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Strong convergence theorem for pseudo-contractive mappings in Hilbert spaces

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ABSTRACT

The purpose of this paper is to construct an Ishikawa type of hybrid algorithm for pseudo-contractive mappings in Hilbert spaces. Our results extend the recent ones announced by Yao et al. [Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997–5002] and many others.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let T be a self-mapping of C. We use $F(T)$ to denote the set of fixed points of T(i.e., $F(T) = \{x \in C : Tx = x\}$).

Definition 1.1 (*[1]*)**.** A mapping $T : C \to C$ is said to be strict pseudo-contraction if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. If $k = 1$, then T is said to be a pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \tag{1.2}$$

is equivalent to,

$$\langle (I - T)x - (I - T)y, x - y\rangle \ge 0, \tag{1.3}$$

for all $x, y \in C$.

The class of strict pseudo-contractions extend the class of nonexpansive mapping. (A mapping T is said to be nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.) That is, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. The pseudo-contractive mapping includes the strict pseudo-contractive mapping.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem [2–4], the split feasibility problem [5–7] and image recovery and signal processing [8–10] etc. However, the Picard sequence $\{T^n x\}_{n=0}^{\infty}$ often

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fails to converge even in the weak topology. Thus averaged iterations prevail. The Mann's iteration is one of the types and is defined by:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\} \subseteq [0, 1]$. Reich [11] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space (see e.g., [12]). From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [13]). Therefore, it is important to develop the theory of iterative methods for strict pseudo-contractions. Indeed, Browder and Petryshyn [1] proved that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad n \geq 0, \tag{1.5}$$

for any starting point $x_0 \in C$, α is a constant such that $k < \alpha < 1$, $\{x_n\}$ converges weakly to a fixed point of strict pseudo-contraction T. Marino and Xu [14] extended the result of Browder and Petryshyn [1] to Mann's iteration (1.4), they proved $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$.

The well known strong convergence theorem for pseudo-contractive mapping was proved by Ishikawa [15] in 1974. More precisely, he got the following theorem.

Theorem 1.1 (*[15]*). *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be a Lipschitzian pseudo-contractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

(i) $\alpha_n \leq \beta_n, n \geq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.1. (i) Since $0 \leq \alpha_n \leq \beta_n \leq 1, n \geq 1$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, the iterative sequence (1.6) could not be reduced to a Mann iterative sequence (1.4). Therefore, the iterative sequence (1.6) has some particular cases.
(ii) The iterative sequence (1.6) is usually called the Ishikawa iterative sequence.
(iii) Chidume and Mutangadura [16] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point of Lipschitzian pseudo-contractive mapping.

In order to obtain a strong convergence theorem for the Mann iteration method (1.4) to nonexpansive mapping, in 2003, Nakajo and Takahashi [17] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 1.2 (*[17]*). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_0 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases} \tag{1.7}$$

where $\{\alpha_n\} \subseteq [0, 1]$ satisfies $\sup_{n\geq 0} \alpha_n < 1$ and $P_{C_n\cap Q_n}x_0$ is the metric projection of H onto $C_n \cap Q_n$. Then $\{x_n\}$ converges strongly to $Px_0 \in F(T)$.

The iterative algorithm (1.7) is often referred as the hybrid algorithm or the CQ algorithm. In this paper, we adopt the former one. Since then, the hybrid algorithm has been studied extensively by many authors (see, for example [18–22]). Specifically, Martinez-Yanes and Xu [23] extend the algorithm (1.7) to the Ishikawa iteration process (1.6). More precisely, they proved the following main theorem:

Theorem 1.3 (*[23]*)**.** *Let C be a closed convex subset of a Hilbert H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.8}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

In 2007, Marino and Xu [14] further generalized the hybrid algorithm from nonexpansive mappings to strict pseudo-contractive mappings. In 2008, Zhou [24] established the hybrid algorithm for pseudo-contractive mapping as follows:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ z_n = (1-\beta_n)x_n + \beta_n Ty_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0. \end{cases} \tag{1.9}$$

He proved that the sequence $\{x_n\}$ defined by (1.9) converges strongly to $P_{F(T)}x_0$.

We observe that the iterative algorithm (1.7)–(1.9) generates a sequence $\{x_n\}$ by projecting x_0 onto the intersection of the suitably constructed closed convex sets C_n and Q_n. Recently, Yao et al. [25] introduced the hybrid iterative algorithm which just involved one closed convex set for pseudo-contractive mapping in Hilbert spaces as follows:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{cases} \tag{1.10}$$

Theorem 1.4 (*[25]*)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\} \in [a, b]$ for some $a, b \in \left(0, \frac{1}{1+L}\right)$. Then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{F(T)}x_0$.*

Motivated and inspired by the above works, in this paper, we generalize the hybrid algorithm (1.10) to the Ishikawa iterative process (1.6).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}, \{\beta_n\}$ be a sequence in $[0, 1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)y_n\rangle \\ \qquad\qquad + 2\alpha_n\beta_n L\|x_n - Tx_n\| \cdot \|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{cases} \tag{1.11}$$

Further, we prove the strong convergence of the hybrid algorithm (1.11) for Lipschitz pseudo-contractive mappings in Hilbert spaces.

