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A novel approach to exponential stability of nonlinear systems with time-varying delays

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ABSTRACT

In this paper, the stability of nonlinear systems with time-varying delays is investigated by means of the concepts of generalized Dahlquist constant, generalized relative Dahlquist constant and relative minimal Lipschitz constant. In detail, two sufficient conditions are derived for the exponential stability of nonlinear systems with time-varying delays and the exponential decay of the solutions is also estimated. Compared with some existing results, our stability conditions are less conservative. Some examples are given to illustrate the effectiveness of the obtained results.

1. Introduction

The stability of the system $\dot{x} = f(x)$ in the finite dimensional spaces was systematically investigated in [1]. Concretely, the system is uniformly stable if the logarithmic norm of linear operator f (i.e., f is a matrix), $\mu(f) \leq 0$; so is the system if the Dahlquist constant of nonlinear Lipschitz operator f, $M(f) \leq 0$. Indeed, being a characteristic quantity characterizing the stability of nonlinear systems, Dahlquist constant is a nonlinear generalization of the logarithmic norm. However, it should be mentioned that the concept of Dahlquist constant makes sense only for a class of special nonlinear operators, i.e., Lipschitz operators. From the theoretical and applied points of view, it is desired that the concept can be generalized to the more general nonlinear case. For this, Peng introduced the generalized Dahlquist constant for general nonlinear operators in Banach spaces to investigate the asymptotic behaviors of systems without time delays in [2]. In addition, Wan introduced the generalized relative Dahlquist constant to investigate the stability of general nonlinear systems without time delays in a Banach space in [3] and proved that the generalized relative Dahlquist constant is more efficient than the generalized Dahlquist constant. Moreover, paper [4] proved that the finite dimensional linear system with delays $\dot{x}(t) = Ax(t)+Bx(t-\tau)$ is exponentially stable if $\mu(A) + \|B\| < 0$, where $\mu(A)$ and $\|B\|$ denote the logarithmic norm of the matrix A and the matrix norm of the matrix B, respectively.

Motivated by this, we attempt to analyze the stability of nonlinear systems with time-varying delays (see (2.1)) by the concepts of generalized Dahlquist constant, generalized relative Dahlquist constant and relative minimal Lipschitz constant. Particularly, two sufficient conditions are obtained for the exponential stability of nonlinear systems with time-varying

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delays, which generalizes and improves the stability criteria in [4,3] and provides the exponential decay estimate of solutions of the system (2.1).

In addition, there comes out many other excellent results on stability of nonlinear systems with time delays [5–16]. Some of such results are obtained using Lyapunov functions [5,7,13]. As we all know, it becomes very skillful to construct a proper Lyapunov function for the given system and Lyapunov method does not usually provide convergent rate of solutions. However, our method does not depend on the construction of any Lyapunov function and presents exponentially convergent rate of solutions. Besides, some existing results rely on restrictive conditions on the coefficient operators of the systems such as Lipschitz continuity [11], strict monotonicity [14], and boundedness [10,16]. However, our method abandons these restrictive conditions on coefficient operators. Compared with these stability results, our stability conditions are less conservative.

Throughout this paper, X is assumed to be a Banach space equipped with the norm $\|\cdot\|$ and Ω an open subset of X. $C([-b, 0], X)$ $(b > 0)$ denotes the set of all continuous functions from $[-b, 0]$ to X.

2. Preliminaries

We consider the following time-delayed system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)) + G(x_t(\tau)), \quad t \geq 0, \quad -b \leq \tau \leq 0 \tag{2.1}$$

where $F : \Omega \subset X \to X$ and $G : X \to X$ are nonlinear continuous operators and $x_t(\cdot) \in C([-b, 0], X)$ is defined by $x_t(\tau) = x(t + \tau)$ for $\tau \in [-b, 0]$.

Definition 1 (*[1]*)**.** An operator $T : \Omega \to X$ is called Lipschitz continuous on Ω if there exists a nonnegative constant M such that $\|Tx - Ty\| \leq M\|x - y\|$ for all $x, y \in \Omega$. The constant

$$L(T) = \sup_{x,y\in\Omega,\ x\neq y} \frac{\|Tx - Ty\|}{\|x - y\|}$$

is said to be the minimal Lipschitz constant of T on Ω. $Lip(\Omega, X)$ denotes the set of all Lipschitz continuous operators from Ω to X.

