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ON ROBUSTNESS OF EXACT CONTROLLABILITY AND EXACT OBSERVABILITY UNDER CROSS PERTURBATIONS OF THE GENERATOR IN BANACH SPACES

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Abstract. This paper is concerned with the exact controllability and exact observability of linear systems in the Banach space setting. It is proved that both the admissibility of control operators and the admissibility of observation operators are invariant to cross perturbations of the generator of a C_0-semigroup. Moreover, under the admissibility invariance premise, the robustness of the exact controllability as well as the exact observability to such cross perturbations is verified. An illustrative example is presented.

1. Introduction

Consider the infinite dimensional linear systems described by the following differential equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), t \geq 0, \\ y(t) = Cx(t), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where the system state $x(t)$ takes values in Banach space X; the coefficient operator $A : D(A) \subset X \to X$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$; the input $u(\cdot)$ and output $y(\cdot)$ take values in Banach spaces U and Y, respectively; the control operator B is bounded from U to the extrapolation space X_{-1}^A of X; and the observation operator C is bounded from X_1^A to Y, where X_1^A denotes the domain $D(A)$ equipped with the graph norm. By [3], the extrapolation space X_{-1}^A is the completion of X under the norm $\|R(\lambda_0, A) \cdot \|$, with $R(\lambda_0, A)$ the resolvent of A at λ_0.

Let $\{T_{-1}(t)\}_{t\geq 0}$ be the extrapolation semigroup of $\{T(t)\}_{t\geq 0}$ on X_{-1}^A. In order to guarantee that the state $x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds$ stays in X for any $x_0 \in X$, $u \in L^p([0,\infty), U)(p \in (1,\infty))$ and $t > 0$, we introduce the notion of an admissible control operator. The control operator B is said to be *admissible* for A if, for any $t \geq 0$, the mapping $M(t) : u \mapsto \int_0^t T_{-1}(t-s)Bu(s)ds$ is continuous from $L^p([0,\infty), U)$ into X. If, in addition, $U = X$, we say B is an *admissible control*

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perturbation for A. For the free system (1.1) ($u = 0$), in order to guarantee the output $y \in L^p([0,\infty), Y)$, we introduce the notion of an admissible observation operator. The observation operator C is said to be *admissible* for A if for some $T > 0$ there exists a constant $K > 0$ such that

$$\int_0^T \|CT(t)x\|_Y^p dt \leq K\|x\|_X^p, \ \forall x \in D(A).$$

If in addition, $Y = X$, we say C is an *admissible observation perturbation* for A. Admissible control and admissible observation are closely related to infinite dimensional linear systems with boundary control and point observation (see [21, 25, 26]). It is proved in [26] that there are some dual relationships of admissible control and admissible observation in the case that X is reflexive and $1 < p < \infty$.

The pair (A, B) with B being admissible for A is said to be *exactly controllable* at $\tau > 0$ if $\Phi_{A,B}(\tau)$ (see Section 2) is surjective. The pair (A, C) with C being admissible for A is said to be *exactly observable* at $\tau > 0$ if there exists a constant $k > 0$ such that $\|\Psi_{A,C}(\tau)x\| \geq k\|x\|$ for any $x \in X$ (see Section 2 for $\Psi_{A,C}$). Exact controllability and exact observability are extensions of controllability and observability from finite dimensional systems theory to infinite dimensions. It is well known that controllability enters into the study of many other important concepts such as stabilizability and optimizability; observability enters into the study of many other important concepts such as detectability and estimatability. In the Hilbert space setting and in the particular case that $p = 2$, it is proved in [24] that exact controllability and exact observability are dual to each other. That is, the exact controllability of (A, B) is equivalent to the exact observability of (A^*, B^*) at the same time τ, where A^* and B^* are the corresponding dual operators. Such duality was first formulated in Dolecki and Russell [2], but it was used for proving the exact controllability of PDE's systems only several years later; see e.g. Lions [15, 16] and Triggiani [23]. Exact controllability and exact observability have received considerable attention in the functional analysis frame (see e.g. [4, 10, 18, 19, 20, 30]), where some necessary and/or sufficient conditions have been given.

There are many references that focus on the robustness of exact controllability and exact observability (namely, the persistence of exact controllability and exact observability under some small perturbations). In the finite dimensional space setting, Lee and Markus [13] have proved that if (A, B) is (exactly) controllable, then there exists an $\epsilon > 0$ such that for all $\|\Delta A\| < \epsilon$ and $\|\Delta B\| < \epsilon$, where ΔA and ΔB are matrices of appropriate dimensions, the linear system $\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t), t \geq 0$, remains (exactly) controllable. This means that the controllability of (A, B) is not affected by "small" perturbations. Since the system is of finite dimensions, the (exact) controllability is also robust by the dual relationship.

In the infinite dimensional space setting, Leiva [14] has considered some class of unbounded perturbations $P : D(A) \to X$ which is not too irregular with respect to A. He proved that if the system $\dot{x}(t) = Ax(t) + B(t)u(t)$, with $B(\cdot)$ being bounded and continuous in the strong operator topology of the space of all the bounded linear operators from U to X, is exactly controllable, then it is as well for all systems $\dot{x}(t) = Ax(t) + Px(t) + B(t)u(t)$ with P near to 0 (w.r.t. some metric). The authors of [1] proved that for bounded control operators in X, the exact controllability of $\dot{x}(t) = Ax(t)+B(t)u(t)$ implies that of the perturbed system $\dot{x}(t) = Ax(t)+\Delta Ax(t)+B(t)u(t)$ for ΔA belonging to a neighborhood of zero in the class of Desch-Schappacher perturbations. The arguments used in [1] relies heavily on estimates using Dyson-Phillips series, involving calculations facilitated only by the boundedness of the control operator on X.

