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A CLASS OF SEMILINEAR SYSTEMS WITH STATE AND INPUT DELAYS

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ABSTRACT. In this paper, we study a class of semilinear systems with state and input delays. We prove that the well-posedness of the considered delay systems, the associated semilinear systems without delays, and the associated extended abstract Cauchy problems are equivalent in the sense of strong solutions. Moreover, some existence conditions of strong solutions of the considered delay systems are deduced. As an application, a nonlinear population dynamics problem with delays in state and input is presented.

1. Introduction

Let X and U be Banach spaces. In this paper, we consider infinite-dimensional semilinear systems with state and output delays described as follows:

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Lx_t + Bu_t, & t \geq 0, \\ x(0) = z, \\ x(t) = \phi(t), \quad u(t) = \xi(t), & t \in [-r, 0], \end{cases} \tag{SLD}$$

where

- $A : X \supset D(A) \to X$ is a closed and densely defined linear operator;
- $\phi \in L^p([-r, 0], X)$, $\xi \in L^p([-r, 0], U), 1 < p < \infty$;
- L and B are nonlinear, Lipschitz continuous operators from $L^p([-r, 0], X)$ to X and from $L^p([-r, 0], U)$ to X, respectively;
- $x : [-r, \infty) \to X$ and $x_t : [-r, 0] \to X$ is defined by $x_t(\sigma) = x(t + \sigma)$ for $\sigma \in [-r, 0]$;
- $u : [-r, \infty) \to U$ and $u_t : [-r, 0] \to U$ is defined by $u_t(\sigma) = u(t + \sigma)$ for $\sigma \in [-r, 0]$.

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Bátkai and Piazzera discussed in [2] the linear free system ($B = 0$) of (SLD), where L is a bounded linear operator from $W^{1,p}([-r,0],X)$ to X. More precisely, they proved the equivalence of the well-posedness of an abstract functional differential equations and associated abstract Cauchy problem in the sense of classical solution. However, they anticipated that it seemed to be impossible to extend their approach to nonlinear equations owing to the incompleteness of the theory of nonlinear semigroups. Moreover, we observe that the initial value problem of the linear free system (SLD) does not necessarily have a solution of any kind. Inspired by the two remarks, Song and Peng [13] studied the semilinear free system (SLD) in the framework of nonlinear semigroups of Lipschitz operators developed by Peng and Xu [10]. They proved that the semilinear free system is equivalent to an associated abstract Cauchy problem in the sense of strong solutions. Moreover, they gave a sufficient condition for the well-posedness of the abstract functional differential equation, that is, A generating a C_0-semigroup.

On the other hand, Hadd, Idrissi and Rhandi [8] considered system (SLD), where L and B are bounded linear operator from $W^{1,p}([-r,0],X)$ to X and from $W^{1,p}([-r,0],U)$ to X, respectively. They studied the mild solution and classical solution. They proved that the classical solution is a mild solution and gave some existence conditions of classical solution and mild solutions by the tool of regular linear system theory developed by Salamon [11] and Weiss [17–19]. Simultaneously, Hadd and Idrissi proved in [7] that the considered linear system (SLD) is equivalent to a regular linear system without delays in the sense of mild solutions.

Motivated by the above statements, in this paper, we study the semilinear delay control systems (SLD) by combining the theory of abstract linear control system and theory of Lipschitzian semigroup developed by Peng and Xu [10] and Song and Peng [13]. The paper is organized as follows. In Sec. 2, we recall the notation and some results on Lipschitzian semigroups on X. In Sec. 3, we recall the theory of abstract linear control systems and left-shift semigroups. In Sec. 4, we study the equivalence of system (SLD), associated nonlinear system without delays (LACP), and associated extended abstract Cauchy problem (CACP). Moreover, some sufficient conditions under which the strong solution exists are deduced. In the final section, an example, that is, the nonlinear Lotka–McKendrick population dynamics problem (NPDP) is presented.

2. Preliminaries on Lipschitzian semigroups

Throughout this section, we assume that X and Y are Banach spaces and C and D are their closed subsets, respectively. A mapping $T : C \to D$ is called Lipschitz continuous if there exists a positive constant $M > 0$ such that

$$\|Tx - Ty\| \leq M\|x - y\| \quad \forall x, y \in C,$$

where the constant M is usually called the Lipschitz constant of T on C. The minimum Lipschitz constant of T on C, denoted by $L(T)$, can be calculated by

$$L(T) = \sup_{\substack{x,y\in C\\ x\neq y}} \frac{\|Tx - Ty\|}{\|x - y\|}.$$

It is not difficult to verify that the nonnegative functional $L(\cdot)$ is a seminorm of the space $\mathrm{Lip}(C, D)$ of Lipschitz operators from C to D.

Definition 2.1 (see [10]). A one-parameter family $(T(t))_{t\geq 0}$ of Lipschitz operators from C into itself is called a Lipschitzian semigroup on C if it possesses the following two properties:

(i) $T(0) = I$ (the identity operator on C), $T(t+s) = T(t)T(s) = T(s)T(t)$ for all t, $s \geq 0$;
(ii) the mapping $t \mapsto T(t)x$ is continuous at $t = 0$ for any $x \in C$.

Moreover, a Lipschitzian semigroup $(T(t))_{t\geq 0}$ is said to be exponentially bounded if there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $L(T(t)) \leq Me^{\omega t}$ for all $t \geq 0$.