Definition 2. Assume $T : \Omega \to X$ to be an operator and $x_0 \in \Omega$. T is relatively Lipschitz continuous at x_0, if there exists a nonnegative constant M such that

$$\|Tx - Tx_0\| \leq M\|x - x_0\| \quad \text{for all } x \in \Omega. \tag{2.2}$$

The constant

$$L(T, x_0) = \sup_{x\in\Omega,\ x\neq x_0} \frac{\|Tx - Tx_0\|}{\|x - x_0\|}$$

is said to be relative minimal Lipschitz constant of T at x_0.

Obviously, each Lipschitz continuous operator T is relatively Lipschitz continuous at any point in its domain $D(T)$ and $L(T) \leq L(T, x)$ for $\forall x \in D(T)$.

Definition 3 (*[1]*)**.** Assume $T \in Lip(\Omega, \Omega)$. The constant

$$M(T) = \lim_{s\to 0^+} \frac{L(I + sT) - 1}{s}$$

is called the Dahlquist constant of T on Ω.

Definition 4 (*[2]*)**.** Let $T : \Omega \to X$ be a general nonlinear operator. Then the constant

$$\alpha(T) = \sup_{x,y\in\Omega,\ x\neq y} \frac{1}{\|x - y\|} \lim_{r\to\infty} (\|(T + rI)x - (T + rI)y\| - r\|x - y\|)$$

is called the generalized Dahlquist constant of T on Ω.

Definition 5 (*[3]*)**.** Let $T : \Omega \to X$ be a general nonlinear operator and $x_0 \in \Omega$. Then the constant

$$\alpha(T, x_0) = \sup_{x\in\Omega,\ x\neq x_0} \frac{1}{\|x - x_0\|} \lim_{r\to\infty} (\|(T + rI)x - (T + rI)x_0\| - r\|x - x_0\|)$$

is called the generalized relative Dahlquist constant of T at x_0.

Lemma 1 ([3]). *Let $F, G : \Omega \to X$ and $x_0 \in \Omega$. The generalized Dahlquist constant and generalized relative Dahlquist constant enjoy respectively the following useful properties:*

$$\alpha(F+G) \le \alpha(F) + \alpha(G) \quad \text{and} \quad \alpha(F+G, x_0) \le \alpha(F, x_0) + \alpha(G, x_0).$$

For Lipschitz continuous operators, we can easily derive the following useful properties.

Lemma 2. *Let $G : \Omega \to X$ be a Lipschitz continuous operator and $x_0 \in \Omega$. We have $\alpha(G) \le L(G)$ and $\alpha(G, x_0) \le L(G, x_0)$.*

Definition 6. Assume x^* to be an equilibrium point of the time-delayed system (2.1). The system (2.1) is said to be exponentially stable on the neighborhood Ω of x^*, if there exist two positive constants λ and M such that

$$\|x(t) - x^*\| \le M e^{-\lambda t} \sup_{-b \le s \le 0} \|x_0(s) - x^*\|, \quad t \ge 0,$$

where $x(t)$ is the solution of the system (2.1) initiated from $x_0(s) \in \Omega$ with $s \in [-b, 0]$.

Moreover, if it has a unique equilibrium point and exponentially stable on the whole space X, the system (2.1) is said to be globally exponentially stable.

3. Main results

We begin with a uniqueness criterion of equilibrium of the system (2.1).

Theorem 1. *If $x^* \in \Omega$ is an equilibrium point of the system (2.1) and $\alpha(F+G, x^*) < 0$, there is no other equilibrium point in Ω than x^*, namely, the equilibrium point of the system (2.1) is unique in Ω.*