However, in partial differential equations the control usually acts on the boundary. Such systems can be reformulated as distributed control systems of the form $\dot{x}(t) = Ax(t)+Bu(t)$, where the control operator is strictly unbounded (see [17, 21]). These cases are beyond the scope of [1] and [14] which considered the case that the control operator B is bounded. Dual to control, the observation is usually of point evaluation, and thereby unbounded. Therefore, it is significant to consider exact controllability and exact observability in the case that the control operator B and/or the observation operator C are unbounded but admissible. It was proved in [29] that for a well-posed linear system with admissible feedback operator the closed-loop system inherits the exact controllability and exact observability of the open-loop system. Furthermore, it was proved that there exists an admissible feedback radius r_0, such that any bounded linear operator $K \in L(Y, U)$ satisfying $\|K\| < r_0$ is an admissible feedback operator for the open-loop system. Hence the closed-loop system keeps the exact observability and exact controllability under the small perturbation of admissible feedback. In this sense, admissible feedback can be regarded as a robust perturbation of exact controllability and exact observability of system (1.1).

Recently, Hadd proved in [7] the admissibility invariance of the control operator to the admissible control perturbation of the generator, and under this premise he proved the robustness of exact controllability. Moreover, he proved in [6, 8] the invariance of admissible observability under admissible observation perturbation. The robustness of exact observability under admissible observation perturbation has been verified by Tucsnak and Weiss [24, Theorem 6.3.2, page 186]. It is easy to see that [6] and [24] are concerned with the admissible control perturbation to exact controllability and the admissible observation perturbation to exact observability, respectively. To the best of the authors' knowledge, for unbounded generator, control and observation operators, there is no literature which discusses the *cross perturbations*, that is, admissible control perturbation to exact observability and admissible observation perturbation to exact controllability (in fact, the work of [14] can be regarded as cross perturbation for the bounded control operator).

This paper is devoted to investigating in the Banach space setting the robustness of exact controllability and exact observability under cross perturbations; that is, if (A, B) ((A, C)) is exactly controllable (exactly observable), then $(A+P, B)$ ($((A_{-1}+\Delta A)|_X, C_\Lambda^A)$) (see Section 2 for C_Λ^A) is exactly controllable (exactly observable) whenever P (ΔA) is small under some metric, where $(A_{-1} + \Delta A)|_X$ is the part of $A + \Delta A$ in X defined by $(A_{-1} + \Delta A)|_X x = A_{-1}x + \Delta A x$ with the domain $D((A_{-1} + \Delta A)|_X) = \{x \in X : (A_{-1} + \Delta A)x \in X\}$. It should be noted that in the Banach space setting, exact controllability and exact observability are not exactly dual to each other. In fact, the admissibility of B for A and B^* for A^* need not be equivalent because in a general Banach space even A^* need not be a generator of a C_0-semigroup. Furthermore, there is no dual form of Λ-extension (see [26]). Hence, the robustness of controllability and observability can't be obtained by the duality from one to the other, and so they are both worth considering. Our procedure is as follows. In Section 2, we give some necessary background on regular linear system, which is the main tool of our paper. In Section 3, we first prove the invariance of admissible controllability and admissible observability under cross perturbations with regularity assumptions, and under such admissibility invariance premise, we prove the robustness of exact controllability and that of exact observability, respectively. Finally, an illustrative example is given.

Throughout this paper, X, U and Y are Banach spaces, $(A, D(A))$ is the generator of the C_0-semigroup $T := (T(t))_{t\geq 0}$ on X, and $1 < p < \infty$. We denote by $L(U,X)$ all the bounded linear operators from U to X and by $M|_X = \{x \in D(M) : Mx \in X\}$ the part of M in X. Denote by I_X the identity operator on X and denote $R^+ = [0, \infty)$. For u, $v \in L^p(R^+, U)$ and $\tau \geq 0$, the τ-concatenation of u and v, denoted by $u \underset{\tau}{\Diamond} v$, is defined by

$$(u \underset{\tau}{\Diamond} v)(t) = \begin{cases} u(t), & t < \tau, \\ v(t-\tau), & t \geq \tau. \end{cases}$$

2. Background on regular linear systems

This section is to recall in a very sketchy way the concept of the regular linear system which is to be used as the main tool in the next section in the sense of Salamon [21] and Weiss [28].

The pair $(T, \Phi) := ((T(t))_{t\geq 0}, (\Phi(t))_{t\geq 0})$ is called an *abstract linear control system* on (X, U) if $\{\Phi(t)\}_{t\geq 0}$ is a one-parameter family of linear bounded operators from $L^p(R^+, U)$ to X satisfying

$$\Phi(t+\tau)(u \underset{\tau}{\Diamond} v) = T(t)\Phi(\tau)u + \Phi(t)v, \forall u, v \in L^p(R^+, U), \forall t, \tau \geq 0. \tag{2.1}$$

By the representation theorem due to Weiss [25], there exists a unique operator $B \in L(U, X_{-1})$, which is admissible for A, such that for any $t \geq 0$ and $u \in L^p(R^+, U)$,

$$\Phi(t)u = \int_0^t T_{-1}(t-s)Bu(s)ds \in X, \tag{2.2}$$

where the integral exists in X_{-1}. Conversely, for any admissible control operator $B \in L(U, X_{-1})$ there corresponds a unique control system (T, Φ). So it is reasonable to denote $\Phi = \Phi_{A,B}$.

The pair $(T, \Psi) = ((T(t))_{t\geq 0}, (\Psi(t))_{t\geq 0})$ is called an *abstract linear observation system* on (X, Y) if $\{\Psi(t)\}_{t\geq 0}$ is a one-parameter family of bounded linear operators from X to $L^p(R^+, Y)$ that satisfies

$$\Psi(t+\tau)x = \Psi(\tau)x \underset{\tau}{\Diamond} \Psi(t)T(\tau)x, \ \forall t, \tau > 0, x \in X. \tag{2.3}$$

By the representation theorem in [26], there exists a unique operator $C \in L(X_1, Y)$, which is admissible for A, such that for any $t \geq 0$ and $x \in D(A)$, $CT(t)x = (\Psi(\infty)x)(t)$, where $\Psi(\infty)$, which is continuous from X to $L^p_{loc}(R^+, Y)$, is the extended output map defined by the strong limit of $\Psi(\tau)$ as $\tau \to \infty$ (see [26]). Conversely, for any admissible observe operator $C \in L(X_1, Y)$ there corresponds a unique abstract linear observation system (T, Ψ). So it is reasonable to denote $\Psi = \Psi_{A,C}$.

