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Fractional Abstract Cauchy Problems

Li Kexue and Peng Jigen

Abstract. This paper is concerned with fractional abstract Cauchy problems with order $\alpha \in (1,2)$. The notion of fractional solution operator is introduced, its some properties are obtained. A generation theorem for exponentially bounded fractional solution operators is given. It is proved that the homogeneous fractional Cauchy problem $(FACP_0)$ is well-posed if and only if its coefficient operator A generates an α-order fractional solution operator. Sufficient conditions are given to guarantee the existence and uniqueness of mild solutions and strong solutions of the inhomogeneous fractional Cauchy problem $(FACP_f)$.

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1. Introduction

In this paper we are concerned with the well-posedness of the homogeneous fractional Cauchy problem

$$(FACP_0) \quad \begin{cases} {}^C D_t^\alpha u(t) = Au(t), \ t \in [0,T], \\ u(0) = 0, \ u'(0) = x, \end{cases}$$

and the existence and uniqueness of the mild solutions and strong solutions of the inhomogeneous fractional abstract Cauchy problem

$$(FACP_f) \quad \begin{cases} {}^C D_t^\alpha u(t) = Au(t) + J_t^{2-\alpha} f(t), \ t \in [0,T], \\ u(0) = 0, \ u'(0) = x, \end{cases}$$

where $1 < \alpha < 2, A : D(A) \subset X \to X$ is a densely defined closed linear operator, X is a Banach space, ${}^C D_t^\alpha$ is the α-order Caputo fractional derivative operator, $J_t^{2-\alpha}$ is the $(2-\alpha)$-order Riemann–Liouville fractional integral

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operator, $f : [0, T] \to X, x \in X$. As a special case, when A is a negative number, $(FACP_f)$ is able to model processes intermediate between exponential decay ($\alpha = 1$) and pure sinusoidal oscillation ($\alpha = 2$), for more details about fractional ordinary differential equations, we refer to [1].

Fractional differential equations has attracted much attention in recent years, see the books of Podlubny [2], Hilfer [3], Kilbas et al. [4] and the papers of Delbosco and Rodino [5], Eidelman and Kochubei [6], Anh and Leonenko [7], Niu and Xie [8]. Fractional derivatives can describe the properties of memory and heredity of materials, which is the major advantage of fractional derivatives compared with integer-order derivatives. Practical problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions. Initial conditions for the Caputo fractional derivatives are expressed in terms of initials of integer order derivatives.

Bazhlekova [9] studied the following fractional order abstract Cauchy problem

$$\begin{cases} \mathbf{D_t^\alpha u(t) = Au(t),\ t \geq 0}, \\ u(0) = x, u^{(k)}(0) = 0, k = 1, \ldots, m-1, \end{cases} \tag{1.1}$$

where $m = \lceil \alpha \rceil$, the smallest integer greater than or equal to $\alpha, \mathbf{D}_t^\alpha$ is the Caputo fractional derivative operator defined by

$$\mathbf{D}_t^\alpha u(t) = D_t^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0) \right).$$

The notion of solution operator is introduced in [9] as follows:

Definition 1.1. Let $\alpha > 0$. A family $\{S_\alpha(t)\}_{t \geq 0} \subset B(X)$ of bounded linear operators on X is called a solution operator for (1.1) if the following three conditions are satisfied:

(a) $S_\alpha(t)$ is strong continuous for $t \geq 0$ and $S_\alpha(0) = I$ (the identity operator on X),
(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$;
(c) For all $x \in D(A), t \geq 0, T_\alpha(t)x$ is a solution of
$$u(t) = x + J_t^\alpha Au(t).$$

Following [10], problem (1.1) is well-posed if and only if it admits a solution operator, just like the first order homogeneous abstract Cauchy problem is well-posed if and only if its coefficient operator generates a strongly continuous semigroup.

The notion of α-times resolvent families (or solution operators) is introduced in Li and Zheng [11]. Li et al. [14] showed that under general conditions, elliptic operators with zero boundary condition can generate fractional resolvent families.

Li et al. [13] considered fractional order evolution equation $D^\alpha u(t) = Au(t); u(0) = u_0, u'(0) = 0$, where A is a differential operator corresponding to a coercive polynomial taking values in a sector of angle less than π, D^α is the Caputo fractional derivative operator, $1 < \alpha < 2$. They showed that such equations are well-posed in the sense that there exists an α-times resolvent family for A.

Chen and Li [12] presented a purely algebraic notion, α-resolvent operator function:

Definition 1.2. Let $\alpha > 0$. A function $S_\alpha : R_+ \to B(X)$ of bounded linear operators on X is called an α-resolvent operator function if the following conditions are satisfied:

(a) $S_\alpha(\cdot)$ is strongly continuous on R_+ and $S_\alpha(0) = I$ (the identity operator on X),
(b) $S_\alpha(t)S_\alpha(s){=}S_\alpha(s)S_\alpha(t)$ for all $t, s \geq 0$,
(c) the functional equation

$$S_\alpha(s)J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s)S_\alpha(t) = J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s)$$

holds for all $t, s \geq 0$.

It is proved that a family $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent operator function if and only if it is a solution operator for a certain fractional Cauchy problem. Moreover, Chen and Li [12] introduced the concept of integrated α-resolvent operator function:

Definition 1.3. Let $\alpha > 0, \beta \geq 0$. A function $S_{\alpha,\beta} : R_+ \to B(X)$ of bounded linear operators on X is called a β-times integrated α-resolvent operator function if the following conditions are satisfied:

(a) $S_{\alpha,\beta}(\cdot)$ is strong continuous on R_+ and $S_{\alpha,\beta}(0) = g_{\beta+1}(0)I$,
(b) $S_{\alpha,\beta}(s)S_\alpha(t){=}S_\alpha(t)S_\alpha(s)$ for all $s, t \geq 0$,
(c) the functional equation

$$S_{\alpha,\beta}(s)J_t^\alpha S_{\alpha,\beta}(t) - J_s^\alpha S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = g_{\beta+1}(s)J_t^\alpha S_{\alpha,\beta}(t) - g_{\beta+1}(t)J_s^\alpha S_{\alpha,\beta}(s)$$

holds for all $s, t \geq 0$.

They gave basic properties and analyticity criteria of fractional resolvent operator functions and discussed the relations between integrated resolvent families and resolvent families.

Li et al. [15] studied the fractional powers of generators of fractional resolvent family, they showed that if $-A$ generates a bounded α-times resolvent family for some $\alpha \in (0, 2]$, then $-A^\beta$ generates an analytic γ-times resolvent family for $\beta \in (0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha}), \gamma \in (0, 2]$. Moreover, they discussed the relations of solutions of fractional Cauchy problems and Cauchy problems of first order.

Umarov [16] presented a fractional generalizations of Duhamel's principle for the Cauchy problem for general inhomogeneous fractional distributed order differential-operator equations of the form

$$\begin{cases} L^\wedge[u] \equiv \int\limits_0^\mu f(\alpha, A)D_*^\alpha u(t)d\wedge(\alpha) = h(t),\ t > 0, \\ u(0) = \varphi_k,\ k = 0, \ldots, m-1, \end{cases}$$

where $\mu \in (m-1, m], A : \mathcal{D} \to X$ is a closed linear operator with domain $D \subset X, X$ is a reflexive Banach space, $f(\alpha, A)$ is a family of operators with the symbol $f(\alpha, z)$ continuous in the variable $\alpha \in [0, \mu]$, analytic in the variable $z \in G \subset C$, $\wedge$ is a finite measure defined on $[0, \mu]$, and D_*^α is the α-order Caputo fractional differential operator.

