain $-(\Delta + |\mathbf{B}|\mathbf{L}^g) = \begin{pmatrix} 9.5 & -4.2 \\ -9.2 & -\omega-7.5 \end{pmatrix}$ and the first-order leading principal minor $\mathbf{A}_{11} = 9.5 > 0$. If $\det(\mathbf{A}) = -9.5(\omega + 7.5) - 38.64 > 0$, i.e., $\omega < -11.5674$, then $-(\Delta + |\mathbf{B}|\mathbf{L}^g)$ is an M-matrix. By Corollary 1, the networks (11) and (12) can reach synchronization. However, by the controlling laws of [7, 8], when $\omega < -17.7$ the networks (11) and (12) can reach synchronization. This implies that the controlling laws of [7, 8] do not give sufficient conditions for synchronization when $\omega \in [-17.7, -11.5674)$. Figure 4 shows the error states $e_i(t)$ of systems (11)–(12) with $\omega = -12$ and initial condition $(x_1(s), x_2(s), y_1(s), y_2(s))^{\mathrm{T}} = (0.4, 0.6, 0.2, -0.1)^{\mathrm{T}}$ for $-1 \le s \le 0$.

Moreover, our results have no symmetry restriction on the coupling matrix. Now we choose $\Omega = \begin{pmatrix} -11 & 1 \\ 2 & \omega \end{pmatrix}$ the same as [8] except one variable ω, then $-(\Delta + |\mathbf{B}|\mathbf{L}^g) = \begin{pmatrix} 8.5 & -1.2 \\ -7.2 & -\omega-7.5 \end{pmatrix}$. By Corollary 1, when $\omega < -8.5165$, $-(\Delta + |\mathbf{B}|)$ is an M-matrix, the networks (11) and (12) can reach synchronization. On the other hand, one can deduce that the networks (11) and (12) can reach synchronization when $\omega < -12.0012$ by the controlling law of [8]. And the controlling laws of [8] do not give sufficient conditions for synchronization when $\omega \in [-12.0012, -8.5165)$. Figure 5 shows the error states $e_i(t)$ between systems with initial condition $(x_1(s), x_2(s), y_1(s), y_2(s))^{\mathrm{T}} = (0.4, 0.6, -0.7, 0.8)^{\mathrm{T}}$ for $-1 \le s \le 0$ when choosing $\omega = -9$.

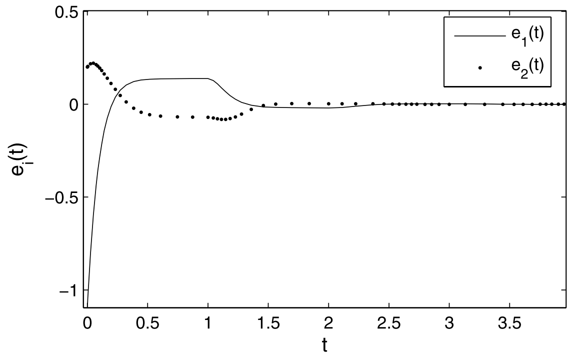

Fig. 5 The error states $e_i(t)$ between systems with $\omega = -9$

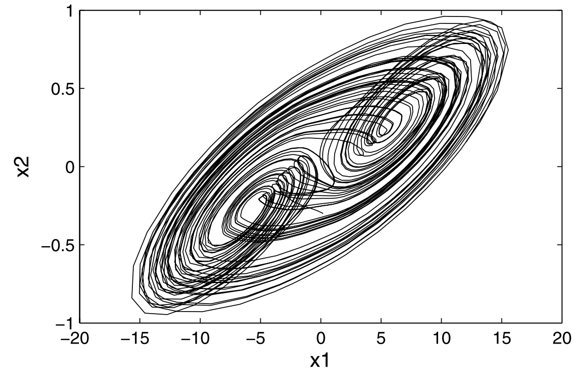

Fig. 6 The chaotic attractor of the system (13)