Assume T, Φ and Ψ are as above. The quadruple $\Sigma = (T, \Phi, \Psi, F)$ is said to be a well-posed linear system on (X, U, Y) if, in addition, $F = (F(t))_{t\geq 0}$ is a family of bounded linear operators from $L^p(R^+, U)$ to $L^p(R^+, Y)$ satisfying

$$F(t+\tau)(u \underset{\tau}{\Diamond} v) = F(\tau)u \underset{\tau}{\Diamond}(\Psi(t)\Phi(\tau)u + F(t)v), \forall u, v \in L^p(R^+, U). \tag{2.4}$$

We say that the well-posed linear system Σ is regular if the limit

$$\lim_{t\to 0}\frac{1}{t}\int_0^t (F(\infty)u_0)(s)ds = Dz \tag{2.5}$$

exists in Y for the constant input $u_0(t) = z$, where $F(\infty)$, which is continuous from $L^p_{loc}(R^+, U)$ to $L^p_{loc}(R^+, Y)$, is the extended input-output map defined by the strong limit of $F(\tau)$ as $\tau \to \infty$ (see [27]). Moreover, we call D to be the *feedthrough operator* of Σ. In this case, we say that the regular linear system $\Sigma = (T, \Phi, \Psi, F)$ is generated by (A, B, C, D), and we denote $\Sigma = \Sigma_{A,B,C,D}$. If $D = 0$, we denote $F = F_{A,B,C}$, $(A, B, C) = (A, B, C, 0)$ and $\Sigma_{A,B,C} = \Sigma_{A,B,C,0}$ for convenience.

In order to introduce the representation theorem of the regular linear system, Weiss [28] introduced an extension of C, called a Λ-*extension* with respect to A, which is defined by

$$C^A_\Lambda x = \lim_{\lambda\to\infty} C\lambda R(\lambda, A)x \tag{2.6}$$

with the domain $D(C^A_\Lambda) = \{x \in X : \text{this above limit exists in } Y\}$.

With the Λ-extension, for the well-posed linear system Σ, the regularity condition (2.5) is equivalent to each of the following two conditions:

a) *Range*$(R(\lambda, A_{-1})B) \subset D(C^A_\Lambda)$ holds for some (and hence for all) $\lambda \in \rho(A)$.
b) For any $u \in U$, $G(\lambda)u$ has a limits when $\lambda \to \infty$, where G is the transfer function associated to $F(\infty)$.

In this case, the transfer function $G = G_{A,B,C,D}$ is given explicitly by

$$G(\lambda) = C^A_\Lambda R(\lambda, A_{-1})B + D,\ Re(\lambda) > w_0(T), \tag{2.7}$$

where $w_0(T)$ is the growth bound of the C_0-semigroup T. We denote $G_{A,B,C} = G_{A,B,C,0}$ for short.

Next, we consider the feedback of the regular linear system; see [22] and [29] for details.

Definition 2.1. Let Σ be a regular linear system on (X, U, Y) with input-output operators $F(t)$. An operator $\Gamma \in L(Y, U)$ is called an admissible feedback operator for Σ if $I - F(\cdot)\Gamma$ have uniformly bounded inverses.

Theorem 2.2. *Let (A, B, C) generate a regular linear system $\Sigma = (T, \Phi, \Psi, F)$ on (X, U, Y) with an admissible feedback operator $\Gamma \in L(Y, U)$. Then the feedback system Σ^Γ is also a regular linear system given by*

$$\Sigma^\Gamma = \begin{pmatrix} T^\Gamma(\cdot) & \Phi^\Gamma(\cdot) \\ \Psi^\Gamma(\cdot) & F^\Gamma(\cdot) \end{pmatrix} = \begin{pmatrix} T(\cdot) + \Phi(\cdot)\Gamma(I - F\Gamma)^{-1}\Psi & \Phi(\cdot)(I - \Gamma F(\cdot))^{-1} \\ (I - \Gamma F(\cdot))^{-1}\Psi & F(I - \Gamma F(\cdot))^{-1} \end{pmatrix}$$

with the generating operators $(A^\Gamma, B^\Gamma, C^\Gamma)$:

$$A^\Gamma = (A_{-1} + B\Gamma C^A_\Lambda)|_X,\ D(A^\Gamma) := \{z \in D(C^A_\Lambda) : (A_{-1} + B\Gamma C^A_\Lambda)z \in X\}$$

and $B^\Gamma = J^{A,A^\Gamma}B, C^\Gamma = C^A_\Lambda$ restricted to $D(A^\Gamma)$, where J^{A,A^Γ} is defined by $J^{A,A^\Gamma}x = \lim_{\lambda\to\infty}(\lambda - A_{-1})^{-1}x$ (in $X^{A^\Gamma}_{-1}$) with $D(J^{A,A^\Gamma}) = \{x \in X^A_{-1}$: the limit $\lim_{\lambda\to\infty}(\lambda - A_{-1})^{-1}x$ exists$\}$. In addition, $D(C^A_\Lambda) = D((C^\Gamma)^{A^\Gamma}_\Lambda)$.

Remark 2.3. Let $\beta \in \rho(A)$. According to [29], the mapping J^{A,A^Γ} defined above is an isomorphism from W to W^Γ, where $W = (\beta - A_{-1})W_1$, with W_1 being the closure of X^A_1 in C^A_Λ, and where W^Γ is defined in the same way. Accordingly, we always consider B as $J^{A,A^{\Gamma}}B$ for short (for example, in Theorem 3.9 we say "B is admissible for $A+P$" instead of "$J^{A,A+P}B$ is admissible for $A+P$").

3. Main results

In this section, we prove the robustness of exact controllability and exact observability to cross perturbations, respectively. As in [7], we have to find an appropriate metric to measure the robustness. To this purpose, we first introduce two lemmas.