Fractional Cauchy Problems are useful in physics to model anomalous diffusion. Zaslavsky [17] introduced the fractional kinetic equation

$$\frac{\partial^\beta g(x,t)}{\partial t^\beta} = Lg(x,t) + p_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{1.2}$$

for Hamiltonian chaos, where $0 < \beta < 1, L$ is the generator of a Feller semigroup $\{T(t)\}_{t\geq 0}$ and $p_0 \in C^\infty(R^1)$ is an arbitrary initial condition. Here $\partial^\beta g(x,t)/\partial t^\beta$ is the inverse Laplace transform of $s^\beta \tilde{g}(x,s)$, where $\tilde{g}(x,s) = \int_0^\infty e^{-st}g(x,t)dt$ is the usual Laplace transform. Stochastic solutions of fractional Cauchy problems are subordinated processes. Baeumer and Meerschaert [18] considered the general case where L is the generator of a Feller semigroup $\{T(t)\}_{t\geq 0}$ associated with some infinitely divisible law on R^d. Assume that g_β is the density of the stable subordinator with Laplace transform $\int_0^\infty e^{-st}g_\beta(t)dt = exp(-s^\beta)$, they proved that if $q(x,t) = T(t)f(x)$ solves

$$\frac{\partial q(x,t)}{\partial t} = Lq(x,t);\ q(x,0) = p_0(x)$$

for all $t > 0$ and $x \in R^d$, then $g(x,t) = S(t)f(x)$ solves the fractional Cauchy problem (1.2), where $S(t)$ is defined by

$$S(t)f = \int_0^\infty T((t/s)^\beta)g_\beta(s)fds.$$

Meerschaert et al. [19] developed classical solutions and stochastic analogues for fractional Cauchy problems in a bounded domain $D \subset R^d$ with Dirichlet boundary conditions. Stochastic solutions are constructed via an inverse stable subordinator whose scaling index corresponds to the order of the fractional time derivative. More precisely, they proved that

$$u(t,x) = \int_0 T_D((t/l)^\beta)f(x)g_\beta(l)dl$$

is the solution of

$$\begin{cases} \frac{\partial^\beta u(t,x)}{\partial t^\beta} & = \Delta u(t,x),\ x \in D,\ t > 0. \\ u(t,x) & = 0,\ x \in \partial D, t > 0, \\ u(0,x) & = f(x),\ x \in D, \end{cases}$$

where $0 < \beta < 1, \partial^\beta/\partial t^\beta$ is the Caputo fractional derivative operator, D is a bounded domain with boundary $\partial D \in C^{1,\beta}, C^{1,\beta}$ is the space consists of functions whose first order partial derivatives are uniformly Hölder continuous with exponent β in $D, T_D(t)$ is the killed semigroup of Brownian motion $\{X_t\}$ in D, E_t is the process inverse to a stable subordinator of index $\beta \in (0,1), g_\beta$ is the density of the stable subordinator with Laplace transform $\int_0^\infty e^{-st}g_\beta(t)dt = exp(-s^\beta)$, and f satisfies some regularity.

The aim of this paper is to develop an operator theory to study fractional Cauchy problems of order $\alpha \in (1, 2)$. In Sect. 2 we present some basic definitions and preliminary facts which will be used throughout the following sections. In Sect. 3, We introduce the notion of fractional solution operator, obtain its some properties, and give a generation theorem for exponentially bounded fractional solution operator. In Sect. 4, we study the homogeneous cauchy problem $(FACP_0)$ and the inhomogeneous fractional Cauchy problem $(FACP_f)$, we prove that problem $(FACP_0)$ is well-posed iff its coefficient operator A generates an α-order fractional solution operator, and obtain sufficient conditions for the existence and uniqueness of problem $(FACP_f)$.

2. Preliminaries

In this section, we introduce some definitions, notations, and preliminary facts which are used throughout this paper. Let $\alpha > 0, m = \lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. Let X be a Banach space, $R_+ = [0, \infty)$. By $L^1((0,T); X)$ we denote the space of all Bochner integrable functions $u : (0,T) \to X$, it is a Banach space with the norm

$$\|u\|_1 = \int_0^T \|u(t)\| dt$$

By $C([0,T]; X)$, resp. $C^1([0,T]; X)$, we denote the space of functions $u : [0,T] \to X$, which are continuous, resp. continuously differentiable.
$C([0,T]; X)$ and $C^1([0,T]; X)$ are Banach spaces endowed with the norms

$$\|u\|_C = \sup_{t\in[0,T]} \|u(t)\|_X, \ \|u\|_{C^1} = \sup_{t\in[0,T]} (\|u(t)\|_X + \|u'(t)\|_X).$$

If A is a linear operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $(\lambda I - A)^{-1}$ is a bounded linear operator in X. $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. Let N denote the sets of natural numbers, $N_0 = N \cup \{0\}$. We use the abbreviation

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau$$

for the convolution. Let $I = (0,T)$, or $I = [0,T]$, or $I = R_+, m \in N, 1 \le p < \infty$. The Sobolev spaces can be defined as following (see [20, Appendix]):

$$W^{m,p}(I; X) = \{u |\ \exists \varphi \in L^1(I; X) : u(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + \frac{t^{m-1}}{(m-1)!} * \varphi(t), \ t \in I\}.$$

Note that, $\varphi(t) = u^{(m)}(t), \ c_k = u^{(k)}(0)$.

Definition 2.1. The α-order Riemann–Liouville fractional integral is defined by

$$J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \ u \in L^1((0,T);X), \tag{2.1}$$

where $\Gamma(\alpha)$ is the Gamma function.
The Riemann–Liouville integral can be written as

$$J_t^\alpha u(t) = (g_\alpha * u)(t), \tag{2.2}$$

where

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Set $J_t^0 u(t) = u(t)$, the integral operators J_t^α satisfy the semigroup property,

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \ \alpha, \beta \geq 0. \tag{2.3}$$

Definition 2.2. The α-order Riemann–Liouville fractional derivative is defined by

$$D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} u(\tau) d\tau, \tag{2.4}$$

where $u \in L^1((0,T);X), \ g_{m-\alpha} * u \in W^{m,1}((0,T);X)$. When $\alpha = m, \ m \in N, N$ denotes the sets of natural numbers, define $D_t^m = \frac{d^m}{dt^m}$.
The Riemann–Liouville derivative operator D_t^α is a left inverse of the integral operator J_t^α but in general not a right inverse, that is,

$$D_t^\alpha J_t^\alpha u = u, \ u \in L^1((0,T);X), \tag{2.5}$$

and

$$(J_t^\alpha D_t^\alpha u)(t) = u(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * u)^{(k)}(0) g_{\alpha+k+1-m}(t), \tag{2.6}$$

where $u \in L^1((0,T);X) \ g_{m-\alpha} * u \in W^{m,1}((0,T);X)$.

Definition 2.3. The α-order Caputo fractional derivative is defined by

$$^C D_t^\alpha u(t) = D_t^\alpha \left[u(t) - \sum_{k=0}^{m-1} u^{(k)}(0) g_{k+1}(t) \right], \tag{2.7}$$

where $u \in L^1((0,T);X) \cap C^{m-1}((0,T);X), \ g_{m-\alpha} * u \in W^{m,1}((0,T);X)$.
The Caputo derivative operator $^C D_t^\alpha$ is also a left inverse of the integral operator J_t^α but in general not a right inverse, that is,

$$^C D_t^\alpha J_t^\alpha u = u, \ u \in L^1((0,T);X), \tag{2.8}$$

and

$$(J_t^\alpha \ ^C D_t^\alpha u)(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0) g_{k+1}(t), \tag{2.9}$$

where $u \in L^1((0,T);X) \cap C^{m-1}((0,T);X)$, $g_{m-\alpha} * u \in W^{m,1}((0,T);X)$.