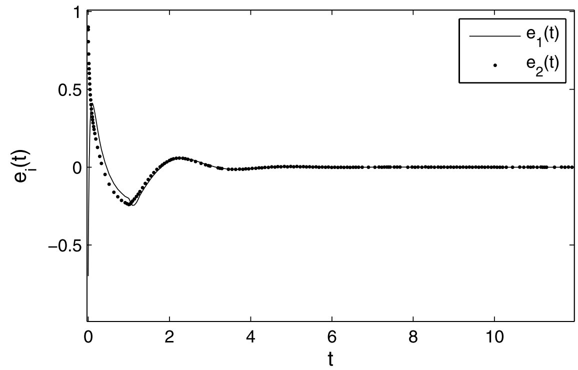

Fig. 7 The error states $e_i(t)$ between systems with $\omega = -11$

Example 3 Consider the following delayed cellular neural network [19]:

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{2} a_{ij} f\big(x_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(x_j(t-1)\big), \quad i = 1, 2, \tag{13}$$

where

$$\mathbf{A} = \begin{pmatrix} 1+\pi/4 & 20 \\ 0.1 & 1+\pi/4 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} -\frac{1.3\sqrt{2}\pi}{4} & 0.1 \\ 0.1 & -\frac{1.3\sqrt{2}\pi}{4} \end{pmatrix},$$

$f(x) = (|x+1| - |x-1|)/2$. Then the assumptions (H) holds with $L_i^f = L_i^g = 1$. Figure 6 shows the chaotic attractor of system (13). The response system has the form as the follows:

$$\dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{2} a_{ij} f\big(y_j(t)\big) + \sum_{j=1}^{2} b_{ij} f\big(y_j(t-1)\big) + u_i(t), \quad i = 1, 2, \tag{14}$$

where $u_i(t)$ is defined as in (4).

First, take $\Omega = \left(\begin{smallmatrix} -24 & 6 \\ 6 & \omega \end{smallmatrix}\right)$, which is same as that in [7] except the variable ω. By Corollary 1, we have that the networks (13) and (14) can reach synchronization when $\omega < -10.0991$. However, according to the controlling law of [7] we have that the networks (11) and (12) can reach synchronization when $\omega \le -31.60$. In other words, the controlling laws of [7] do not give sufficient conditions for synchronization when $\omega \in [-31.60, -10.0991)$. The state errors $e_i(t)$ between systems (13) and (14) are shown in Fig. 7 with $\omega = -11$ and initial condition $x_1(s) = 0.3$, $x_2(s) = 0.3$, and $y_1(s) = -0.5$, $y_2(s) = 0.6$ for $-1 \le s \le 0$.

Second, choose $\Omega = \left(\begin{smallmatrix} -24.3 & -0.9 \\ -0.8 & \omega \end{smallmatrix}\right)$ the same as that in [8] except the variable ω. By Corollary 1, one can obtain that the networks (13) and (14) can reach synchronization for $\omega < -3.5194$. However, one can obtain that the networks (11) and (12) can reach synchronization when $\omega < -23.2293$ by the controlling law of [8]. It is clear that our method can give sufficient conditions for synchronization over a larger parameter range. Figure 8 shows the states $e_i(t)$ between systems (13) and (14) with $\omega = -4$ and initial condition $x_1(s) = 0.3$, $x_2(s) = 0.3$, and $y_1(s) = -0.7$, $y_2(s) = 0.8$ for $-1 \le s \le 0$.

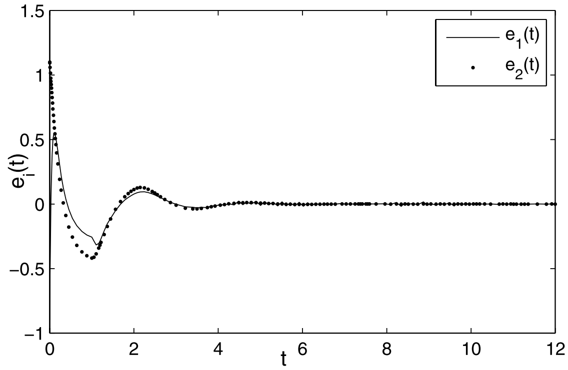

Fig. 8 The error states $e_i(t)$ between systems with $\omega = -4$

5 Conclusion

In this paper, the global exponential synchronization of two chaotic neural networks via state or output coupling has been studied based on M-matrix theory. Our results require neither the symmetry of coupling matrix, nor differentiability of varying-time delays. Therefore, a more extensive application domain for the chaotic synchronization of cellular neural networks is provided. Finally, number simulations are shown to verify the effectiveness of the obtained results.