Lemma 3.1. *Assume that (A,C) generates an abstract linear observation system on (X,Y). Let $w\in R$, $M\geq 1$ such that $\|T(t)\|\leq Me^{wt}$ for $t\geq 0$. Then, we have that*

$$\|\Psi_{A,C}(\tau)\|\leq k(\tau,\tau_0)\|\Psi_{A,C}(\tau_0)\|,\forall\tau>0, \tag{3.1}$$

where

$$k(\tau,\tau_0)=\begin{cases} Me^{w\tau}\frac{e^{w\tau_0}}{e^{w\tau_0}-1}, & w>0,\\ M(1+\frac{\tau}{\tau_0}), & w=0,\\ \frac{M}{1-e^{w\tau_0}}, & w<0.\end{cases}$$

Proof. Similarly to the proof of [26, Proposition 2.3], it is easy to show that

$$\|\Psi_{A,C}(n\tau_0)\|\leq\|\Psi_{A,C}(\tau_0)\|M\sum_{k=0}^{n-1}e^{wk\tau_0},\ \forall n\in\mathbb{N}.$$

Let $\tau>0$. Without loss of generality, assume that $\tau\in((n-1)\tau_0,n\tau_0]$ for a certain $n\in\mathbb{N}$. Then, by the above inequality it follows that

$$\|\Psi_{A,C}(\tau)\|\leq\|\Psi_{A,C}(n\tau_0)\|\leq\begin{cases} \|\Psi_{A,C}(\tau_0)\|Me^{w\tau}\frac{e^{w\tau_0}}{e^{w\tau_0}-1}, & w>0,\\ \|\Psi_{A,C}(\tau_0)\|M(1+\frac{\tau}{\tau_0}), & w=0,\\ \|\Psi_{A,C}(\tau_0)\|\frac{M}{1-e^{w\tau_0}}, & w<0.\end{cases}$$

This completes the proof. □

Lemma 3.2. *Assume that (A,B,C) generates a regular linear system on (X,U,Y). Let $\tau_0>0$ and let $w\in R$, $M\geq 1$ such that $\|T(t)\|\leq Me^{wt}$ for $t\geq 0$. Then, for any $\tau>0$ we have*

$$\|F_{A,B,C}(\tau)\|\leq k(\tau,\tau_0)(1+\|\Phi_{A,B}(\tau_0)\|)\|C\|_{\mathscr{R}_{A,B}(\tau_0)} \tag{3.2}$$

and

$$\|F_{A,B,C}(\tau)\|\leq k(\tau,\tau_0)(1+\|\Psi_{A,C}(\tau_0)\|)\|B\|_{\mathscr{W}_{A,C}(\tau_0)}, \tag{3.3}$$

where $\|C\|_{\mathscr{R}_{A,B}(\tau_0)}=\|\Psi_{A,C}(\tau_0)\|+\|F_{A,B,C}(\tau_0)\|$, $\|B\|_{\mathscr{W}_{A,C}(\tau_0)}=\|\Phi_{A,B}(\tau_0)\|+\|F_{A,B,C}(\tau_0)\|$, and $k(\tau,\tau_0)$ is the same as in Lemma 3.1.

Proof. Similarly to the proof of [27, Proposition 2.1], it can be proved that, $\forall n\in\mathbb{N}$,

$$\|F_{A,B,C}(n\tau_0)\|\quad\leq\quad\|F_{A,B,C}(\tau_0)\|+\|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\|M\sum_{l=1}^{n-1}e^{w(l-1)\tau_0}.$$

Let $\tau \in ((n-1)\tau_0, n\tau_0]$ for some $n \in \mathbb{N}$, *w.l.g.* Then, (1) in the case that $w > 0$ we have

$$\begin{aligned}\|F_{A,B,C}(\tau)\| &\leq \|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\| M \frac{e^{wn\tau_0}}{e^{w\tau_0}-1} \\ &\leq \|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\| M e^{w\tau_0} \frac{e^{w\tau}}{e^{w\tau_0}-1} \\ &\leq (\|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\|) M e^{w\tau_0} \frac{e^{w\tau}}{e^{w\tau_0}-1} \\ &\leq \|C\|_{\mathscr{R}_{A,B}} M e^{w\tau_0}(1 + \|\Phi_{A,B}(\tau_0)\|) \frac{e^{w\tau}}{e^{w\tau_0}-1};\end{aligned}$$

(2) in the case that $w = 0$ we have

$$\begin{aligned}\|F_{A,B,C}(\tau)\| &\leq \|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\| M n \\ &\leq \|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\| M (1 + \tau/\tau_0) \\ &\leq (\|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\|) M (1 + \tau/\tau_0) \\ &\leq \|C\|_{\mathscr{R}_{A,B}} (1 + \|\Phi_{A,B}(\tau_0)\|) M (1 + \tau/\tau_0);\end{aligned}$$

and (3) in the case that $w < 0$ we have

$$\begin{aligned}\|F_{A,B,C}(\tau)\| &\leq \|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\| M \frac{1}{1-e^{w\tau_0}} \\ &\leq (\|F_{A,B,C}(\tau_0)\| + \|\Psi_{A,C}(\tau_0)\|\|\Phi_{A,B}(\tau_0)\|) M \frac{1}{1-e^{w\tau_0}} \\ &\leq \|C\|_{\mathscr{R}_{A,B}} (1 + \|\Phi_{A,B}(\tau_0)\|) M \frac{1}{1-e^{w\tau_0}}.\end{aligned}$$

The three cases cover all possibilities, so the inequality (3.2) is proved. The inequality (3.3) can be proved similarly. The proof is therefore completed. □

With the above two lemmas, we can obtain the following completeness result of a linear space, which is important for us to introduce a metric to describe the robustness of exact observability.

Theorem 3.3. *If* (A, B) *generates an abstract linear control system on* (X, U)*, then*

$$\mathscr{R}_{A,B} = \{R \in L(X_1^A, Y) : (A, B, R) \text{ generates a regular linear system}\}$$

is a Banach space endowed with the norm

$$\|R\|_{\mathscr{R}_{A,B}(\tau_0)} := \|\Psi_{A,R}(\tau_0)\| + \|F_{A,B,R}(\tau_0)\|, \tag{3.4}$$

for any fixed $\tau_0 > 0$.