The Laplace transform formula for the Riemann–Liouville fractional integral is defined by

$$L\{J_t^\alpha u(t)\} = \frac{1}{\lambda^\alpha}\widehat{u}(\lambda). \tag{2.10}$$

where $\widehat{u}(\lambda)$ is the Laplace of u defined by

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt,\ Re\lambda > \omega, \tag{2.11}$$

where $Re\lambda$ represents the real part of the complex number λ.

Definition 2.4. The Mittag-Leffler functions are defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta > 0,\ z \in C,$$

where C denotes the set of complex numbers.

The Mittag-Leffler function are related to the Laplace integral

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{1/\alpha},\ \omega > 0,$$

and to the following asymptotic formulas as $z \to \infty$. If $0 < \alpha < 2, \beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z),\ |\arg z| \le \frac{1}{2}\alpha\pi, \tag{2.12}$$

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z),\ |\arg(-z)| < (1-\frac{1}{2}\alpha)\pi, \tag{2.13}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\alpha-\beta n)} + O(|z|^{-N}),\ z \to \infty.$$

3. Fractional Solution Operator

In this section, we introduce the notion of fractional solution operator, obtain its some properties, give a generation theorem for exponentially bounded fractional solution operator and present two examples illustrating the abstract theory.

Definition 3.1. Let $1 < \alpha < 2$, a family $\{T_\alpha(t)\}_{t\ge 0} \subset B(X)$ of all bounded linear operators on X is called an α-order fractional solution operator if it satisfies the following assumptions:

(1) $T_\alpha(t)$ is strongly continuous on R_+ and $\lim_{t\to 0+}\frac{T_\alpha(t)}{t}x = x$ for all $x \in X$;
(2) $T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s)$ for all $t, s \ge 0$;
(3) $T_\alpha(s)J_t^\alpha T_\alpha(t) - J_s^\alpha T_\alpha(s)T_\alpha(t) = sJ_t^\alpha T_\alpha(t) - tJ_s^\alpha T_\alpha(s)$ for all $t, s \ge 0$.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0+} \frac{T_\alpha(t)x - tx}{t^{\alpha+1}} \text{ exists}\}$$

and

$$Ax = \Gamma(\alpha+2) \lim_{t\to 0+} \frac{T_\alpha(t)x - tx}{t^{\alpha+1}}, \text{ for } x \in D(A)$$

is the generator of the α-order fractional solution operator $T_\alpha(t)$, $D(A)$ is the domain of A.

Remark 3.2. Define $T'_\alpha(0)x = \frac{dT_\alpha(t)x}{dt}|_{t=0}$. From (1) of Definition 3.1, it is evident that $T_\alpha(0) = 0$, then we have $T'_\alpha(0) = I$ (the identity operator on X).

Remark 3.3. The fractional integrals in Definition 3.1 are understood strongly in the sense of Bochner.

Remark 3.4. It is clear to see that the notion of fractional solution operator is just the special case of β-times integrated α-resolvent operator function for $\beta = 1$. However, it also should be pointed out that the equality $\lim_{t\to 0+} \frac{T_\alpha(t)}{t} x = x$ ($\forall x \in X$) in Definition 3.1 is essential in the proof of properties (b), (c), (d) of Propositon 3.7, which are necessary for the equivalency of the well-posedness of Cauchy problem $(FACP_0)$ to the existence of an α-order fractional solution operator with coefficient operator A as the generator.

Definition 3.5. If an α-order fractional solution operator $T_\alpha(t)$ satisfies

$$\|T_\alpha(t)\| \le Me^{\omega t},\ t \ge 0. \tag{3.1}$$

for some constants $\omega \ge 0$ and $M \ge 1$, then it is said to be exponentially bounded. An operator A is said to belong $\mathcal{C}^\alpha(M,\omega)$ if A generates a fractional solution operator $T_\alpha(t)$ satisfying (3.1). Denote $\mathcal{C}^\alpha(\omega) = \bigcup\{\mathcal{C}^\alpha(M,\omega); M \ge 1\}$.

Proposition 3.6. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ on X. Then $\sup_{t\in[0,T]} \|T_\alpha(t)\| < \infty$ for every $T > 0$.*

Proof. Any given $T > 0$, defined a mapping $S : X \to C([0,T]; X)$ by $(Sx)t = T_\alpha(t)x,\ t \in [0,T]$. It is easy to show that S is linear and closed, hence by the closed graph theorem S is bounded, there exists a constant $M > 0$ such that $\sup_{t\in[0,T]} \|T_\alpha(t)x\| \le M\|x\|$ for all $x \in X$. Therefore, by the uniform boundedness theorem it follows that $\sup_{t\in[0,T]} \|T_\alpha(t)\| < \infty$. □

Proposition 3.7. *Let $T_\alpha(t)$ be an α-order fractional solution operator on X, and let A be its generator. Then*

(a) *$T_\alpha(t)$ commutes with A, which means that $T_\alpha(t)(D(A)) \subset D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$ and $t \ge 0$.*
(b) *for all $x \in D(A)$ and $t \ge 0$,*

$$T_\alpha(t)x = tx + AJ_t^\alpha T_\alpha(t)x.$$

(c) *for all* $x \in D(A)$ *and* $t \geq 0$,

$$T_\alpha(t)x = tx + J_t^\alpha T_\alpha(t)Ax,$$

and

$$T_\alpha(\cdot)x \in C^1(R_+; X).$$

(d) *A is closed and densely defined.*

Proof. (a) Let $x \in D(A)$, for $t \geq 0,\ s \geq 0$, by (2) of Definition 3.1,

$$(T_\alpha(s) - s)T_\alpha(t)x = T_\alpha(t)(T_\alpha(s)x - sx),$$

hence

$$\Gamma(\alpha+2)\lim_{s\to 0+}\frac{T_\alpha(s)T_\alpha(t)x - sT_\alpha(t)x}{s^{\alpha+1}} = T_\alpha(t)Ax.$$

That is, $T_\alpha(t)x \in D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.
(b) For all $x \in X$ and $s \geq 0$,

$$\begin{aligned}
&\Gamma(\alpha+2)\frac{J_s^\alpha T_\alpha(s)x}{s^{\alpha+1}} - x\\
&\quad= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)s^{\alpha+1}}\int_0^s (s-\tau)^{\alpha-1}T_\alpha(\tau)xd\tau - x\\
&\quad= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)s}\int_0^1 (1-\tau)^{\alpha-1}T_\alpha(s\tau)xd\tau - x\\
&\quad= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)s}\int_0^1 (1-\tau)^{\alpha-1}T_\alpha(s\tau)xd\tau\\
&\qquad - \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau xd\tau\\
&\quad= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau\left(\frac{T_\alpha(s\tau)}{s\tau}x - x\right)d\tau.
\end{aligned} \tag{3.2}$$

By (1) of Definition 3.1, we have

$$\lim_{s\to 0+}\frac{T_\alpha(s\tau)}{s\tau}x = x,$$

Apply dominated convergence theorem to (3.2) to conclude that

$$\lim_{s\to 0+}\Gamma(\alpha+2)\frac{J_s^\alpha T_\alpha(s)x}{s^{\alpha+1}} = x. \tag{3.3}$$