Obversely, a Lipschitzian semigroup is a more general type of C_0-semigroup [6], semigroups of contractions [3, 9], semigroups of ω-type [9], and uniformly k-Lipschitzian semigroups [5]. Unlike C_0-semigroups, until now, it is unknown whether a Lipschitzian semigroup is exponentially bounded or not. However, Peng and Xu [10] gave an equivalent characterization for exponential boundedness of Lipschitzian semigroups, that is, Lipschitzian semigroup $(T(t))_{t\geq 0}$ is exponential bounded if and only if $\limsup_{t\to 0^+} L(T(t)) < \infty$.

Definition 2.2 (see [10]). Let $(T(t))_{t\geq 0}$ be a Lipschitzian semigroup on C and let

$$D(A) = \left\{ x \in C : \text{the limit } \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}.$$

If $D(A)$ is nonempty, then we say that $(T(t))_{t\geq 0}$ possesses a generator A, which is defined by

$$A : D(A) \subset C \to X, \quad Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

for any $x \in D(A)$.

Consider the following nonlinear abstract Cauchy problem:

$$\begin{cases} x'(t) = Ax(t), & t \geq 0, \\ x(0) = z, \end{cases} \tag{ACP}$$

where $A : D(A) \subset C \to X$ is a nonlinear operator. By virtue of [14] we recall that a function $x(\cdot) \in C(\mathbb{R}^+, X)$ is called a *strong solution* of (ACP) if

(i) $x(\cdot)$ is Lipschitz continuous on every compact interval of $[0, \infty)$;
(ii) $x(t)$ is differentiable for almost every $t > 0$;
(iii) $x'(t) = Ax(t)$ for a.e. $t > 0$; $x(0) = z$.

Clearly, $x(\cdot)$ is a strong solution of (ACP) if and only if

$$x(t) \in D(A) \text{ for a.e. } t > 0, \quad x(t) = z + \int_0^t Ax(s)ds, \quad t \geq 0.$$

Definition 2.3. The abstract Cauchy problem (ACP) is well posed if, for arbitrary $z \in D(A)$, (ACP) has a strong solution $x(\cdot, z)$ with the initial value z and there exists a locally bounded function $p(t) > 0$ such that strong solutions depend on the initial values by the relation as follows:

$$\|x(t, z_1) - x(t, z_2)\| \leq p(t)\|z_1 - z_2\|,$$

where $x(\cdot, z_1)$ and $x(\cdot, z_2)$ are the strong solution of (ACP) with initial values z_1 and z_2, respectively.

Remark 2.4. By the definition of strong solution, (ACP) cannot have a strong solution whenever $x \overline{\in} \overline{D(A)}$. So, in order to define the well-posedness, the existence of the solutions and $x \in D(A)$ are necessary. Moreover, the well-posedness implies the uniqueness of the strong solution whenever an initial value is given.

Definition 2.5 (see [1])**.** A space X is said to satisfy the Radon–Nikodym property if every Lipschitz continuous function $F : \mathbb{R} \to X$ is almost everywhere differentiable.

Lemma 2.6 (see [13])**.** *Let X satisfy the Radon–Nikodym property. Then the Cauchy problem* (ACP) *is well-posed if and only if A generates an exponentially bounded Lipschitzian semigroup on $\overline{D(A)}$.*

Lemma 2.7 (see [12])**.** *Let A be the infinitesimal generator of C_0-semigroup $(T(t))_{t \geq 0}$ on X satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some $\omega \in \mathbb{R}$ and $M \geq 1$. If $K \in \operatorname{Lip}(X, X)$, then* (ACP) *is the generator of an exponentially bounded Lipschitzian semigroup $(S(t))_{t \geq 0}$ on X satisfying $L(T(t)) \leq Me^{(\omega + ML(K))t}$ for all $t \geq 0$. Moreover,*

$$S(t)x = T(t)x + \int_0^t T(t - s)KS(s)x\, ds \quad \forall x \in X.$$

Under the assumptions of Lemma 2.7, we derive from the proof of Lemma 2.6 (see [13]) that $u(t, x) = S(t)x$ is the strong solution of the abstract Cauchy problem $\dot{u}(t) = (A + K)u(t)$ with $u(0) = x$.

3. Preliminaries on abstract linear control systems

In this section, we briefly recall main concepts associated to abstract linear control systems in the sense of Salamon [11] and Weiss [16]. Throughout this section, Z and U are Banach spaces, $(A, D(A))$ is the generator of C_0-semigroup $T := (T(t))_{t\geq 0}$ on Z, and $1 < p < \infty$. We denote by $\mathbb{R}^+$ the interval $[0, \infty)$. For u, $v \in L^p(\mathbb{R}^+, U)$ and $\tau \geq 0$, the τ-concatenation of u and v, denoted by $u \diamondsuit_\tau v$, is defined by

$$(u \diamondsuit_\tau v)(t) = \begin{cases} u(t), & t < \tau, \\ v(t-\tau), & t \geq \tau. \end{cases}$$

The extrapolation space Z_{-1} of Z is the completion of Z under the norm $\|R(\lambda_0, A) \cdot \|$, where $R(\lambda_0, A)$ is the resolvent of A at λ_0. Denote by $\{T_{-1}(t)\}_{t\geq 0}$ the extrapolation semigroup of $\{T(t)\}_{t\geq 0}$ on Z_{-1}, where A_{-1} denotes its generator (for more details of extrapolation space and extrapolation semigroup, see [6]). Sometimes, to avoid confusion, we denote Z_{-1} by Z_{-1}^A to make clear that it is induced by A.