Proof. It follows from the above two lemmas that for any τ_1, τ_2,

$$\|R\|_{\mathscr{R}_{A,B}(\tau_1)} \leq k(\tau_1, \tau_2)(2 + \|\Phi_{A,B}(\tau_2)\|)\|R\|_{\mathscr{R}_{A,B}(\tau_2)}, \forall R \in \mathscr{R}_{A,B},$$

which shows that $\{\|\cdot\|_{\mathscr{R}_{A,B}(\tau)}\}_{\tau>0}$ is an equivalent norm family. To complete the proof, it suffices to show that $\mathscr{R}_{A,B}$ is complete under some norm $\|R\|_{\mathscr{R}_{A,B}(\tau)}$. Similarly to the proof of [9, Proposition 1], it is easy to show that $\mathscr{R}_{A,B}$ is a Banach space under the norm $\|R\|_{\mathscr{R}_{A,B}(1)}$. □

In order to derive the "dual" conclusion, we introduce the following lemma due to Said Hadd [7]:

Lemma 3.4. *Assume that* (A, B) *generates an abstract linear control system on* (X, U). *Let* $(T(t))_{t\geq 0}$ *be the* C_0*-semigroup generated by* A *satisfying* $\|T(t)\| \leq Me^{wt}$ *for some* $w \in R$. *Then, we have*

$$\|\Phi_{A,B}(\tau)\| \leq k(\tau, \tau_0)\|\Phi_{A,B}(\tau_0)\|, \ \tau, \tau_0 > 0,$$

where $k(\tau, \tau_0)$ *is defined as in Lemma* 3.1.

Theorem 3.5. *If* (A, C) *generates an abstract linear observation system on* (X, Y), *then*

$$\mathscr{W}_{A,C} = \{W \in L(U, X_{-1}) : (A, W, C) \text{ generates a regular linear system}\}$$

is a Banach space endowed with the norm

$$\|W\|_{\mathscr{W}_{A,C}(\tau_0)} := \|\Phi_{A,W}(\tau_0)\| + \|F_{A,W,C}(\tau_0)\| \tag{3.5}$$

for any fixed $\tau_0 > 0$.

Proof. By Lemma 3.2 and Lemma 3.4, it follows that $\forall \tau, \tau_0 > 0$,

$$\|R\|_{\mathscr{W}_{A,C}(\tau)} \leq k(\tau, \tau_0)(2 + \|\Psi_{A,C}(\tau_0)\|)\|R\|_{\mathscr{W}_{A,C}(\tau_0)},$$

which indicates that $\{\| \cdot \|_{\mathscr{W}_{A,C}(\tau)}\}_{\tau>0}$ is a family of equivalent norms. So, to conclude the theorem, it is sufficient to show that $\mathscr{W}_{A,C}$ is completed under some norm $\| \cdot \|_{\mathscr{W}_{A,C}(\tau)}$. In the same way as with the proof of [9, Proposition 1], we can show that $\mathscr{W}_{A,C}$ is a Banach space for the norm $\|R\|_{\mathscr{W}_{A,C}(1)}$, as expected. □

The following famous lemma, which says that the subset of surjective operators is open in the space of bounded linear operators, gives an important tool to prove the robustness of exact controllability to cross perturbations. For convenience, we define the set

$$\mathfrak{S}(E, F) = \{\Xi \in L(E, F) : \ \Xi \text{ is surjective}\},$$

for Banach spaces E and F.

Lemma 3.6 ([11, page 227])**.** *Let* E *and* F *be Banach spaces. Then,* $\mathfrak{S}(E, F)$ *is an open set in* $L(E, F)$*; i.e., given* $\Pi \in \mathfrak{S}(E, F)$, *there exists* $\alpha > 0$ *such that*

$$\{\Xi \in L(E, F) : \|\Pi - \Xi\| < \alpha\} \subset \mathfrak{S}(E, F).$$

The constant α *is called a radius of surjectivity of* Π.

In order to prove our main results, the following two lemmas are also needed.

Lemma 3.7 ([7, Lemma 3.2])**.** *If* $(A, \Delta A)$ *generates an abstract linear control system on* (X, X), *then* $(A, \Delta A, I_X)$ *generates a regular linear system with admissible feedback operator* I_X.

Lemma 3.8. *If* (A, P) *generates an abstract observation system on* (X, X), *then* (A, I_X, P) *generates a regular linear system with admissible feedback operator* I_X.

Proof. For $u \in W^{1,p}_{0,loc}(R^+, X) = \{v \in W^{1,p}_{loc}(R^+, X) : v(0) = 0\}$, we have that $\Phi_{A,I_X}(t)u = \int_0^t T(t-r)u(r)dr \in D(A)$, $t \geq 0$. Setting $F(\infty)u = P\Phi_{A,I_X}(\cdot)u$, by the proof of [6, Proposition 3.3], we obtain that for any $\tau > 0$, $u \in W^{1,p}_{0,loc}(R^+, X)$,

$$\begin{aligned} \|F(\infty)u\|_{L^p([0,\tau],X)} &= \|P\Phi_{A,I_X}(\cdot)u\|_{L^p([0,\tau],X)} \\ &= \left\| P\int_0^{\cdot} T(\cdot - s)u(s)ds \right\|_{L^p([0,\tau],X)} \\ &\leq \tau^{1/q}\|\Psi_{A,P}(\tau)\|\|u\|_{L^p([0,\tau],X)}, \end{aligned} \tag{3.6}$$

where q satisfies $1/p + 1/q = 1$.

Let $F(\tau) = P_\tau F(\infty)$, where P_τ is defined by $(P_\tau f)(t) = f(t)$ when $0 \leq t \leq \tau$ and $(P_\tau f)(t) = 0$ when $t > \tau$. It is easy to check that for $u, v \in W^{1,p}_{0,loc}(R^+, X)$, F satisfies (2.4). Since $W^{1,p}_{0,loc}(R^+, X)$ is dense in $L^p_{loc}(R^+, X)$, $F(\infty)$ can be extend continuously to $L^p(R^+, X)$, and by the same technique as in the proof of [9, Theorem 3] we obtain that F satisfies (2.4) as well as (2.5) in $L^p(R^+, X)$. Thus (A, I_X, P) generates a regular linear system.