References

1. Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–823 (1990)
2. Chen, G., Dong, X.: From Chaos to Order: Methodologies Perspectives and Application. Singapore, World Scientific (1998)
3. Wagg, D.J.: Partial synchronization of nonlinear chaotic systems via adaptive control, with applications to modelling coupled nonlinear systems. Int. J. Bifurc. Chaos **12**(3), 561–570 (2002)
4. Yang, T., Chua, L.O.: Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. IEEE Trans. Circuits Syst. I **44**, 976–988 (1997)
5. Steinmetz, P.N., Roy, A., Fitzgerald, P.J., Hsiao, S.S., Johnson, K.O., Niebur, E.: Attention modulates synchronized neuronal firing in primate somatosensory cortex. Nature **404**, 187–190 (2000)
6. Tsuda, I.: Chaos itinerancy as a dynamical basis of hermeneutics in brain and mind. World Futures **32**, 167–184 (1991)
7. Cheng, C., Liao, T., Hwang, C.: Exponential synchronization of a class of chaotic neural networks. Chaos Solitons Fractals **24**, 197–206 (2005)
8. He, W., Cao, J.: Exponential synchronization of chaotic neural networks: a matrix measure approach. Nonlinear Dyn. **55**, 55–65 (2009)
9. Park, J.H.: Adaptive synchronization of a unified chaotic system with an uncertain parameter. Int. J. Nonlinear Sci. Numer. Simul. **6**(2), 201–207 (2005)
10. Bernardo, M.D.: An adaptive control and synchronization of continuous-time chaotic systems. Int. J. Bifurc. Chaos **6**, 557–568 (1996)
11. Fotsin, H.B., Daafouz, J.: Adaptive synchronization of uncertain chaotic colpitts oscillators based on parameter identification. Phys. Lett. A **339**, 304–315 (2005)
12. Khadra, A., Liu, X.Z., Shen, X.: Impulsively synchronizing chaotic systems with delay and applications to secure communication. Automatica **41**(9), 1491–1502 (2005)
13. Earl, M.G., Strogatz, S.H.: Synchronization in oscillator networks with delayed coupling: a stability criterion. Phys. Rev. E **67**, 036204 (2003)
14. Grassi, G., Mascolo, S.: Synchronization high dimensional chaotic systems via eigenvalue placement with application to cellular neural networks. Int. J. Bifurc. Chaos **9**(4), 705–711 (1999)
15. Lu, H., He, Z.: Synchronization of chaotic systems based on system partition approaches. Phys. Lett. A **219**, 271–276 (1996)
16. Pecora, L.M., Carroll, T.L.: Master stability functions for synchronization coupled systems. Phys. Rev. Lett. **80**, 2109–2112 (1998)
17. Lei, Y., Xu, W., Zheng, H.: Synchronization of two chaotic nonlinear gyros using active control. Phys. Lett. A **343**, 153–158 (2005)
18. Lu, H.: Chaotic attractors in delayed neural networks. Phys. Lett. A **298**, 109–116 (2002)
19. Gilli, M.: Strange attractors in delayed cellular neural networks. IEEE Trans. Circuits Syst. I **40**, 166–173 (1993)
20. Zou, F., Nossek, J.A.: Bifurcation and chaos in cellular neural networks. IEEE Trans. Circuits Syst. I **40**(3), 166–173 (1993)
21. Lu, W., Chen, T.: Synchronization of coupled connected neural networks with delays. IEEE Trans. Circuits Syst. I **51**(12), 2491–2503 (2004)
22. Yu, W., Cao, J.: Synchronization control of stochastic delayed neural networks. Physica A **373**, 252–260 (2007)
23. Horn, R.A., Johnson, C.R.: Topic in Matrix Analysis, pp. 112–116. Cambridge Univ. Press, Cambridge (1994)
24. Tokumarn, H., Adachi, N., Amemiya, T.: Macroscopic stability of interconnected systems. In: 6th IFAC Congress Papers ID44.4, pp. 1–7 (1975)