Definition 3.1 (see [16])**.** The pair $(T, \Phi) := ((T(t))_{t\geq 0}, (\Phi(t))_{t\geq 0})$ is called an abstract linear control system on (Z, U) if $(\Phi(t))_{t\geq 0}$ is a family of linear bounded operators from $L^p(\mathbb{R}^+, U)$ to Z such that

$$\Phi(t+\tau)(u \diamondsuit_\tau v) = T(t)\Phi(\tau)u + \Phi(t)v$$

for $u, v \in L^p(\mathbb{R}^+, U)$ and $t, \tau \geq 0$.

By the Weiss representation theorem [16], there exists a unique operator $B \in L(U, Z_{-1})$, such that

$$\Phi(t)u = \int_0^t T_{-1}(t-s)Bu(s)ds \in Z \tag{3.1}$$

for any $t \geq 0$ and $u \in L^p(\mathbb{R}^+, U)$.

Motivated by this, we define the notion admissible control operator.

Definition 3.2 (see [16])**.** The control operator B is said to be admissible for A if, for any $t \geq 0$, the mapping

$$\Phi(t) : u \mapsto \int_0^t T_{-1}(t-s)Bu(s)ds$$

is continuous from $L^p([0, \infty), U)$ into Z.

Tucsnak and Weiss proved [15,16] that B is admissible if $\Phi(T)u \in Z$ for a certain $T > 0$ and for all $u \in L^p([0, \infty), U)$. Moreover, they proved that B is admissible for A if and only if (A, B) generates an abstract linear control system.

Now we introduce an abstract linear control system related to the left-shift semigroup on $L^p([-r,0],Z)$, which is defined by

$$(S_Z(t)g)(\theta) = \begin{cases} g(t+\theta), & t+\theta \le 0, \\ 0, & \text{if not,} \end{cases}$$

with the generator given by

$$Q_Z = \frac{d}{d\theta}, \quad D(Q_Z) = \{g \in W^{1,p}([-r,0],Z) : g(0) = 0\}.$$

To do this, we define the family $(\Phi_Z(t))_{t\ge 0}$ of bounded linear operators from $L^p(\mathbb{R}^+,Z)$ to $L^p([-r,0],Z)$ by

$$(\Phi_Z(t)u)(\theta) = \begin{cases} u(t+\theta), & t+\theta \ge 0, \\ 0, & \text{if not,} \end{cases}$$

for any $u \in L^p(\mathbb{R}^+,Z)$ and $\theta \in [-r,0]$. It is easy to see that (S_Z,Φ_Z) is an abstract linear control system on $(L^p([-r,0],Z),\, Z)$ with the control operator

$$\beta_Z \in L(Z,(L^p([-r,0],Z))_{-1}^{Q_Z})$$

such that

$$(\lambda - (Q_Z)_{-1})\beta_Z = e_\lambda, \tag{3.2}$$

where $e_\lambda : Z \to L^p([-r,0],Z)$ is a continuous linear operator defined by $(e_\lambda x)(\sigma) := e^{\lambda\sigma}x$ for any $x \in Z$ and $\sigma \in [-r,0]$. For more details of abstract linear control system (S_Z,Φ_Z), we refer the reader to [8,18].

4. Main results

This section is devoted to the well-posedness of system (SLD). To study this problem, some preparations are needed. First, we introduce the notion of a strong solution of (SLD), which is the extension of a strong solution of the free system developed by Song and Peng [13].

Definition 4.1. For a given function $u : [-r,\infty) \to U$, a function $x(\cdot) \in C([0,\infty),X)$ is called a strong solution of (SLD) if

(i) $x(t)$ and x_t are Lipschitz continuous on every compact interval of $[0,\infty)$;

(ii) $x(t)$ are differentiable, $x(t) \in D(A)$, and $x_t \in W^{1,p}([-r,0],X)$ for almost every $t \in (0,\infty)$;

(iii) $x(0) = x$, $x_0 = \phi$, $u_0 = \xi$, and $u \in W^{1,p}([-r,\infty),U)$; $\dot{x} = Ax(t) + Lx_t + Bu_t$ holds for almost all $t \ge 0$.

We define the operator $\mathcal{A}$ in $X \times L^p([-r,0],X)$ as follows:

$$\mathcal{A} = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1,0], X) : f(0) = x \right\}.$$

It is clear that $\mathcal{A}$ is closed and densely defined.

Definition 4.2. The delay system (SLD) is said to be well-posed if

(i) for every $(z,\phi)^T \in D(\mathcal{A})$ and $u \in W^{1,p}([-r,\infty), U)$ satisfying $u_0 = \xi$, there exists a unique strong solution $x(\cdot, z, \phi, u)$ and

(ii) these strong solutions continuously depend on the initial values and inputs in the sense that there exists a locally bounded function $q(t) > 0$ such that

$$\begin{aligned} &\|x(t, z_1, \phi_1, u_1) - x(t, z_2, \phi_2, u_2)\| \\ &\quad \leq q(t)\Big(\|z_1 - z_2\| + \|\phi_1 - \phi_2\| + \|u_1 - u_2\|_{L^p([-r,\infty),U)}\Big), \quad t \in [0,\infty) \end{aligned}$$

where $x(\cdot, z_1, \phi_1, u_1)$ and $x(\cdot, z_2, \phi_2, u_2)$ are strong solutions of (SLD) with initial values $(z_1, \phi_1)^T$ and $(z_2, \phi_2)^T$ from $D(\mathcal{A})$ and inputs u_1 and u_2 from $W^{1,p}([-r,\infty), U)$, respectively.