It is clear by definition that $\|F(\tau)u\|_{L^p([0,\tau],X)} = \|F(\infty)u\|_{L^p([0,\tau],X)}$ for any $u \in L^p([0,\tau], U)$. Hence, by (3.6) and by the fact that $W^{1,p}_{0,loc}(R^+, X)$ is dense in $L^p_{loc}(R^+, X)$, we have that $\|F(\tau)\| \leq \tau^{1/q}\|\Psi_{A,P}(\tau)\|$, $\tau > 0$. It follows from [26, Proposition 2.3 and Remark 2.4] that $\|\Psi_{A,P}(\tau)\|$ is bounded on any bounded interval. So, $\|F(\tau)\| \to 0$ as $\tau \to 0$. Therefore, (A, I_X, P) generates a regular linear system with admissible feedback operator I_X. □

With the above preparation work, we can prove our main results.

Theorem 3.9. *If (A, B, P) generates a regular linear system on (X, U, X), then $(A + P, B)$ generates an abstract linear control system. In addition, if (A, B) is exactly controllable at $\tau > 0$, then there exists a constant $\Theta_0 > 0$ such that for any P satisfying $\|P\|_{\mathscr{R}_{A,B}(\tau)} < \Theta_0$, the pair $(A + P, B)$ is exactly controllable at $\tau > 0$.*

Proof. Let $\mathbb{B} := (I_X, B) : X \times U \to X_{-1}$, $\mathbb{C} = \begin{pmatrix} P \\ 0 \end{pmatrix} : X \to X \times U$. Then, it follows that $\mathbb{B}$ is admissible for A with the input map $\Phi_{A,\mathbb{B}} = (\Phi_{A,I_X}, \Phi_{A,B})$, and $\mathbb{C}$ is admissible for A with the output map $\Psi_{A,\mathbb{C}} = \begin{pmatrix} \Psi_{A,P} \\ 0 \end{pmatrix}$. Obviously, $D(\mathbb{C}^A_\Lambda) = D(P^A_\Lambda)$ and $\mathbb{C}^A_\Lambda = \begin{pmatrix} P^A_\Lambda \\ 0 \end{pmatrix}$. Let $F_1(\cdot) = \begin{pmatrix} F_{A,I_X,P}(\cdot) & F_{A,B,P}(\cdot) \\ 0 & 0 \end{pmatrix}$. Then, it is easy to verify that $(F_1(t))_{t \geq 0}$ is a family of bounded linear operators from $L^p(R^+, X \times U)$ to $L^p(R^+, X \times U)$ and that $(T, \Phi_{A,\mathbb{B}}, \Psi_{A,\mathbb{C}}, F_1)$ satisfies (2.1), (2.3) and (2.4). Moreover, it is easy to see that the transfer function of F_1 is $G_1 = \begin{pmatrix} G_{A,I_X,P} & G_{A,B,P} \\ 0 & 0 \end{pmatrix}$, which satisfies the fact that $G_1(\lambda)(u,v)^T \to 0$ as $\lambda \to \infty$ for all $(u,v)^T \in X \times U$, by Lemma 3.8 and the assumption that (A, B, P) generates a regular linear system. So, $(A, \mathbb{B}, \mathbb{C})$ generates a regular linear system given by

$$\Sigma_{A,\mathbb{B},\mathbb{C}} := \begin{pmatrix} T & (\Phi_{A,I_X}, \Phi_{A,B}) \\ \begin{pmatrix} \Psi_{A,P} \\ 0 \end{pmatrix} & \begin{pmatrix} F_{A,I_X,P} & F_{A,B,P} \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{3.7}$$

with $F_{A,\mathbb{B},\mathbb{C}} = F_1$.

Since by Lemma 3.8 I_X is an admissible feedback operator for $\Sigma_{A,I_X,P}$, it follows that $I_{X\times U} - F_{A,\mathbb{B},\mathbb{C}}(\cdot) = \begin{pmatrix} I - F_{A,I_X,P}(\cdot) & -F_{A,B,P}(\cdot) \\ 0 & I_U \end{pmatrix}$ has uniformly bounded inverses, and so $I_{X\times U}$ is an admissible feedback operator for $\sum_{A,\mathbb{B},\mathbb{C}}$. Thus, in terms of Theorem 2.2, $A^{I_{X\times U}} = (A_{-1} + \mathbb{B}\mathbb{C}_\Lambda)|_X = A + P$, B is admissible for $A + P$, and

$$(\Phi_{A+P,I_X}, \Phi_{A+P,B}) = (\Phi_{A,I_X}, \Phi_{A,B}) \begin{pmatrix} \mathfrak{A} & \mathfrak{A}F_{A,B,P} \\ 0 & I_U \end{pmatrix},$$

where $\mathfrak{A} = (I_X - F_{A,I_X,P})^{-1}$. Obviously,

$$\begin{aligned} \Phi_{A+P,B} &= \Phi_{A,I_X}(I_X - F_{A,I_X,P})^{-1}F_{A,B,P} + \Phi_{A,B} \\ &= \Phi_{A+P,I_X}F_{A,B,P} + \Phi_{A,B}. \end{aligned} \tag{3.8}$$

We now prove the robustness of exact controllability. It follows from (3.8) that

$$\|\Phi_{A+P,B}(\tau) - \Phi_{A,B}(\tau)\| \le \|\Phi_{A,I_X}(\tau)\|\|(I_X - F_{A,I_X,P}(\tau))^{-1}\|\|F_{A,B,P}(\tau)\|,$$

and by (3.6) it follows that $\|F_{A,I_X,P}(\tau)\| \le \tau^{1/q}\|\Psi_{A,I_X}(\tau)\|$, where $q > 1$ satisfies $1/p + 1/q = 1$. Hence, for $\|P\|_{\mathscr{R}_{A,B}(\tau)}\| < \tau^{-1/q}$, we have

$$\|\Phi_{A+P,B}(\tau) - \Phi_{A,B}(\tau)\| \le \frac{\|\Phi_{A,I_X}(\tau)\|\|F_{A,B,P}(\tau)\|}{1 - \tau^{1/q}\|\Psi_{A,P}(\tau)\|} \le \frac{\|\Phi_{A,I_X}(\tau)\|\|P\|_{\mathscr{R}_{A,B}(\tau)}}{1 - \tau^{1/q}\|P\|_{\mathscr{R}_{A,B}(\tau)}}.$$