Thus, using (3) of Definition 3.1 and 3.3, we get

$$\begin{aligned} AJ_t^\alpha T_\alpha(t)x &= \Gamma(\alpha+2)\lim_{s\to 0+}\frac{T_\alpha(s)J_t^\alpha T_\alpha(t)x - sJ_t^\alpha T_\alpha(t)x}{s^{\alpha+1}} \\ &= \Gamma(\alpha+2)\lim_{s\to 0+}\frac{J_s^\alpha T_\alpha(s)(T_\alpha(t)x-tx)}{s^{\alpha+1}} \\ &= T_\alpha(t)x - tx, \end{aligned} \tag{3.4}$$

therefore (b) holds.
(c) For $x\in D(A)$, the limit

$$\lim_{s\to 0+}\frac{T_\alpha(s)x-sx}{s^{\alpha+1}}$$

exists, then the function

$$f(s)=\lim_{s\to 0+}\frac{T_\alpha(s)x-sx}{s^{\alpha+1}}$$

is bounded for sufficiently small $s>0$. For $t\geq 0$, by dominated convergence theorem, we get

$$\begin{aligned} T_\alpha(t)x - tx &= AJ_t^\alpha T_\alpha(t)x \\ &= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\lim_{s\to 0+}\frac{T_\alpha(s)-s}{s^{\alpha+1}}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)x\,d\tau \\ &= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\lim_{s\to 0+}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)\frac{T_\alpha(s)x-sx}{s^{\alpha+1}}d\tau \\ &= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)\lim_{s\to 0+}\frac{T_\alpha(s)x-sx}{s^{\alpha+1}}d\tau \\ &= J_t^\alpha T_\alpha(t)Ax. \end{aligned} \tag{3.5}$$

From (3.5) it follows that $T_\alpha(t)x$ is differentiable for $t\geq 0$ and

$$\begin{aligned} \frac{d}{dt}T_\alpha(t)x &= \frac{d}{dt}(tx + J_t^\alpha T_\alpha(t)Ax) \\ &= x + \frac{d}{dt}J_t^\alpha T_\alpha(t)Ax \\ &= x + J_t^{\alpha-1}T_\alpha(t)Ax \\ &= x + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}T_\alpha(\tau)Ax\,d\tau \\ &= x + \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}T_\alpha(t\tau)Ax\,d\tau. \end{aligned} \tag{3.6}$$

Now use dominated convergence theorem to (3.6) to conclude that

$$T_\alpha(\cdot)x\in C^1(R_+;X),\ x\in D(A).$$

(d) Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. Using (3.5) and dominated convergence theorem, we have

$$\begin{aligned} T_\alpha(t)x - tx &= \lim_{n\to\infty} T_\alpha(t)x_n - tx_n \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)Ax_n d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau) y d\tau \\ &= J_t^\alpha T_\alpha(t)y. \end{aligned} \tag{3.7}$$

Using (3.7) and (3.3), we have

$$\begin{aligned} Ax &= \Gamma(\alpha+2) \lim_{t\to 0+} \frac{T_\alpha(t)x - tx}{t^{\alpha+1}} \\ &= \Gamma(\alpha+2) \lim_{t\to 0+} \frac{J_t^\alpha T_\alpha(t)y}{t^{\alpha+1}} \\ &= y. \end{aligned}$$

The closeness of A is proved.
For every $x \in X$, set $x_t = J_t^\alpha T_\alpha(t)x$, from the proof of (b) it follows that $x_t \in D(A)$, and $\Gamma(\alpha+2)\frac{J_t^\alpha T_\alpha(t)x}{t^{\alpha+1}} \to x$ as $t \to 0+$. So A is densely defined. □

From Definition 3.1, it is clear to see that an α-order fractional solution operator possesses unique generator. The following proposition shows that the converse conclusion is also true.

Proposition 3.8. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ on X, then $T_\alpha(t)$ is unique.*

Proof. If $T_\alpha(t)$ and $S_\alpha(t)$ are both α-order fractional solution operator generated by A, then for $x \in D(A)$, by property (c) of Proposition 3.7, we have

$$\begin{aligned} t * T_\alpha(t)x &= (S_\alpha(t) - J_t^\alpha S_\alpha(t)A) * T_\alpha(t)x \\ &= S_\alpha(t) * T_\alpha(t)x - g_\alpha(t) * S_\alpha(t) * AT_\alpha(t)x \\ &= S_\alpha(t) * T_\alpha(t)x - S_\alpha(t) * g_\alpha(t) * AT_\alpha(t)x \\ &= S_\alpha(t) * (T_\alpha(t)x - g_\alpha(t) * AT_\alpha(t)x) \\ &= t * S_\alpha(t)x, \end{aligned}$$

from Titchmarsh's theorem (see [21, p. 166]), it follows that $T_\alpha(t)x = S_\alpha(t)x$ for each $x \in D(A), t \geq 0$. Hence $T_\alpha(t) = S_\alpha(t)$ by the density of $D(A)$. □

Lemma 3.9. ([22, Corollary 1.2]) *Let G be a Banach space. Let $a \geq 0$ and $r : (a,\infty) \to G$ be an infinitely differentiable function. For $M \geq 0, \omega \in (-\infty, a]$ the following assertions are equivalent.*

(i) $\|(\lambda-\omega)^{n+1} r^{(n)}(\lambda)/n!\| \leq M, \lambda > a, n \in N_0$.
(ii) *There exists a function $F : [0,\infty) \to G$ satisfying $F(0) = 0$ and*

$$\limsup_{h\downarrow 0} (1/h)\|F(t+h) - F(t)\| \leq Me^{\omega t} \ (t \geq 0) \tag{3.8}$$

such that

$$r(\lambda) = \lambda \int_0^\infty e^{-\lambda t} F(t) dt \ (\lambda > a). \tag{3.9}$$

Moreover, r has an analytic extension to $\{\lambda \in C : Re\lambda > \omega\}$ which is given by (3.9) if $Re\lambda > 0$.

It is well-known that both strongly continuous semigroup and strongly continuous cosine function are necessarily exponentially bounded. However, whether an α-order fractional solution operator $T_\alpha(t)$ is exponentially bounded is unknown in general. In fact, if $T_\alpha(t)$ is exponentially bounded, we have the following generation theorem.

Theorem 3.10. *Let X be a Banach space. Let A be a closed linear operator with dense domain $D(A) \subset X$. Then $A \in \mathcal{C}^\alpha(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and*

$$\|\frac{d^n}{d\lambda^n}(\lambda^{\alpha-2} R(\lambda^\alpha, A))\| \le \frac{Mn!}{(\lambda - \omega)^{n+1}}, \ \lambda > \omega, \ n \in N_0. \tag{3.10}$$

Proof. (Necessary) Suppose $A \in \mathcal{C}^\alpha(M, \omega)$ and $T_\alpha(t)$ is the fractional solution operator generated by A. For $x \in X, \lambda > \omega$, we define

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt. \tag{3.11}$$

Since $\|T_\alpha(t)\| \le Me^{\omega t}$, $R(\lambda)$ is well-defined for every λ satisfying $\lambda > \omega$. By properties (b), (c) of Proposition 3.7 and the identity (2.10), it follows that

$$\lambda^\alpha R(\lambda)x - \lambda^{\alpha-2}x = AR(\lambda)x, \ x \in X,$$

$$\lambda^\alpha R(\lambda)x - \lambda^{\alpha-2}x = R(\lambda)Ax, \ x \in D(A),$$

Thus, $\lambda^\alpha I - A$ is invertible and $R(\lambda) = \lambda^{\alpha-2} R(\lambda^\alpha, A)$, that is

$$\{\lambda^\alpha : \lambda > \omega\} \subset \rho(A) \tag{3.12}$$

and

$$\lambda^{\alpha-2} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt, \ \lambda > \omega, \ x \in X. \tag{3.13}$$

From (3.13), we have

$$\frac{d}{d\lambda}(\lambda^{\alpha-2} R(\lambda^\alpha, A))x = \frac{d}{d\lambda}\int_0^\infty e^{-\lambda t} T_\alpha(t) x dt = -\int_0^\infty t e^{-\lambda t} T_\alpha(t) x dt.$$