Assume that a linear operator $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$ is closed and densely defined on a Banach space X. Let $\mathfrak{B}$ be a bounded linear operator from a Banach space U to X_{-1}, and let $\mathfrak{K} \in \mathrm{Lip}(X, X)$. We consider the following semilinear control system:

$$\begin{cases} \dot{x}(t) = \mathfrak{A}_{-1}x(t) + \mathfrak{K}x(t) + \mathfrak{B}u(t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{IACP}$$

Lemma 4.3. *For system* (IACP)*, the operator*

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A}_{-1} & \mathfrak{B}\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with

$$D(\mathfrak{M}) = \Big\{ (x,f)^T \in X \times W^{1,p}(\mathbb{R}^+, U) : \mathfrak{A}x + \mathfrak{B}f(0) \in X \Big\}$$

is closed and densely defined on $X \times L^p(\mathbb{R}^+, U)$.

Proof. First, we prove the closedness. Let $(x_n, f_n)^T \in D(\mathfrak{M})$ be a sequence converging to $(x,f)^T)$ and

$$\mathfrak{M}\begin{pmatrix} x_n \\ f_n \end{pmatrix} = \begin{pmatrix} \mathfrak{A}_{-1}x_n + \mathfrak{B}f_n(0) \\ \frac{df_n}{d\sigma} \end{pmatrix}$$

converges to $(y,g)^T$.

In particular, by the closedness of the differential operator $d/d\sigma$, we see that the sequence (f_n) converges to f in the norm topology of the Sobolev space $W^{1,p}(\mathbb{R}^+, U)$. Hence, we have that $f \in W^{1,p}(\mathbb{R}^+, U)$ and

$df/d\sigma = g$. Since the operator $\delta_0 : W^{1,p}(\mathbb{R}^+, U) \to U$ is bounded, we have $\delta_0 f_n \to \delta_0 f = f(0)$. Combining this with the fact that $\mathfrak{B}$ is bounded from U to X_{-1}, we obtain that $\mathfrak{B}\delta_0 f_n \to \mathfrak{B}f(0)$ in the sense of $\|\cdot\|_{-1}$. Moreover, it is easy to obtain $\mathfrak{A}_{-1} \in L(X, X_{-1})$ and $\mathfrak{A}_{-1}x_n + \mathfrak{B}f_n(0) \to y$; therefore, $\mathfrak{A}_{-1}x_n + \mathfrak{B}f_n(0) \to y$ in the sense of $\|\cdot\|_{-1}$. Thus, we obtain $\mathfrak{A}_{-1}x + \mathfrak{B}f(0) = y \in X$. Hence $(x, f)^T \in D(\mathfrak{M})$, $\mathfrak{M}(x, f)^T = (y, g)^T$, and the operator $\mathfrak{M}$ is closed.

Now we prove the density of $D(\mathfrak{M})$. Let $(y, g)^T \in X \times L^p(\mathbb{R}^+, U)$ and $\epsilon > 0$. Since $D(\mathfrak{A})$ and $W^{1,p}(\mathbb{R}^+, U)$ are dense in X and $L^p(\mathbb{R}^+, U)$, respectively, we can find $x \in D(\mathfrak{A})$ and $f \in W^{1,p}(\mathbb{R}^+, U)$ such that

$$\|x - y\| < \epsilon \quad \text{and} \quad \|g - f\| < \epsilon.$$

Now let h be equal to f on $(0, \infty)$ and zero at the point zero. We obtain

$$\left\| \begin{pmatrix} y \\ g \end{pmatrix} - \begin{pmatrix} x \\ h \end{pmatrix} \right\| \le \left\| \begin{pmatrix} y - x \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ g - f \end{pmatrix} \right\| \le 2\epsilon.$$

Therefore, the domain $D(\mathfrak{M})$ is dense. The proof is complete. □

A function $x(\cdot) \in C(\mathbb{R}^+, X)$ is called a *strong solution* of system (IACP) if

(i) $x(\cdot)$ is Lipschitz continuous on any compact interval of $\mathbb{R}^+$;
(ii) $x(t)$ is differentiable a.e. $t > 0$;
(iii) $\mathfrak{A}_{-1}x(t) + \mathfrak{B}u(t) \in X$ and $\dot{x}(t) = \mathfrak{A}_{-1}x(t) + \mathfrak{K}x(t) + \mathfrak{B}u(t)$ a.e. $t > 0$; $x(0) = x_0$, $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0) \in X$, and $u \in W^{1,p}(\mathbb{R}^+, U)$.