Let α be the radius of surjectivity of $\Phi_{A,B}(\tau)$ and set

$$\Theta_0 = \min\left\{\tau^{-1/q}, \frac{\alpha}{\alpha\tau^{1/q} + \|\Phi_{A,I_X}(\tau)\|}\right\}.$$

Then, we have that $\|\Phi_{A+P,B}(\tau) - \Phi_{A,B}(\tau)\| < \alpha$ if $\|P\|_{\mathscr{R}_{A,B}(\tau)} < \Theta_0$. Hence, by Lemma 3.6 it follows that $(A + P, B)$ is exactly controllable at τ for any P that satisfies $\|P\|_{\mathscr{R}_{A,B}(\tau)} < \Theta_0$. The proof is completed. $\square$

Remark 3.10. It should be mentioned that when the considered spaces are Hilbert spaces and $p = 2$, the admissibility part of the above theorem is contained in [24, Proposition 5.5.2]. In fact, by [29, (7.14)] and Theorem 2.2 it follows that for any $x \in (\beta - A_{-1})D(\mathbb{C}_\Lambda^A) = (\beta - A_{-1})D(P_\Lambda^A)$, $J^{A,A+P}x = (\beta - (A+P)_{-1})(\beta - A_{-1})^{-1}x + \mathbb{B}^{I_{X\times U}}\mathbb{C}_\Lambda^A(\beta - A_{-1})x = (\beta - (A+P)_{-1})(\beta - A_{-1})^{-1}x + C_\Lambda^A(\beta - A_{-1})x$. This implies that $J^{A,A+P}$ is the same as the corresponding J defined in [24, Proposition 5.52].

Theorem 3.11. *If $(A, \Delta A, C)$ generates a regular linear system on (X, X, Y), then $((A_{-1} + \Delta A)|_X, C_\Lambda^A)$ generates an abstract linear observation system. In addition, if (A, C) is exactly observable at $\tau > 0$, then there exists a constant $k > 0$ such that for any ΔA satisfying $\|\Delta A\|_{\mathscr{W}_{A,C}(\tau)} < k$, the pair $((A_{-1} + \Delta A)|_X, C_\Lambda^A)$ is exactly observable at $\tau > 0$.*

Proof. Consider the operators $\mathscr{B} := (\Delta A, 0) : X \times Y \to X_{-1}$ and $\mathscr{C} = \begin{pmatrix} I_X \\ C \end{pmatrix}$: $X \to X \times Y$. It is easy to show that $\mathscr{B}$ is admissible for A with the input mapping $\Phi_{A,\mathscr{B}} = (\Phi_{A,\Delta A}, 0)$ and $\mathscr{C}$ is admissible for A with the output mapping $\Psi_{A,\mathscr{C}} = \begin{pmatrix} \Psi_{A,I_X} \\ \Psi_{A,C} \end{pmatrix}$. Obviously, $D(\mathscr{C}_\Lambda^A) = D(C_\Lambda^A)$ and $\mathscr{C}_\Lambda^A = \begin{pmatrix} I_X \\ C_\Lambda^A \end{pmatrix}$ in terms of (2.6). Let $F_2 = \begin{pmatrix} F_{A,\Delta A,I_X} & 0 \\ F_{A,\Delta A,C} & 0 \end{pmatrix}$. Then, it is easy to verify that $(F_2(t))_{t\ge 0}$ is a family of bounded linear operators from $L^p(R^+, X\times Y)$ to $L^p(R^+, X\times Y)$ and that $(T, \Phi_{A,\mathscr{B}}, \Psi_{A,\mathscr{C}}, F_2)$ satisfies (2.1), (2.3) and (2.4). Moreover, it is easy to see that the transfer function $\begin{pmatrix} G_{A,\Delta A,I_X} & 0 \\ G_{A,\Delta A,C} & 0 \end{pmatrix}$ of F_2 satisfies $\begin{pmatrix} G_{A,\Delta A,I_X} & 0 \\ G_{A,\Delta A,C} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \to 0$ as $\lambda\to 0$ for all $(u,v)^T \in X\times Y$. Thus $(A,\mathscr{B},\mathscr{C})$ generates a regular linear system given by

$$\Sigma_{A,\mathbb{B}_1,\mathbb{C}_1} := \begin{pmatrix} T & (\Phi_{A,\Delta A}, 0) \\ \begin{pmatrix} \Psi_{A,I_X} \\ \Psi_{A,C} \end{pmatrix} & \begin{pmatrix} F_{A,\Delta A,I_X} & 0 \\ F_{A,\Delta A,C} & 0 \end{pmatrix} \end{pmatrix} \tag{3.9}$$

with $F_2 = F_{A,\mathscr{B},\mathscr{C}}$.

By Lemma 3.7, it is not difficult to see that I_X is an admissible feedback operator for $(A, \Delta A, I_X)$; that is, $I - F_{A,\Delta A,I_X}$ has a uniformly bounded inverse, and $I_{X\times Y} - F_{A,\mathscr{B},\mathscr{C}} = \begin{pmatrix} I - F_{A,\Delta A,I_X} & 0 \\ -F_{A,\Delta A,C} & I_Y \end{pmatrix}$ does as well. Thus, by definition $I_{X\times Y}$ is an admissible feedback operator for $\sum_{A,\mathscr{B},\mathscr{C}}$. In terms of Theorem 2.2, $A^{I_{X\times Y}} = (A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X = \{x \in D(\mathscr{C}_\Lambda^A) :\ (A_{-1}+\Delta A)x \in X)\}$. This means that $(A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X \subset (A_{-1}+\Delta A)|_X$. Since both the operators generate C_0-semigroups, for λ large enough we have that $\lambda \in \rho((A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X) \cap \rho((A_{-1}+\Delta A)|_X)$ and