2. Preliminaries

In this section, we collect some useful results which will be used in the following section.

We use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

It is well known that a Hilbert space H satisfies the Opial's condition [26], i.e., for each sequence $\{x_n\}$ in H which converges weakly to a point $x \in H$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in H, y \neq x$.

Recall that given a closed convex subset of C of a real Hilbert space H, the nearest point projection P_C form H onto C assigns to each $x \in C$ its nearest point denoted $P_C x$ in C from x to C, that is, $P_C x$ is the unique point in X with the property

$$\|x - P_C x\| \le \|x - y\|, \quad \text{for all } y \in C.$$

The following Lemmas 2.1 and 2.2 are well known.

Lemma 2.1. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \le 0, \quad \text{for all } y \in C.$$

Lemma 2.2. *Let H be a real Hilbert space, then for all $x, y \in H$*

$$\|x - y\|^2 = \|x\|^2 - \|y^2\| - 2\langle x - y, y\rangle.$$

Lemma 2.3 ([27]). *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudo-contractive mapping, then*

(i) *$F(T)$ is closed convex subset of C.*
(ii) *$I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I - T)x_n \to 0$, then $(I - T)z = 0$.*

Lemma 2.4 ([23]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \le \|u - q\|, \quad \forall n \ge 1.$$

Then $\{x_n\}$ converges strongly to q.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $L \ge 1$ such that $F(T) \ne \emptyset$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying: (i) $b \le \alpha_n < \alpha_n(L+1)(1+\beta_n L) < a < 1$, for some $a, b \in (0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$. Then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{F(T)}x_0$.*

Proof. By Lemma 2.3(i), we see that $F(T)$ is closed and convex. Hence, $P_{F(T)}$ is well defined. In fact, it is easy to check that C_n is closed and convex. Next, we prove that $F(T) \subseteq C_n$ for all n.

Let $p \in F(T)$. With the help of Lemma 2.2 and (1.3), we have

$$\begin{aligned}
\|x_n - p - \alpha_n(I-T)y_n\|^2 &= \|x_n - p\|^2 - 2\alpha_n\langle (I-T)y_n, x_n - p - \alpha_n(I-T)y_n\rangle - \|\alpha_n(I-T)y_n\|^2 \\
&= \|x_n - p\|^2 - \|\alpha_n(I-T)y_n\|^2 - 2\alpha_n\langle (I-T)y_n - (I-T)p, y_n - p\rangle \\
&\quad - 2\alpha_n\langle (I-T)y_n, x_n - y_n - \alpha_n(I-T)y_n\rangle \\
&\le \|x_n - p\|^2 - \|\alpha_n(I-T)y_n\|^2 - 2\alpha_n\langle (I-T)y_n, x_n - y_n - \alpha_n(I-T)y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n(I-T)y_n\|^2 \\
&\quad - 2\alpha_n\langle (I-T)y_n, x_n - y_n - \alpha_n(I-T)y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I-T)y_n\|^2 \\
&\quad - 2\langle x_n - y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle \\
&\quad + 2\alpha_n\langle (I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle \\
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I-T)y_n\|^2 \\
&\quad + 2|\langle x_n - y_n - \alpha_n(I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle|.
\end{aligned} \tag{3.1}$$

We will give an estimation to the last item of (3.1), then

$$\begin{aligned}
&|\langle x_n - y_n - \alpha_n(I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle| \\
&\quad = \alpha_n|\langle x_n - Tz_n - (I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle| \\
&\quad = \alpha_n|\langle x_n - Tx_n + Tx_n - Tz_n - (I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle| \\
&\quad = \alpha_n|\langle (I-T)x_n - (I-T)y_n, y_n - x_n + \alpha_n(I-T)y_n\rangle + \langle Tx_n - Tz_n, y_n - x_n + \alpha_n(I-T)y_n\rangle| \\
&\quad \le \alpha_n(L+1)\|x_n - y_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| + \alpha_n L\|x_n - z_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| \\
&\quad = \alpha_n(L+1)\|x_n - y_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| + \alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| \\
&\quad \le \frac{\alpha_n(L+1)}{2}\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I-T)y_n\|^2\right) + \alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\|.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned}\|x_n - p - \alpha_n(I-T)y_n\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I-T)y\|^2 \\ &\quad + \alpha_n(L+1)(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I-T)y_n\|^2) + 2\alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| \\ &\le \|x_n - p\|^2 + 2\alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\|. \end{aligned} \tag{3.3}$$

Notice that

$$\|x_n - p - \alpha_n(I-T)y_n\|^2 = \|x_n - p\|^2 - 2\alpha_n\langle x_n - p, (I-T)y_n\rangle + \|\alpha_n(I-T)y_n\|^2. \tag{3.4}$$

Therefore, from (3.3) and (3.4), we get

$$\|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - p, (I-T)y_n\rangle + 2\alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\|$$

i.e. $p \in C_{n+1}$ if $p \in C_n$. From induction we have thus that $F(T) \subseteq C_n$ for all n.