Definition 4.4. The abstract Cauchy problem (IACP) is well-posed if for $x_0 \in X$ and $u(0) \in U$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0) \in X$ and $u \in W^{1,p}(\mathbb{R}^+, U)$, problem (IACP) has a strong solution $x(t, x_0, u)$ with the initial value x_0 and input u, and there exists a locally bounded function $p(t) > 0$ such that the strong solutions depend on the initial values and inputs. This means that for $x_0, y_0 \in X$ and $u(0), v(0) \in U$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0)$, $\mathfrak{A}_{-1}y_0 + \mathfrak{B}v(0) \in X$ and $u, v \in W^{1,p}(\mathbb{R}^+, U)$, we have

$$\|x(t, x_0, u) - x(t, y_0, v)\| \le p(t)(\|x_0 - y_0\| + \|u - v\|_{L^p(\mathbb{R}^+, U)}),$$

where $x(t, x_0, u)$ and $x(t, y_0, v)$ are strong solutions of (IACP) with initial values x and y and inputs u and v, respectively.

Lemma 4.5. *Assume that a Banach space X satisfies the Radon–Nikodym property. Let $\tilde{A}$ generate a C_0-semigroup and $\tilde{K}$ be a Lipschitz operator on X. Then, for any $x \in D(\tilde{A})$, the strong solution of the Cauchy problem*

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) + \tilde{K}x(t), & t \ge 0, \\ x(0) = x \end{cases}$$

satisfies

$$x(t) = T(t)x + \int_0^t T(t-s)\tilde{K}x(s)ds.$$

Proof. The proof immediately follows from Lemma 2.7. □

Theorem 4.6. *System (IACP) is well-posed if and only if the system*

$$\begin{cases} \dot{\mathscr{X}}(t) = \begin{pmatrix} \mathfrak{A}_{-1} & \mathfrak{B}\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \mathscr{X}(t) + \begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix} \mathscr{X}(t), \\ \mathscr{X}(0) = \begin{pmatrix} x_0 \\ u \end{pmatrix}, \end{cases} \tag{EACP}$$

is well-posed. This is equivalent to the fact that

$$\begin{pmatrix} \mathfrak{A}_{-1} & \mathfrak{B}\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix}$$

generates a Lipschitzian semigroup on $X \times L^p(\mathbb{R}^+, U)$.

Proof. Assume that problem (EACP) is well-posed. Then for $(x_0, u)^T \in D(\mathfrak{M})$, i.e., $x_0 \in X$ and $u \in W^{1,p}(\mathbb{R}^+, U)$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0) \in X$, $(x_0, u)^T \in D(\mathfrak{M})$, there exists a unique strong solution $(x(\cdot), g(\cdot))^T$ satisfying $x(0) = x_0$, $g(0) = u$, and

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} \mathfrak{A}_{-1} & \mathfrak{B}\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \begin{pmatrix} x(t) \\ g(t) \end{pmatrix} + \begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ g(t) \end{pmatrix} \tag{4.1}$$

for a.e. $t \geq 0$. Hence we derive that $\mathfrak{A}_{-1}x(t) + \mathfrak{B}g(t)(0) \in X$ for a.e. $t \geq 0$, and

$$\frac{d}{dt}x(t) = \mathfrak{A}_{-1}x(t) + \mathfrak{B}g(t)(0) + \mathfrak{K}x(t)$$

for a.e. $t \geq 0$. Moreover, taking the second component of (4.1), we obtain

$$\frac{d}{dt}g(t) = \frac{d}{d\sigma}g(t) \quad \text{for a.e. } t \geq 0.$$

Similarly to the proof in [8], it is not difficult to prove that $u \in W^{1,p}(\mathbb{R}^+, U)$ implies that $u_{\cdot} \in C^1(\mathbb{R}^+, L^p(\mathbb{R}^+, U))$ and $\frac{d}{dt}u_t = \frac{d}{d\sigma}u_t$ for any $t \geq 0$. Moreover, observe that $g(0) = u$. Let $w(t) = g(t) - u_t$; then

$$\begin{cases} \frac{d}{dt}w(t) = \frac{d}{d\sigma}w(t), & t \geq 0, \\ w(0) = 0. \end{cases} \tag{4.2}$$

Since (4.2) is the abstract Cauchy problem associated to the generator of left-shift semigroup on $L^p(\mathbb{R}^+, U)$ with initial value 0, we obtain that $w(t) =$

0 for a.e. $t \geq 0$ and, in particular, $g(t)(0) = u_t(0) = u(t)$ for a.e. $t \geq 0$. Thus, $\mathfrak{A}_{-1}x(t) + \mathfrak{B}u(t) \in X$ and

$$\frac{d}{dt}x(t) = \mathfrak{A}_{-1}x(t) + \mathfrak{B}u(t) + \mathfrak{K}x(t) \tag{4.3}$$

for a.e. $t \geq 0$. On the other hand, $x(\cdot)$ is Lipschitz continuous on any compact interval of $\mathbb{R}^+$. By definition, $x(\cdot)$ is the strong solution of (IACP).

For given $x_0, y_0 \in X$, $u(0), v(0) \in U$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0), \mathfrak{A}_{-1}y_0 + \mathfrak{B}v(0) \in X$, and $u, v \in W^{1,p}(\mathbb{R}^+, U)$, we have $(x_0, u)^T \in D(\mathfrak{M})$. Therefore, by the well-posedness of system (EACP), we obtain

$$\begin{aligned}\|x(t,x_0,u) - x(t,y_0,v)\| &\leq \left\| \begin{pmatrix} x(t,x_0,u) \\ g(t,x_0,u) \end{pmatrix} - \begin{pmatrix} x(t,y_0,u) \\ g(t,y_0,v) \end{pmatrix} \right\| \\ &\leq p(t)\big(\|x_0 - y_0\| + \|u - v\|\big).\end{aligned}$$

This implies the well-posedness of (IACP).