Proof. Assume that $u \in \Omega$ is any other equilibrium point of the system (2.1) different from x^*, namely, $(F+G)(x^*) = (F+G)(u) = 0$, $u \ne x^*$. This implies that

$$\begin{aligned} \alpha(F+G, x^*) &= \sup_{x \in \Omega,\ x \ne x^*} \frac{1}{\|x - x^*\|} \lim_{r \to +\infty} (\|((F+G) + rI)x - ((F+G) + rI)x^*\| - r\|x - x^*\|) \\ &\ge \frac{1}{\|u - x^*\|} \lim_{r \to +\infty} (\|((F+G) + rI)u - ((F+G) + rI)x^*\| - r\|u - x^*\|) \\ &= 0, \end{aligned}$$

which contradicts $\alpha(F+G, x^*) < 0$. Hence $u = x^*$, that is, we have proved the uniqueness of equilibrium point of the system (2.1) in Ω. □

Lemma 3 ([17]). *If $a > c > 0$, then for every nonnegative real number b, the equation*

$$0 = \lambda - a + c e^{\lambda b}$$

has a unique positive solution.

Lemma 4 ([4]). *If $\mu(A) + \|B\| < 0$, the finite dimensional linear system*

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t - \tau), \quad t \ge 0$$

is exponentially stable, where $\mu(A)$ and $\|B\|$ denote the logarithmic norm of the matrix A and the matrix norm of the matrix B, respectively.

Theorem 2. *Suppose that Ω is a neighborhood of an equilibrium x^* of the system (2.1). If $G : X \to X$ is Lipschitz continuous and F, G satisfy $\alpha(F) + L(G) < 0$, the system (2.1) is exponentially stable on Ω. Furthermore, the exponential decay estimate is determined by*

$$\|x(t) - y(t)\| \le e^{-\lambda t} \sup_{-b \le \tau \le 0} \|x_0(\tau) - y_0(\tau)\| \quad \text{for all } t \ge 0, \tag{3.1}$$

where $x(t)$ and $y(t)$ are two solutions of (2.1) initiated from $x_0(\cdot), y_0(\cdot) \in C([-b, 0], \Omega)$ respectively and λ is the unique positive solution of the equation

$$0 = \lambda + \alpha(F) + L(G) e^{b\lambda}.$$

Proof. Since $x(t)$ and $y(t)$ are two solutions of the system (2.1) with the initial values $x_0(\cdot), y_0(\cdot) \in C([-b, 0], X)$ respectively, we have

$$[e^{rt}(x(t) - y(t))]' = e^{rt}[(F + rI)x(t) - (F + rI)y(t) + G(x_t(\tau)) - G(y_t(\tau))]$$

for all $t \geq 0$ and $\tau \in [-b, 0]$. Let $t \geq s \geq 0$, we derive

$$e^{rt}(x(t) - y(t)) = e^{rs}(x(s) - y(s)) + \int_s^t e^{ru}[(F + rI)x(u) - (F + rI)y(u) + G(x_u(\tau)) - G(y_u(\tau))]du.$$

Then

$$\begin{aligned} e^{rt}\|x(t) - y(t)\| - e^{rs}\|x(s) - y(s)\| &\leq \int_s^t e^{ru}[\|(F + rI)x(u) - (F + rI)y(u)\| + \|G(x_u(\tau)) - G(y_u(\tau))\|]du \\ &\leq \int_s^t e^{ru}[\|(F + rI)x(u) - (F + rI)y(u)\| + L(G)\|x_u(\tau) - y_u(\tau)\|]du. \end{aligned}$$

For every $t \geq 0$, we have

$$(e^{rt}\|x(t) - y(t)\|)' \leq e^{rt}[\|(F + rI)x(t) - (F + rI)y(t)\| + L(G)\|x_t(\tau) - y_t(\tau)\|].$$

Moreover,

$$\|x(t) - y(t)\|' \leq [\|(F + rI)x(t) - (F + rI)y(t)\| - r\|x(t) - y(t)\| + L(G)\|x_t(\tau) - y_t(\tau)\|].$$

Let $r \to \infty$ and we have

$$\|x(t) - y(t)\|' \leq \alpha(F)\|x(t) - y(t)\| + L(G) \sup_{-b \leq \tau \leq 0} \|x_t(\tau) - y_t(\tau)\|, \quad \text{for all } t \geq 0.$$