Proceeding by induction we get

$$\frac{d^n}{d\lambda^n}(\lambda^{\alpha-2} R(\lambda^\alpha, A))x = (-1)^n \int_0^\infty t^n e^{-\lambda t} T_\alpha(t) x dt.$$

Therefore,

$$\|\frac{d^n}{d\lambda^n}(\lambda^{\alpha-2}R(\lambda^\alpha,A))x\| = \|\int_0^\infty t^n e^{-\lambda t}T_\alpha(t)xdt\|$$
$$\leq M\int_0^\infty t^n e^{(\omega-\lambda)t}\|x\|dt$$
$$\leq \frac{Mn!}{(\lambda-\omega)^{n+1}}\|x\|. \tag{3.14}$$

(Sufficiency) By Lemma 3.9, for $M \geq 1, \omega \geq 0$, there exists a function $S : [0,\infty) \to B(X)$ satisfying $S(0) = 0$, and

$$\limsup_{h\downarrow 0}(1/h)\|S(t+h) - S(t)\| \leq Me^{\omega t}\ (t \geq 0) \tag{3.15}$$

such that

$$\lambda^{\alpha-3}R(\lambda^\alpha,A) = \int_0^\infty e^{-\lambda t}S(t)dt\ (\lambda > \omega). \tag{3.16}$$

From (3.16), we see that $S(t)$ commutes with A and

$$\int_0^\infty e^{-\lambda t}S(t)dt = \frac{1}{\lambda^3} + \frac{1}{\lambda^\alpha}\int_0^\infty e^{-\lambda t}S(t)Adt, \tag{3.17}$$

Making inverse Laplace transform to (3.17) we obtain that

$$S(t)x = \frac{t^2x}{2} + J_t^\alpha S(t)Ax, \tag{3.18}$$

where $x \in D(A), t \geq 0$.
From (3.18) and note that $\alpha \in (1,2)$, we get

$$\frac{d}{dt}S(t)x = tx + \frac{d}{dt}\int_0^t g_\alpha(t-s)S(s)Axds$$
$$= tx + \int_0^t \frac{d}{dt}g_\alpha(t-s)S(s)Axds$$
$$= tx + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}S(s)Axds. \tag{3.19}$$

Set $G(t)x = \int_0^t (t-s)^{\alpha-2}S(s)Axds$, then

$$\frac{d}{dt}S(t)x = tx + G(t)x \tag{3.20}$$

we shall prove that $G(t)x$ is continuous on R_+ for every $x \in D(A)$.
According to (3.15), there exists a sufficiently small $\delta > 0$ such that for $h \in (0,\delta)$,

$$\|S(t+h)-S(t)\| \leq Mhe^{\omega t},\ t \geq 0. \tag{3.21}$$

For $\Delta t \in (0, \delta)$ and $x \in D(A)$, we have

$$\begin{aligned}\|G(t+\Delta t)x - G(t)x\| &= \| \int_0^{t+\Delta t} (t+\Delta t - s)^{\alpha-2} S(s)Ax \\ &\quad - \int_0^t (t-s)^{\alpha-2} S(s)Ax\| \\ &\leq \int_0^t \left((t-s)^{\alpha-2} - (t+\Delta t - s)^{\alpha-2}\right) \|S(s)\|\|Ax\| ds \\ &\quad + \int_t^{t+\Delta t} (t+\Delta t - s)^{\alpha-2} \|S(s)\|\|Ax\| ds. \end{aligned} \tag{3.22}$$

By dominated convergence theorem,

$$\int_0^t \left((t-s)^{\alpha-2} - (t+\Delta t - s)^{\alpha-2}\right) \|S(s)\|\|Ax\| ds \to 0 \tag{3.23}$$

as $\Delta t \to 0$.
By (3.21), for $s \in [t, t+\Delta t]$ we have

$$\|S(s)\| \leq \|S(s)-S(t)\| + \|S(t)\| \leq M\Delta t e^{\omega t} + \|S(t)\|. \tag{3.24}$$

By (3.24),

$$\int_t^{t+\Delta t} (t+\Delta t - s)^{\alpha-2} \|S(s)\|\|Ax\| ds \leq \frac{(\Delta t)^{\alpha-1}(M\Delta t e^{\omega t} + \|S(t)\|)\|Ax\|}{\alpha-1} \to 0 \tag{3.25}$$

as $\Delta t \to 0$.
Put (3.23) and (3.25) into (3.22) to conclude that $G(t+\Delta t)x - G(t)x \to 0$ as $\Delta t \to 0+$. By the same way we can show that $G(t+\Delta t)x - G(t)x \to 0$ as $\Delta t \to 0-$. Hence $G(t)x$ is continuous on R_+ for every $x \in D(A)$. Since $D(A)$ is dense in X, from (3.20) it follows that $S(t)$ is continuous differentiable on R_+.
Define $T_\alpha(t)x = \frac{d}{dt}S(t)x = S'(t)x, t \geq 0, x \in X$. Then $T_\alpha(t)$ is continuous on R_+. By (3.19), it is clear that $\frac{d}{dt}S(t)|_{t=0} = 0$ and for $x \in D(A)$, we have

$$\frac{T_\alpha(t)}{t}x = x + \frac{t^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^1 s(1-s)^{\alpha-2} \frac{S(ts)}{ts} Axds. \tag{3.26}$$

Note that $1 < \alpha < 2$ and $S(0) = S'(0) = 0$, applying dominated convergence theorem to (3.26), we obtain

$$\lim_{t \to 0+} \frac{T_\alpha(t)}{t}x = x,\ x \in X. \tag{3.27}$$

By (3.15), we see that

$$\|T_\alpha(t)\| \leq Me^{\omega t}. \tag{3.28}$$

From the fact that $S(t)$ commutes with A and A is closed, it follows that $T_\alpha(t)$ commutes with A. Since $T_\alpha(t)x = \frac{d}{dt}S(t)x$ and $S(0) = 0$, we have $S(t)x = \int_0^t T_\alpha(\tau)xd\tau$, this together with (3.19) yields

$$T_\alpha(t)x = tx + J_t^\alpha T_\alpha(t)Ax. \tag{3.29}$$

The closedness of A and the density of $D(A)$ imply that for all $x \in X$

$$T_\alpha(t)x = tx + AJ_t^\alpha T_\alpha(t)x. \tag{3.30}$$

Since $T_\alpha(t)$ commutes with A and A is closed, from (3.30) it follows that

$$\begin{aligned} J_s^\alpha T_\alpha(s)T_\alpha(t)x &= tJ_s^\alpha T_\alpha(s)x + J_s^\alpha T_\alpha(s)AJ_t^\alpha T_\alpha(t)x \\ &= tJ_s^\alpha T_\alpha(s)x + AJ_s^\alpha T_\alpha(s)J_t^\alpha T_\alpha(t)x \\ &= tJ_s^\alpha T_\alpha(s)x + T_\alpha(s)J_t^\alpha T_\alpha(t)x - sJ_t^\alpha T_\alpha(t)x. \end{aligned} \tag{3.31}$$

We next show that $T_\alpha(t)$ commutes with $T_\alpha(s)$. For all $x \in D(A)$ and $t, s \geq 0$, by (3.29) we have

$$\begin{aligned} T_\alpha(t)T_\alpha(s)x &= tT_\alpha(s)x + g_\alpha(t) * T_\alpha(t)AT_\alpha(s)x \\ &= tT_\alpha(s)x + g_\alpha(t) * AT_\alpha(t)T_\alpha(s)x, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} T_\alpha(s)T_\alpha(t)x &= tT_\alpha(s)x + g_\alpha(t) * T_\alpha(s)AT_\alpha(t)x \\ &= tT_\alpha(s)x + g_\alpha(t) * AT_\alpha(s)T_\alpha(t)x, \end{aligned} \tag{3.33}$$