Now we prove that the well-posedness of (IACP) implies the well-posedness of (EACP). Assume that x is the strong solution of system (IACP) with $x(0) \in X$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0) \in X$ and $u \in W^{1,p}(\mathbb{R}^+, U)$. This implies that $u_{\cdot} \in C^1(\mathbb{R}^+, L^p(\mathbb{R}^+, U))$.

Let $g(t) = u_t = S(t)u = S(t)g(0)$, where $(S(t))_{t\geq 0}$ is the left-shift semigroup on $L^p(\mathbb{R}^+, U)$ generated by $Q = \frac{d}{d\sigma}$ with domain $D(Q) = W^{1,p}(\mathbb{R}^+, U)$. It is not difficult to see that $g(\cdot)$ is Lipschitz continuous on any compact interval. In fact, $g(0) = u \in D(Q)$; therefore, for $a \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned}\|g(t_1) - g(t_2)\| &= \|S(t_1)u - S(t_2)u\| \\ &= \left\| \int_{t_1}^{t_2} S(\sigma)u'd\sigma \right\| \leq \max_{0\leq s\leq b} \|S(s)\|\|u'\||t_1 - t_2|.\end{aligned}$$

On the other hand, the well-posedness of (IACP) implies (4.3). Hence, $\mathscr{X}(\cdot) = (x(\cdot), g(\cdot))^T$ is the strong solution of system (EACP). For given $x_0, y_0 \in X$ and $u(0), v(0) \in U$ satisfying $\mathfrak{A}_{-1}x_0 + \mathfrak{B}u(0), \mathfrak{A}_{-1}y_0 + \mathfrak{B}v(0) \in X$ and $u, v \in W^{1,p}(\mathbb{R}^+, U)$, we have

$$\begin{aligned}&\|\mathscr{X}(t,x_0,u) - \mathscr{X}(t,y_0,v)\| \\ &\quad = \|x(t,x_0,u) - x(t,y_0,v)\| + \|u_t - v_t\|_{L^p(\mathbb{R}^+,U)} \\ &\quad \leq \|x(t,x_0,u) - x(t,y_0,v)\| + \|u - v\|_{L^p(\mathbb{R}^+,U)} \\ &\quad \leq (p(t)+1)(\|x_0 - y_0\| + \|u - v\|_{L^p(\mathbb{R}^+,U)}).\end{aligned}$$

Hence (EACP) is well-posed. The proof is complete. □

Corollary 4.7. *Assume that* $(\mathfrak{A}, \mathfrak{B})$ *generates an abstract linear control system. Then, for* $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^+, U)$ *satisfying* $\mathfrak{A}x_0 + \mathfrak{B}u(0) \in X$*, system*

(IACP) *has a unique strong solution. In this case, the strong solution is given by*

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)\mathfrak{B}u(s)ds + \int_0^t T(t-s)Kx(s)ds. \tag{4.4}$$

Proof. Since $(\mathfrak{A}, \mathfrak{B})$ generates an abstract linear control system, by [14, Theorem 3.1], $\mathfrak{M}$ generates a C_0-semigroup on the space $X \times L^p(\mathbb{R}^+, U)$. From Lemma 2.7 we derive that

$$\mathfrak{M} + \begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix}$$

generates an exponentially bounded Lipschitzian semigroup on the space $X \times L^p(\mathbb{R}^+, U)$. For any $T > 0$, let

$$u_T(t) = \begin{cases} u(t), & t \in [0, T], \\ 0, & t > T. \end{cases}$$

Then $u_T \in W^{1,p}(\mathbb{R}^+, U)$. By Lemma 2.6, the semilinear system (EACP) with input u_T is well-posed and has a unique strong solution for any $T > 0$; therefore, system (EACP) with input u has a unique strong solution. The combination of this and Theorem 4.6 implies that (SLD) has a unique strong solution.

By Lemma 4.5 and the proof of Theorem 4.6, the strong solution of (IACP) is equal to the first component of the strong solution of (EACP) given by

$$\mathfrak{X}(t) = \mathscr{T}(t)\begin{pmatrix} x_0 \\ u \end{pmatrix} + \int_0^t \mathscr{T}(t-s)\begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix}\mathscr{X}(s)ds. \tag{4.5}$$

Taking the first component of (4.5), we obtain (4.4). The proof is complete. □

With the above preparations, we consider the system

$$\begin{cases} \dot{\mathscr{X}}(t) = \begin{pmatrix} \mathcal{A}_{-1} & 0 \\ 0 & (Q_U)_{-1} \end{pmatrix}\mathscr{X}(t) \\ \qquad\qquad + \begin{pmatrix} 0 & L & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathscr{X}(t) + \begin{pmatrix} 0 \\ 0 \\ \beta_U \end{pmatrix}u(t), \\ \mathscr{X}(0) = \begin{pmatrix} z \\ \phi \\ \xi \end{pmatrix}. \end{cases} \tag{LACP}$$

By Theorem 4.6, for $u \in W^{1,p}([-r, \infty), U)$, system (LACP) is equivalent to the system

$$\begin{cases} \dfrac{d\mathscr{Y}(t)}{dt} = \begin{pmatrix} \mathscr{A}_{-1} & \mathscr{B}\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \mathscr{Y}(t) + \begin{pmatrix} \mathscr{K} & 0 \\ 0 & 0 \end{pmatrix} \mathscr{Y}(t), \\ \mathscr{Y}(0) = \begin{pmatrix} \mathscr{X}(0) \\ u \end{pmatrix}, \end{cases} \tag{CACP}$$

where

$$\mathscr{A} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & Q_U \end{pmatrix}, \quad \mathscr{K} = \begin{pmatrix} 0 & L & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_U \end{pmatrix}.$$

Next, we show that systems (SLD) and (CACP) are equivalent.