Combined with Halanay's inequality in [18, P. 389–391] and Lemma 3, this implies that

$$\|x(t) - y(t)\| \leq e^{-\lambda t} \sup_{-b \leq \tau \leq 0} \|x_0(\tau) - y_0(\tau)\| \quad \text{for all } t \geq 0,$$

where λ is the unique positive solution of the equation

$$0 = \lambda + \alpha(F) + L(G)e^{b\lambda}.$$

Particularly, for the equilibrium x^*, the system (2.1) has

$$\|x(t) - x^*\| \leq e^{-\lambda t} \sup_{-b \leq \tau \leq 0} \|x_0(\tau) - x^*\| \quad \text{for all } t \geq 0.$$

Consequently, the system (2.1) is exponentially stable on Ω. □

Remark 1. Obviously, Theorem 2 provides a new method to stability analysis of nonlinear dynamic systems with time-varying delays in general Banach space. In fact, it is well known that the generalized Dahlquist constant of nonlinear operators and the minimal Lipschitz constant of nonlinear Lipschitz operators are nonlinear extensions of logarithmic norm of the matrices and matrix norm, respectively. It implies that Theorem 2 is not only the nonlinear, but also infinite dimensional extensions of Lemma 4.

Example 1. Consider the following time-delayed system:

$$\frac{du(t)}{dt} = -\frac{u^3(t)}{2} - u(t) + \frac{1}{2}u(t + \tau), \tag{3.2}$$

where $u(t + \cdot) \in C([-b, 0], \mathbb{R})$.

Let $\Omega = X = \mathbb{R}$ and $F(r) = -\frac{r^3}{2} - r$, $G(r) = \frac{r}{2}$ for all $r \in \mathbb{R}$. Clearly, $u^* = 0$ is the equilibrium point of the system (3.2). By computing, we can easily conclude that F is not Lipschitz continuous on $\mathbb{R}$, $\alpha(F) = -1$ and G is a Lipschitz continuous operator with $L(G) = \frac{1}{2}$. Obviously,

$$\alpha(F) + L(G) = -1 + \frac{1}{2} = -\frac{1}{2} < 0.$$

According to Theorem 2, we know that u^* is globally exponentially stable in $\mathbb{R}$ and the exponential decay estimate is governed by

$$|u(t)| \leq e^{-\lambda t} \sup_{-b \leq \tau \leq 0} |\phi(\tau)|,$$

where $u(t)$ is a solution of the system (3.2) initiated from $\phi(\cdot) \in C([-b, 0], \mathbb{R})$ and λ is the unique positive solution of the equation

$$0 = \lambda - 1 + \frac{1}{2}e^{b\lambda}.$$

Example 2. Consider the heat equation with time delays

$$\begin{cases} \frac{\partial}{\partial t} y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) + B(y_t(x,\tau)), & 0 < x < 1, t > 0, -b \le \tau \le 0 \\ y(0,t) = y(1,t) = 0, \\ y_0(x,\tau) = y(x,\tau), \end{cases} \tag{3.3}$$

where $B : L^2(0,1) \to \mathbb{R}$ is Lipschitz continuous, $y_t(x,\tau) = y(x,t+\tau)$ and $y(x,t+\cdot) \in C([-b,0],\mathbb{R})$.

We take $Y = L^2(0,1)$ and $A = \mathrm{d}^2/\mathrm{d}x^2$ with $D(A) = H_0^1(0,1) \cap H^2(0,1)$. It is obvious that A generates the compact strongly continuous semigroup $(T(t))_{t\ge 0}$ given by

$$T(t)y = \sum_{n=1}^{\infty} 2\mathrm{e}^{-n^2\pi^2 t} \cos n\pi x \int_0^1 \cos n\pi s\, y(s)\mathrm{d}s.$$

In addition, we can easily compute

$$\alpha(A) = \sup_{y\in D(A),\ y\ne 0} \frac{\langle Ay, y\rangle}{\|y\|} = -\pi^2 \tag{3.4}$$

where $\|\cdot\|$ is L^2-norm and $\langle\cdot,\cdot\rangle$ denotes inner product. According to Theorem 2, we conclude that the system (3.3) is exponentially stable on $D(A)$ if $0 \le L(B) \le \pi^2$.

Practically, there are many systems whose equilibrium points are known, for example, pattern recognition systems whose stored patterns are equilibria of the systems. For these systems, we can make full use of the information on known equilibrium points to investigate their asymptotic behaviors.