From (3.32) and (3.33) we see that both $w_1(t) = T_\alpha(t)T_\alpha(s)x$ and $w_2(t) = T_\alpha(s)T_\alpha(t)x$ are solutions of

$$u(t) = tT_\alpha(s)x + g_\alpha(t) * Au(t) \tag{3.34}$$

Hence for $x \in D(A)$ we have

$$\begin{aligned} tT_\alpha(s)x * w_1(t) &= (w_2(t) - g_\alpha(t) * Aw_2(t)) * w_1(t) \\ &= w_2(t) * w_1(t) - g_\alpha(t) * Aw_2(t)) * w_1(t) \\ &= w_2(t) * (w_1(t) - g_\alpha(t) * Aw_1(t)) \\ &= w_2(t) * tT_\alpha(s)x. \end{aligned} \tag{3.35}$$

From (3.35) it follows that

$$tT_\alpha(s)x * (w_1(t) - w_2(t)) = 0, \tag{3.36}$$

by Titchmarsh's theorem (see [21, p. 166]) we get $w_1(t) - w_2(t) = 0$, that is $T_\alpha(t)T_\alpha(s)x = T_\alpha(t)T_\alpha(s)x$. By density of $D(A)$, for all $t, s \geq 0$, we obtain

$$T_\alpha(t)T_\alpha(s) = T_\alpha(t)T_\alpha(s). \tag{3.37}$$

By (3.27), (3.28), (3.31), (3.37) and the fact that $T_\alpha(t)$ is strongly continuous on R_+, we conclude that $A \in \mathcal{C}^\alpha(M, \omega)$. Therefore, the proof of Theorem 3.10 is completed. □

Theorem 3.11. *Let $1 < \alpha < 2$. Then $A \in \mathcal{C}^\alpha(M,\omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there is a strongly continuous vector-valued function $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}, M \geq 1, t \geq 0$, and such that*

$$\lambda^{\alpha-2}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt,\ \lambda > \omega,\ x \in X. \tag{3.38}$$

Proof. Suppose there exists a function $T(t)$ and a linear operator A satisfy the conditions above. For $\lambda > \omega, R(\lambda^\alpha, A)$ is differentiable any number of times for $\lambda > \omega$. Then, by differentiating $\lambda^{\alpha-2}R(\lambda^\alpha, A)x$ n-times for λ, we obtain

$$\frac{d^n}{d\lambda^n}(\lambda^{\alpha-2}R(\lambda^\alpha, A))x = (-1)^n \int_0^\infty t^n e^{-\lambda t}T(t)xdt.$$

Hence, we have

$$\|\frac{d^n}{d\lambda^n}(\lambda^{\alpha-2}R(\lambda^\alpha, A))\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}},\ \lambda > \omega,\ n \in N_0.$$

From Theorem 3.10, it follows that $A \in \mathcal{C}^\alpha(M,\omega)$. Let $T_\alpha(t)$ be the corresponding α-order fractional solution operator. Then $T_\alpha(t)$ and $T(t)$ both satisfy (3.38), by the uniqueness theorem for Laplace transforms, we see that $T_\alpha(t) = T(t)$. The converse has already proven in Theorem 3.10. □

Remark 3.12. Note that for $\alpha = 2$, Theorem 3.11 is consistent with the characterization by Laplace transform of strongly continuous 1-times integrated cosine function (see Arendt-Kellermann [23]).

Proposition 3.13. *Let $1 < \alpha < 2$. If A is the generator of an exponentially bounded α-times resolvent family $S_\alpha(t)$, then A is the generator of an exponentially bounded α-order fractional solution operator $T_\alpha(t)$.*

Proof. Assume there are constants $M \geq 1, \omega \geq 0$ such that $\|S_\alpha(t)\| \leq Me^{\omega t}$, then by relation (2.6) in [9] we have

$$(\omega^\alpha, \infty) \subset \rho(A)$$

and

$$\lambda^{\alpha-1}R(\lambda^\alpha, A) = \int_0^\infty e^{-\lambda t}S_\alpha(t)xdt, \lambda > \omega, x \in X. \tag{3.39}$$

Define

$$T_\alpha(t)x = \int_0^t S_\alpha(\tau)xd\tau,\ x \in X.$$

It is clear that $T_\alpha(t)$ is exponentially bounded. By the convolution property of Laplace transforms and (3.39), we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}T_\alpha(t)xdt &= \int_0^\infty e^{-\lambda t}(1*S_\alpha)(t)xd\tau \\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}S_\alpha(t)xdt \\ &= \lambda^{\alpha-2}R(\lambda^\alpha, A).\end{aligned}$$

From Theorem 3.10, it follows that A is generator of the exponentially bounded α-order fractional solution operator $T_\alpha(t)$. □

Example 3.14. Consider the equation

$$\begin{cases} {}^CD_t^\alpha u(t,x) = Au(t,x), \ t \geq 0, \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = 0, \ \frac{\partial u}{\partial t}u(t,x)|_{t=0} = f(x), \end{cases} \tag{3.40}$$

where $\alpha \in (1,2), A := e^{i\theta}\frac{\partial^2}{\partial x^2}$ $(\theta \in [0,\pi))$
is defined in $X = L^2(0,1)$ with domain $D(A) = \{\varphi \in W^{2,2}(0,1), \varphi(0) = \varphi(1) = 0\}, f(x) = \sum_{n=1}^\infty c_n \sin n\pi x, n \in N$.
It is easy to see that A has eigenvalues $-e^{i\theta}n^2\pi^2$ with eigenfunctions $\sin n\pi x, n \in N$. The solution of (3.40) is

$$u(t,x) = \sum_{n=0}^\infty tE_{\alpha,2}(-e^{i\theta}n^2\pi^2t^\alpha)c_n \sin n\pi x.$$

From the asymptotic formulas of the Mittag-Leffler function (2.12) and (2.13),
it follows that $A \in \mathcal{C}^\alpha(0)$ if and only if $|\theta| \leq (1-\frac{\alpha}{2})\pi$.

Example 3.15. Assume that A is a self-adjoint operator on a Hilbert space H and A is bounded above; i.e., $(Ax,x) \leq \omega\|x\|^2$ for all $x \in D(A)$ and some $\omega \in R$, where $(\cdot)$ is the inner product in H. Then for $\alpha \in (1,2), A$ generates an exponentially bounded α-order fractional solution operator.

Proof. Since A satisfies the conditions mentioned above, we have from Example 3.14.16 in [24] that A generates a cosine function. Thus, by Theorem 3.1 in [9], A generates an exponentially bounded α-times resolvent family for $\alpha \in (1,2)$. It follows from Proposition 3.13 that A generates an exponentially bounded α-order fractional solution operator. □

4. Fractional Cauchy Problems

In this section, we devoted to building the relationship between fractional solution operator and the Cauchy problem $(FACP_0)$. Besides, we give sufficient conditions to guarantee the existence and the uniqueness of mild solutions and strong solutions of the Cauchy problem $(FACP_f)$.

Definition 4.1. A function $u \in C([0,T];X)$ is called a mild solution of $(FACP_0)$ if $J_t^\alpha u(t) \in D(A)$ and $u(t) = tx + AJ_t^\alpha u(t), t \in [0,T]$.

Definition 4.2. A function $u \in C([0,T];X)$ is called a strong solution of $(FACP_0)$ if $u \in C([0,T];D(A)) \cap C^1([0,T];X)$, $g_{2-\alpha} * (u - tx) \in C^2([0,T];X)$ and $(FACP_0)$ holds.

Definition 4.3. A function $u \in C([0,T];X)$ is called a mild solution of $(FACP_f)$ if $J_t^\alpha u(t) \in D(A)$ and $u(t) = tx + AJ_t^\alpha u(t) + t * f(t), t \in [0,T]$.