Theorem 4.8. *The well-posedness of system* (CACP) *is equivalent to the well-posedness of system* (SLD).

Proof. Assume that (CACP) is well-posed. By Theorem 4.6, (LACP) is well-posed. Let $\mathscr{X}(t) = (x(t), y(t), z(t))^T$. By definition, for $\mathscr{X}(0) \in X \times L^p([-r,0],X) \times L^p([-r,0],U)$ satisfying

$$\begin{pmatrix} \mathcal{A}_{-1} & 0 \\ 0 & (Q_U)_{-1} \end{pmatrix} \mathscr{X}(0) + \begin{pmatrix} 0 \\ 0 \\ \beta_U \end{pmatrix} u(0) \in X \times L^p([-r,0],X) \times L^p([-r,0],U)$$

with $u \in W^{1,p}(\mathbb{R}^+, U)$, we derive

$$\begin{pmatrix} \mathcal{A}_{-1} & 0 \\ 0 & (Q_U)_{-1} \end{pmatrix} \mathscr{X}(t) + \begin{pmatrix} 0 \\ 0 \\ \beta_U \end{pmatrix} u(t) \in X \times L^p([-r,0],X) \times L^p([-r,0],U)$$

and

$$\dot{\mathscr{X}}(t) = \begin{pmatrix} \mathcal{A}_{-1} & 0 \\ 0 & (Q_U)_{-1} \end{pmatrix} \mathscr{X}(t) + \begin{pmatrix} 0 & L & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathscr{X}(t) + \begin{pmatrix} 0 \\ 0 \\ \beta_U \end{pmatrix} u(t) \tag{4.6}$$

for a.e. $t \geq 0$. This implies $(x(t), y(t))^T \in D(\mathcal{A})$ for a.e. $t \geq 0$. Hence,

$$\frac{dy(t)}{dt} = \frac{dy(t)}{d\sigma}, \text{ a.e. } t \geq 0, \quad y(t)(0) = x(t),\ t \geq 0, \quad y(0) = \phi.$$

Thus, $y(t) = x_t$ for a.e. $t \geq 0$. Furthermore, by (4.6), we obtain

$$\frac{dz(t)}{dt} = (Q_U)_{-1} z(t) + \beta_U u(t) \tag{4.7}$$

for a.e. $t \geq 0$. Fix $\lambda \in \rho(Q_U)$. Premultiplying (4.7) by $R(\lambda, Q_U)$, we derive

$$R(\lambda, Q_U)\frac{dz(t)}{dt} = -z(t) + \lambda R(\lambda, Q_U) z(t) + R(\lambda, (Q_U)_{-1})\beta_U u(t) \tag{4.8}$$

for a.e. $t \geq 0$. Observe that (3.2) implies

$$R(\lambda, (Q_U)_{-1})\beta_U = e_\lambda.$$

Substituting it in (4.8), we obtain

$$R(\lambda, Q_U)\left(\lambda - \frac{dz(t)}{dt}\right) = z(t) - e_\lambda u(t) \in D(Q_U) \tag{4.9}$$

for a.e. $t \geq 0$. By the same procedure, we obtain $z(0) - e_\lambda u(0) \in D(Q_U)$; therefore, $z(0) \in W^{1,p}([-r, 0], U)$. Since $z(0) = \xi$, $u = \xi$ on $[-r, 0]$, and $u \in W^{1,p}(\mathbb{R}^+, U)$, we obtain $u \in W^{1,p}([-r, \infty), U)$. It follows from [8, Lemma 1] that $u_\cdot \in C^1(\mathbb{R}^+, L^p([-r, 0], U)$. The combination of (4.9) and the assumptions implies

$$\frac{dz(t)}{dt} = \frac{dz(t)}{d\sigma}, \text{ a.e. } t \geq 0, \quad z(t)(0) = u(t), \quad z(0) = \phi.$$

Let $w(t) = z(t) - u_t$. Then

$$\frac{dw(t)}{dt} = \frac{dw(t)}{d\sigma}, \text{ a.e. } t \geq 0, \quad w(t)(0) = 0, \quad w(0) = 0.$$

Similarly to [13, Proposition 2], we obtain $z(t) = u_t$ for a.e. $t \geq 0$. Consequently, $x(\cdot) : [-r, \infty) \to X$ is a strong solution of system (SLD). The continuous dependence of strong solutions of (SLD) on their initial values and inputs is immediately followed by the continuous dependence of (LACP).

Conversely, assume that (SDL) is well-posed; then we can obtain the well-posedness of (CACP) by taking the converse procedure of the former proof. The proof is complete. □

Theorem 4.8 asserts that the equivalence of systems (SLD) and system (CACP) are in the sense of strong solution. This means that we can convert system (SLD) into an extended abstract Cauchy problem. From the proof of Theorem 4.8 and the proof techniques of Corollary 4.7, we obtain the following corollary.