From the definition of $\{x_n\}$ that $x_n = P_{C_n}x_0$. This implies that $\|x_n - x_0\| \le \|z - x_0\|$, for all $z \in C_n$. Since $F(T) \subseteq C_n$ we have $\|x_n - x_0\| \le \|p - x_0\|$, for any $p \in F(T)$. In particular,

$$\|x_n - x_0\| \le \|q - x_0\|, \quad q = P_{F(T)}x_0. \tag{3.5}$$

Hence $\{x_n\}$ is bounded. Since T is L-Lipschitz continuous, then $\{y_n\}$, $\{Tx_n\}$ and $\{Ty_n\}$ are all bounded.

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}x_0} \in C_{n+1} \subseteq C_n$, we have

$$\langle x_n - x_0, x_{n+1} - x_n\rangle \ge 0. \tag{3.6}$$

By Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0 - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \end{aligned} \tag{3.7}$$

which implies that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|,$$

for all n. Then $\{\|x_n - x_0\|\}$ is a nondecreasing sequence, and notice that $\{\|x_n - x_0\|\}$ is also bounded. Hence, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. At the same time, letting $n \to \infty$ in the right side of inequality (3.7), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} \in C_{n+1} \subseteq C_n$, we have

$$\begin{aligned}\|\alpha_n(I-T)y_n\|^2 &\le 2\alpha_n\langle x_n - x_{n+1}, (I-T)y_n\rangle + 2\alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| \\ &\le 2\alpha_n\|x_n - x_{n+1}\|\,\|y_n - Ty_n\| + 2\alpha_n\beta_n L\|x_n - Tx_n\|\,\|y_n - x_n + \alpha_n(I-T)y_n\| \\ &\longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\le (L+1)\|x_n - y_n\| + \|y_n - Ty_n\| \\ &\le \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n L(L+1)\|x_n - z_n\| + \|y_n - Ty_n\| \\ &= \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n\beta_n L(L+1)\|x_n - Tx_n\| + \|y_n - Ty_n\|, \end{aligned}$$

that is

$$\|x_n - Tx_n\| \le \frac{1}{1-a}\|y_n - Ty_n\| \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.3(ii), $I - T$ is demiclosed at zero. Together with the fact that $\{x_n\}$ is bounded, which guarantees that every weak limit point of $\{x_n\}$ is a fixed point of T. That is $\omega_w(x_n) \subseteq F(T)$. Therefore, by inequality (3.5) and Lemma 2.4, we know $\{x_n\}$ converges strongly to $q = P_{F(T)}x_0$. This completes the proof. □

Remark 3.1. If let $\beta_n = 0$, for all n in (1.11), then the hybrid algorithm (1.11) reduces to (1.10). So Theorem 1.4 of Yao et al. [25] is a special case of our Theorem 3.1.

Recall that a mapping A is said to be monotone or accretive, if $\langle Ax - Ay, x - y\rangle \ge 0$, for all $x, y \in H$. The pseudo-contractive mapping is strongly related to the monotone mapping. It is well known that A is monotone or accretive mapping if and only if $(I - A)$ is pseudo-contractive mapping. Hence, the fixed points of pseudo-contractive mapping actually is the zero of monotone or accretive mapping. Due to Theorem 3.1, we have the following corollaries which generalize the corresponding results of Yao et al. [25].

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying:* (i) $b \leq \alpha_n < 2\alpha_n(1 + \beta_n) < 1$, *for some* $b \in (0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{F(T)}x_0$.*

Corollary 3.3. *Let $A : H \to H$ be a L-Lipschitz monotone mapping for which $A^{-1}(0) \neq \emptyset$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying:* (i) $b \leq \alpha_n < \alpha_n(L + 1)(1 + \beta_n L) < a < 1$, *for some* $a, b \in (0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence $\{x_n\}$ generated by the following:*

$$\begin{cases} y_n = x_n - \alpha_n A z_n, \\ z_n = x_n - \beta_n A x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n A y_n\|^2 \leq 2\alpha_n \langle x_n - z, A y_n\rangle + 2\alpha_n\beta_n L\|Ax_n\| \cdot \|y_n - x_n + \alpha_n A y_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

Converges strongly to $P_{A^{-1}}x_0$.

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