Definition 4.4. A function $u \in C([0,T];X)$ is called a strong solution of $(FACP_f)$ if $u \in C([0,T];D(A)) \cap C^1([0,T];X)$, $g_{2-\alpha} * (u - tx) \in C^2([0,T];X)$ and $(FACP_f)$ holds.

Lemma 4.5. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ on X. Let $x \in X$ and $u \in C([0,T];X)$ be a mild solution of $(FACP_0)$. Then $u(t) = T_\alpha(t)x$ is a unique mild solution of $(FACP_0)$ on $[0,T]$.*

Proof. By property (b) of Proposition 3.7 and Definition 4.1, it follows that

$$\begin{aligned} t * u(t) &= (T_\alpha(t) - Ag_\alpha(t) * T_\alpha(t)) * u(t) \\ &= T_\alpha(t) * u(t) - T_\alpha(t) * Ag_\alpha(t) * u(t) \\ &= T_\alpha(t) * (u(t) - Ag_\alpha(t) * u(t)) \\ &= t * T_\alpha(t)x, \end{aligned}$$

we have $u(t) = T_\alpha(t)x, \;\; t \in [0,T]$. Let $u(t), \; v(t)$ be two mild solution of $(FACP_0)$ and let $w(t) = u(t) - v(t)$, then $w(t) = AJ_t^\alpha w(t)$, hence $w(t) = 0$. □

Lemma 4.6. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ on X. Let $x \in D(A)$, then $u(t) = T_\alpha(t)x$ is a unique strong solution of $(FACP_0)$ on $[0,T]$.*

Proof. By property (c) of Proposition 3.7, for $x \in D(A)$,

$$T_\alpha(\cdot)x \in C^1([0,T];X). \tag{4.1}$$

From (1) of Definition 3.1, we have $T_\alpha(0) = 0$. Then

$$T_\alpha(0)x = 0, \; T'_\alpha(0)x = \lim_{t \to 0+} \frac{T_\alpha(t)x}{t} = x. \tag{4.2}$$

Since $T_\alpha(t)$ is strongly continuous on R_+ and A is closed, it follows that

$$T_\alpha(\cdot)x \in C([0,T];D(A)), \; x \in D(A). \tag{4.3}$$

Using (2.5), (2.7) and properties (c), (a) of Proposition 3.7, we have

$$\begin{aligned} {}^C D_t^\alpha T_\alpha(t)x &= D_t^\alpha (T_\alpha(t)x - T_\alpha(0)x - tT'_\alpha(0)x) \\ &= D_t^\alpha (T_\alpha(t)x - tx) \\ &= D_t^\alpha J_t^\alpha T_\alpha(t)Ax \\ &= D_t^\alpha J_t^\alpha AT_\alpha(t)x \\ &= AT_\alpha(t)x. \end{aligned} \tag{4.4}$$

It follows from (2.3) and property (c) of Proposition 3.7 that

$$\begin{aligned} J_t^{2-\alpha}(T_\alpha(t)x - tx) &= J_t^{2-\alpha} J_t^{\alpha} T_\alpha(t)Ax \\ &= J_t^2 T_\alpha(t)Ax, \end{aligned}$$

then

$$\frac{d^2}{dt^2} J_t^{2-\alpha}(T_\alpha(t)x - tx) = T_\alpha(t)Ax. \tag{4.5}$$

From (4.1)–(4.5), it follows that $u(t) = T_\alpha(t)x$ is a strong solution of $(FACP_0)$ for all $x \in D(A)$ and $t \in [0,T]$. Note that every strong solution of $(FACP_0)$ is also a mild solution, hence strong solutions of $(FACP_0)$ are unique. □

Definition 4.7. The problem $(FACP_0)$ is said to be well-posed if for any $x \in D(A)$ there exists a unique strong solution $u(t;x)$ on $[0,\infty)$, and $\{x_n\} \subset D(A), x_n \to 0$ imply that $u(t;x_n) \to 0$ as $n \to \infty$ in X, uniformly on compact intervals.

To study the relationship between well-posedness of $(FACP_0)$ and existence of fractional solution operator for coefficient operator A, we consider the Volterra equation

$$u(t) = tx + \int_0^t g_\alpha(t-s)Au(s)ds, \ t \in [0,T]. \tag{4.6}$$

Definition 4.8. A function $u \in C([0,T];X)$ is called
(a) strong solution of (4.6) if $u \in C([0,T];D(A))$ and (4.6) holds on $[0,T]$.
(b) mild solution of (4.6) if $J_t^\alpha u(t) \in D(A)$ and $u(t) = tx + AJ_t^\alpha u(t)$ on $[0,T]$.

Definition 4.9. Equation (4.6) is called well-posed if for every $x \in D(A)$ there is a unique strong solution $u(t;x)$ on R_+ of (4.6), and $\{x_n\} \subset D(A), x_n \to 0$ imply $u(t;x_n) \to 0$ in X uniformly on compact intervals.

Remark 4.10. Applying (2.9), we see that the Cauchy problem $(FACP_0)$ is well-posed in sense of Definition 4.7 if and only if (4.6) is well-posed in sense of Definition 4.9.

Theorem 4.11. *The fractional abstract Cauchy problem $(FACP_0)$ is well-posed if and only if A generates an α-order fractional solution operator $T_\alpha(t)$.*

Proof. (Sufficiency). Suppose A generates an α-order fractional solution operator $T_\alpha(t)$. By Lemma 4.6, for every $x \in D(A)$ the function $u(t) = T_\alpha(t)x$ is a unique strong solution of $(FACP_0)$. Continuous dependence of the solutions on x follows from uniform boundedness of $T_\alpha(t)$ on compact intervals of R_+ by Proposition 3.6.

(Necessity). Assume that $(FACP_0)$ is well-posed, then (4.6) is well-posed by Remark 4.10. For every $x \in D(A)$, by $u(t;x)$ we denote a unique strong solution of (4.6). We Define a mapping $T_\alpha(t) : D(A) \to D(A)$ by $T_\alpha(t)x = u(t;x), x \in D(A), t \geq 0$. From the uniqueness of the solutions of (4.6) it follows that $T_\alpha(t)$ is well defined. Obviously, $T_\alpha(t)$ is linear. By the definition of $T_\alpha(t)$, it follows that $T_\alpha(t)x \in C(R_+; D(A))$ and $T_\alpha(0)x = 0,\ T'_\alpha(0)x = x$ for every $x \in D(A)$. By density of $D(A)$, we have

$$T_\alpha(0)x = 0, \quad T'_\alpha(0)x = \lim_{t\to 0}\frac{T_\alpha(t)}{t}x = x,\ x \in X. \tag{4.7}$$

We show that $T_\alpha(t)$ is uniformly bounded on compact intervals of R_+. If this is false then there is a sequence $\{t_n\} \subset [0,T]$ and $\{y_n\} \subset D(A), \|y_n\| = 1$, such that $\|T(t_n)y_n\| \geq n$ for every $n \in N$. Let $x_n = \frac{y_n}{n}$, then $x_n \in D(A), x_n \to 0$, by the definition of $T_\alpha(t)$ we get the contradiction $1 \leq \|T_\alpha(t_n)x_n\| = \|u(t_n; x_n)\| \to 0$ as $n \to \infty$. So $T_\alpha(t)$ is uniformly bounded on compact intervals of R_+. This implies that $T_\alpha(t)$ can be extended to all of $X, T_\alpha(t)x$ is continuous for every $x \in X$. By (a) of Definition 4.8, it follows that for every $x \in D(A)$,

$$T_\alpha(t)x = tx + J_t^\alpha AT_\alpha(t)x = tx + AJ_t^\alpha T_\alpha(t)x,\ t \geq 0. \tag{4.8}$$