Corollary 4.9. *Assume that A generates a C_0-semigroup on X. Then system* (SLD) *is well-posed. In particular, for $(z, \phi)^T \in D(\mathcal{A})$ and $u \in W^{1,p}_{\text{loc}}([-r, \infty), U)$, system* (SLD) *has a unique strong solution $x(\cdot)$ satisfying*

$$x(t) = T(t)z + \int_0^t T(t-s)(Lx_s + Bu_s)ds.$$

5. Application

As an application of the results obtained in Sec. 4, we consider the following nonlinear Lotka–McKendrick population dynamics problem with delay in state and control:

$$\begin{cases} \dfrac{\partial x(t,a)}{\partial t} = -\mu(a)x(t,a) - \displaystyle\int_{-r}^{0} K(a, x(t+s,a))ds \\ \qquad\qquad + \displaystyle\int_{-r}^{0} G(a, u(t+s,a))ds, \qquad t, a \geq 0, \\ x(t,0) = \displaystyle\int_{0}^{\infty} \beta(a)x(t,a)da, \qquad\qquad t \geq 0, \\ x(s,a) = \phi(s,a), \qquad s \in [-r, 0], \quad a \geq 0, \end{cases} \tag{NPDP}$$

where $r > 0$, and $\mu, \beta \in L^\infty(\mathbb{R}^+)$ are nonnegative functions. Equation (NPDP) represents the density of the population at time t and age a. The function μ is the death rate caused by nature death and β is the fertility rate. The function

$$\int_{-r}^{0} K(\cdot, x(\cdot + s, \cdot))ds$$

represents the death caused by pregnancy. Here we assume that gestation period dependents on time t and age a. The function

$$\int_{-r}^{0} G(\cdot, x(\cdot + s, \cdot))ds$$

represents the death caused by harvesting.

Take $X = U = L^1(\mathbb{R}^+)$ and define the operator

$$A := -\frac{\partial}{\partial a} + M_\mu$$

with domain

$$D(A) = \{x \in W^{1,1}(\mathbb{R}^+),\ x(0) = \Gamma_\beta x\},$$

where

$$\Gamma_\beta x := \int_{0}^{\infty} \beta(a)x(a)da.$$

From [4], it is known that $(A, D(A))$ generates a C_0-semigroup on X. Let

$$Lf := \int_{-r}^{0} K(\cdot, f(s,\cdot))ds \quad \text{for any} \quad f \in L^p([-r,0], X])$$

and

$$Bg := \int_{-r}^{0} G(\cdot, g(s,\cdot))ds \quad \text{for any} \quad g \in L^p([-r,0],X]).$$

Assume that K and G satisfy the following conditions:

$$|K(a, f(s,a)) - K(a, g(s,a))| \leq |w(a)| \cdot |f(s,a) - g(s,a)|$$

for a.e. $a \geq 0$ and a.e. $s \in [-r,0]$ and

$$|G(a, \mathfrak{F}(s,a)) - G(a, \mathfrak{G}(s,a))| \leq |v(a)| \cdot \|\mathfrak{F}(s,a) - \mathfrak{G}(s,a)\|$$

for a.e. $a \geq 0$ and a.e. $s \in [-r,0]$, where $w, v \in L^\infty(\mathbb{R}^+)$. In particular, we can chose K and G such that $K(a, f(s,a)) = w(a)f(s,a)$ and $G(a, \mathfrak{F}(s,a)) = v(a)\mathfrak{F}(s,a)$ for any $a \geq 0$ and $s \in [-r,0]$. Then Eq. (NPDP) takes the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Lx_t + Bu_t, \\ x(s) = \phi(s,\cdot), \quad u(s) = \xi(s,\cdot), \quad s \in [-r,0]. \end{cases}$$

We can prove that L and B are Lipschitz continuous functions. Indeed,

$$\begin{aligned} \|Lf - Lg\| &= \left\| \int_{-r}^{0} K(\cdot, f(s,\cdot))ds - \int_{-r}^{0} K(\cdot, g(s,\cdot))ds \right\| \\ &= \int_0^\infty \left| \int_{-r}^{0} K(a, f(s,a))ds - \int_{-r}^{0} K(a, f(s,a))ds \right| da \\ &\leq \int_0^\infty |w(a)| \int_{-r}^{0} |f(s,a) - g(s,a)| ds\, da \\ &= \int_{-r}^{0} \int_0^\infty |w(a)| \cdot |f(s,a) - g(s,a)| da\, ds \\ &= r^{1/q} \|w\|_{L^\infty(\mathbb{R}^+)} \cdot \|f - g\| \end{aligned}$$

for any $f, g \in L^p([-r,0],X)$, where $q = p/(p-1)$. Similarly, we can obtain

$$\|B\mathfrak{F} - B\mathfrak{G}\| \leq r^{1/q} \|v\|_{L^\infty(\mathbb{R}^+)} \cdot \|\mathfrak{F} - \mathfrak{G}\|$$

for any $\mathfrak{F}, \mathfrak{G} \in L^p([-r,0],X)$. By Corollary 4.9, system (NPDP) is well-posed and has a unique strong solution for any $u \in W^{1,p}_{\text{loc}}([-r,\infty),X)$ and $(x(0),\phi)^T \in D(A) \times W^{1,p}([-r,0],X)$ satisfying $\phi(0) = x(0)$.

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