Theorem 3. *Let x^* be an equilibrium of the system (2.1) and Ω its neighborhood. If $G : X \to X$ is relatively Lipschitz continuous at x^* and nonlinear coefficient operators F and G satisfy*

$$\alpha(F, x^*) + L(G, x^*) < 0, \tag{3.5}$$

then x^* is exponentially stable on Ω. Moreover, the exponential decay estimation of any solution $x(t)$ initiated from $x_0 = \phi(\cdot) \in C([-b,0],\Omega)$ satisfies

$$\|x(t) - x^*\| \le \mathrm{e}^{-\lambda t} \sup_{-b\le\tau\le 0} \|\phi(\tau) - x^*\| \quad \text{for all } t \ge 0, \tag{3.6}$$

where λ is the unique positive solution of the equation

$$0 = \lambda + \alpha(F, x^*) + L(G, x^*)\mathrm{e}^{b\lambda}.$$

Proof. Since $x(t)$ is a solution of the system (2.1) initiated from $x_0 = \phi(\cdot)$, analogous to the proof of Theorem 2, we can easily conclude that

$$\|x(t) - x^*\| \le \alpha(F, x^*)\|x(t) - x^*\| + L(G, x^*) \sup_{-b\le\tau\le 0} \|x_t(\tau) - x^*\|. \tag{3.7}$$

The combination of $\alpha(F, x^*) + L(G, x^*) < 0$ and (3.7) implies that the equilibrium point x^* is exponentially stable on Ω and that the exponentially decay estimation of the solution $x(t)$ is determined by the inequality (3.6). □

Remark 2. When G is relatively Lipschitz continuous at x^*, according to Lemmas 1 and 2, we have

$$\alpha(F + G, x^*) \le \alpha(F, x^*) + \alpha(G, x^*) \le \alpha(F, x^*) + L(G, x^*).$$

Therefore, from Theorem 1 we can conclude that $\alpha(F, x^*) + L(G, x^*) < 0$ ensures that the equilibrium point x^* is not only exponentially stable but also unique on Ω. Moreover, paper [3] discussed the stability of the system (2.1) with $G = 0$ by a generalized relative Dahlquist constant. Consequently, Theorem 3 is nonlinear time-delayed perturbed extension of paper [3]. It should be mentioned that our method does not require coefficient operators F and G to be Lipschitz continuous [11], strict monotonic [14] and bounded [10,16]. Consequently, compared with these stability results, our stability conditions are less conservative.

Remark 3. On the one hand, Theorem 3 is an extension of Theorem 2 because Theorem 3 does not require G to be Lipschitz continuous. On the other hand, Theorem 3 is an improvement of Theorem 2. That is, even for the system with Lipschitz continuous operator G, Theorem 3 is more efficient than Theorem 2 because $\alpha(F, x^*) + L(G, x^*)$ may be strictly less than $\alpha(F) + L(G)$. In what follows, we give an illustrative simple example.

Example 3. Consider the following time-delayed system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \sin^2(u(t)) - \frac{4\pi u(t)}{\pi^2+4} + ku(t+\tau) \tag{3.8}$$

where $0 < k < \frac{4\pi}{\pi^2+4} - \sup_{r\neq 0} \frac{\sin^2 r}{r}$ is a constant and $u(t+\cdot) \in C([-b, 0], \mathbb{R})$.

Let $F(r) = \sin^2 r - \frac{4\pi r}{\pi^2+4}$ and $G(r) = kr$, for $r \in \mathbb{R}$. Obviously, $u^* = 0$ is an equilibrium point. Moreover, G is Lipschitz continuous on $\mathbb{R}$ because of

$$L(G) = \sup_{r\in\mathbb{R}} |G'(r)| = k < \infty.$$

By computing, we have

$$\alpha(F) = 1 - \frac{4\pi}{\pi^2+4} > 0 \quad \text{and} \quad \alpha(F, 0) = \sup_{r\neq 0} \frac{\sin^2 r}{r} - \frac{4\pi}{\pi^2+4} < 0.$$

We have $\alpha(F) + L(G) > 0$, but $\alpha(F, 0) + L(G, 0) = \sup_{r\neq 0} \frac{\sin^2 r}{r} - \frac{4\pi}{\pi^2+4} + k < 0$.

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