$$\begin{aligned} R(\lambda, (A_{-1}+\Delta A)|_X) &= R(\lambda, (A_{-1}+\Delta A)|_X)(\lambda - (A_{-1}+\Delta A)|_X) \\ &\quad \times R(\lambda, (A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X) \\ &= R(\lambda, (A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X), \end{aligned}$$

which means $D((A_{-1} + \mathscr{B}\mathscr{C}_\Lambda^A)|_X) = D((A_{-1}+\Delta A)|_X)$. So we get that $A^{I_{X\times Y}} = (A_{-1}+\Delta A)|_X$. Hence, by Theorem 2.2 it follows that $C_\Lambda^{A^{I_{X\times Y}}} = C_\Lambda^A$ is admissible for $(A+\Delta A)|_X$ and

$$\begin{pmatrix} \Psi_{(A_{-1}+\Delta A)|_X, I_X} \\ \Psi_{(A_{-1}+\Delta A)|_X, C_\Lambda^A} \end{pmatrix} = \begin{pmatrix} (I_X - F_{A,\Delta A,I_X})^{-1} & 0 \\ F_{A,\Delta A,C}(I_X - F_{A,\Delta A,I_X})^{-1} & I_Y \end{pmatrix} \begin{pmatrix} \Psi_{A,I_X} \\ \Psi_{A,C} \end{pmatrix}.$$

Obviously,

$$\Psi_{(A_{-1}+\Delta A)|_X, C_\Lambda^A} = F_{A,\Delta A,C}(I_X - F_{A,\Delta A,I_X})^{-1}\Psi_{A,I_X} + \Psi_{A,C}. \tag{3.10}$$

Below we prove the robustness of exact observability. From the proof of Lemma 3.7 (see [7]), we can derive that

$$\|F_{A,\Delta A,I_X}\| \le \tau^{1/p}\|\Phi_{A,\Delta A}(\tau)\|, \tag{3.11}$$

with which we can further prove that, for $\|\Delta A\|_{\mathscr{W}_{A,C}(\tau)} < \tau^{-1/p}$,

$$\|F_{A,\Delta A,C}(\tau)(I_X - F_{A,\Delta A,I_X}(\tau))^{-1}\Psi_{A,I_X}(\tau)\| \le \frac{\|\Psi_{A,I_X}(\tau)\|\|\Delta A\|_{\mathscr{W}_{A,C}(\tau)}}{1-\tau^{1/p}\|\Delta A\|_{\mathscr{W}_{A,C}(\tau)}}.$$

By the assumption that (A, C) is exactly observable at τ, there exists a constant $k_0 > 0$ such that

$$\|\Psi_{A,C}(\tau)\|\|x\| \geq k_0\|x\|, \ x \in X.$$

Let $0 < \alpha_0 < k_0$ and let $k = \min\{\tau^{-1/p}, \frac{k_0-\alpha_0}{(k_0-\alpha_0)\tau^{1/p}+\Phi_{A,I}(\tau)}\}$. Then, we obtain that, when $\|\Delta A\|_{\mathscr{W}_{A,C}(\tau)} < k$,

$$\begin{aligned} & \|\Psi_{(A_{-1}+\Delta A)|_X,C}(\tau)x\| \\ \geq\ & \|\Psi_{A,C}(\tau)x\| - \|F_{A,\Delta A,C}(\tau)(I_X - F_{A,\Delta A,I_X}(\tau))^{-1}\Psi_{A,I_X}(\tau)x\| \\ \geq\ & \alpha_0\|x\|, \end{aligned}$$

for all $x \in X$, which means that $((A + \Delta A)|_X, C_\Lambda^A)$ is exactly observable at τ. The proof is completed. □

Example 3.12. As an illustrative example, we consider the Schrödinger equation with boundary perturbation described by

$$\left\{ \begin{array}{ll} w_t(x,t) + i\Delta w(x,t) = 0, & x \in \Omega, t > 0, \\ w(x,t) = 0, & x \in \Gamma_1, t \geq 0, \\ w(x,t) = Lw(x,t), & x \in \Gamma_0, t \geq 0, \\ y(x,t) = i\frac{\partial(\Delta^{-1}w)}{\partial\nu}, & x \in \Gamma_0, t \geq 0, \end{array} \right. \tag{3.12}$$

where $\Omega \subset R^n$, $n \geq 2$, is an open bounded region with smooth C^3-boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, Γ_0 and Γ_1 are disjoint parts of the boundary relatively open in $\partial\Omega$ and $\text{int}(\Gamma_0) \neq \Omega$, ν is the unit normal vector of Γ_0 pointing towards the exterior of Ω, u is the input function (or control) and y is the output function (or output). Let $H = H^{-1}(\Omega)$ be the state space and let $U = L^2(\Gamma_0)$ be the control (input) or observation (output) space. L is a bounded linear operator from H to U. The system with $u(t)$ replacing $Lw(x,t)$ has been considered by Guo and Shao in [5], where the system (3.12) is cast into the abstract form as (1.1) with well-defined operators A, B and C, and it is proved that this system is a regular linear system with feedthrough operator 0 on (H, U, U). By definition, it is not hard to obtain that (A, BL, C) generates a regular linear system on (H, H, U). Moreover, through simple calculation, we can obtain the relation $\|BL\|_{\mathscr{W}_{A,C}(\tau)} \leq \|L\|\|B\|_{\mathscr{W}_{A,C}(\tau)}$. By [17], it can be proved that $D((A_{-1} + BL)|_H) \subset \mathscr{Z} := D(A) + Range((\alpha - A_{-1})^{-1}B) \subset D(C_\Lambda^A)$, where $\alpha \in \rho(A)$ (observe that $\mathscr{Z}$ is independent on α), and then the system (3.12) can be converted to the observation system $((A_{-1}+BL)|_H, C_\Lambda^A)$. According to [12], system (3.12) with $L = 0$ is exactly observable at some $\tau > 0$. Therefore, by Theorem 3.11, $((A_{-1} + BL)|_H, C_\Lambda^A)$, hence (3.12), is an abstract linear observation system and is exactly observable at τ whenever $\|L\|$ is small enough.

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