For every $x \in X$, since $T_\alpha(t)$ is bounded and A is a closed and densely defined we have $J_t^\alpha T_\alpha(t)x \in D(A)$ and $AJ_t^\alpha T_\alpha(t) = T_\alpha(t) - t$ is strongly continuous for $t \geq 0$, hence $u(t; x) = T_\alpha(t)x$ is a mild solution of (4.6). Next, we prove the mild solutions of (4.6) are unique. In fact, let $u_1,\ u_2 \in C(R_+; X)$ be two mild solutions of (4.6). Then $u = u_1 - u_2 \in C([0,\infty); X)$ and $u(t) = AJ_t^\alpha u(t)$ for all $t \geq 0$. Let $v(t) = J_t^\alpha u(t)$, then $v(t)$ is a strong solution of (4.6) with $v(0) = v'(0) = 0$. It is obvious that $u = 0$ is a strong solution of (4.6) with $x = 0$, by uniqueness of the strong solutions, we have $v(t) = 0$, then $u(t) = D_t^\alpha v(t) = 0$. For $x \in D(A)$, both $u(t; Ax)$ and $Au(t; x)$ are mild solutions of (4.6) with tx replaced by tAx, therefore

$$T_\alpha(t)Ax = u(t; Ax) = Au(t; x) = AT_\alpha(t)x,\ t \geq 0. \tag{4.9}$$

For all $x \in D(A), t, s \geq 0$, by (4.8) and (4.9) it follows that

$$\begin{aligned} T_\alpha(t)T_\alpha(s)x &= tT_\alpha(s)x + g_\alpha(t) * T_\alpha(t)AT_\alpha(s)xd\tau \\ &= tT_\alpha(s)x + g_\alpha(t) * AT_\alpha(t)T_\alpha(s)xd\tau, \end{aligned}$$

and

$$\begin{aligned} T_\alpha(s)T_\alpha(t)x &= tT_\alpha(s)x + g_\alpha(t) * T_\alpha(s)T_\alpha(t)Ax \\ &= tT_\alpha(s)x + g_\alpha(t) * AT_\alpha(s)T_\alpha(t)x. \end{aligned}$$

By (a) of Definition 4.8, both $T_\alpha(\cdot)T_\alpha(s)x$ and $T_\alpha(s)T_\alpha(\cdot)x$ are strong solutions of (4.6) with initial conditions $T_\alpha(0)T_\alpha(s)x = T_\alpha(s)T_\alpha(0)x = 0$ and $T'_\alpha(0)T_\alpha(s)x = T_\alpha(s)T'_\alpha(0) = T_\alpha(s)x$, hence $T_\alpha(t)T_\alpha(s)x = T_\alpha(s)T_\alpha(t)x$ for every $x \in D(A)$ by the well-posedness of (4.6). Since $D(A)$ is dense in X, we have

$$T_\alpha(t)T_\alpha(s) = T_\alpha(s)T_\alpha(t),\ t, s \geq 0. \tag{4.10}$$

Finally, we prove that $T_\alpha(t)$ satisfies (3) of Definition 3.1. For $x \in D(A)$, it follows from (4.8) and (4.9) that

$$\begin{aligned} T_\alpha(s)J_t^\alpha T_\alpha(t) - sJ_t^\alpha T_\alpha(t) &= AJ_s^\alpha T_\alpha(s)J_t^\alpha T_\alpha(t) \\ &= J_s^\alpha T_\alpha(s)AJ_t^\alpha T_\alpha(t) \\ &= J_s^\alpha T_\alpha(s)T_\alpha(t) - tJ_s^\alpha T_\alpha(s). \end{aligned} \tag{4.11}$$

Density of $D(A)$ implies that (4.11) holds for all $x \in X$. By (4.7), (4.10) and (4.11), it follows that $T_\alpha(t)$ is an α-order fractional resolvent. The proof is therefore complete. □

Theorem 4.12. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ on X. Then for every $f \in L^1([0,T];X)$ the problem $(FACP_f)$ has a unique mild solution u given by*

$$u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-\tau)f(\tau)d\tau, \quad t \in [0,T]. \tag{4.12}$$

Proof. Uniqueness: Let $u_1, u_2 \in C([0,T];X)$ be two mild solutions of $(FACP_f)$. Then $w := u_1 - u_2 \in C([0,T];X)$ and $AJ_t^\alpha w(t) = w(t)$ for all $t \in [0,T]$. It follows from Lemma 4.5 that $w \equiv 0$.

Existence: We have seen that $T_\alpha(\cdot)x$ is a mild solution of the homogeneous fractional Cauchy problem $(FACP_0)$. It remains to show that $v(t) = \int_0^t T_\alpha(t-s)f(s)ds$ is a mild solution of $(FACP_f)$. Since $T_\alpha(t)$ is strongly continuous on R_+ and $f \in L^1([0,T];X)$, by Proposition 1.3.4 in [22], we have $v \in C([0,T];X)$. Using properties (d), (b) of Proposition 3.7 we obtain

$$\begin{aligned} AJ_t^\alpha v(t) &= A(g_\alpha * T_\alpha * f) \\ &= (Ag_\alpha * T_\alpha) * f \\ &= (T_\alpha * f)(t) - t * f(t) \\ &= v(t) - t * f(t). \end{aligned}$$

The proof is therefore completed. □

Theorem 4.13. *Let A be the generator of an α-order fractional solution operator $T_\alpha(t)$ and $x \in D(A)$. Assume that one of the following two conditions is satisfied:*

(i) *$f \in D(A), f \in C([0,T];X)$ and $Af \in C([0,T];X)$, or*
(ii) *$g_{2-\alpha} * f \in W^{2,1}([0,T];X), f \in W^{1,1}([0,T];X)$.*

Then $(FACP_f)$ has a unique strong solution u defined by

$$u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-s)f(s)ds, \quad t \geq 0. \tag{4.13}$$

Proof. For the uniqueness, let $u_1, u_2 \in C([0,T];X)$ be two strong solutions of $(FACP_f)$. Then $w := u_1 - u_2 \in C([0,T];X), w(0) = w'(0) = 0$ and

${}^{C}D_t^{\alpha}w(t) = Aw(t)$ for all $t \in [0,T]$. It follows from Lemma 4.6 that $w \equiv 0$. For the existence, let

$$v(t) = \int_0^t T_\alpha(t-s)f(s)ds.$$

Case (i): Since $f \in D(A)$, then from property (c) of proposition 3.7 and Proposition 1.3.4 in [24] it follows that

$$v \in C^1([0,T];X) \tag{4.14}$$

and

$$v'(t) = \int_0^t T'_\alpha(t-s)f(s)ds. \tag{4.15}$$

It is clear that $v(0) = v'(0) = 0$. We have seen that $T_\alpha(t)x$ is a strong solution of the homogeneous Cauchy problem $(FACP_0)$ for $x \in D(A)$. Then we only need to show that $v(t)$ is a strong solution of $(FACP_f)$ with $x = 0$.

Since $v \in C([0,T];X), A$ is closed, then

$$Av(t) = \int_0^t T_\alpha(t-s)Af(s)ds. \tag{4.16}$$

Since $Af \in C([0,T];X)$, Proposition 1.3.4 in [24] shows that

$$Av \in C([0,T];X). \tag{4.17}$$

By Definition 2.3, we have

$$\begin{aligned} {}^{C}D_t^{\alpha}v(t) &= D_t^{\alpha}(v(t) - v(0) - v'(0)t) \\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-r)^{1-\alpha}v(r)dr. \\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\Big(\frac{d}{dt}\int_0^t (t-r)^{1-\alpha}v(r)dr\Big). \end{aligned} \tag{4.18}$$

Let

$$S(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t (t-r)^{1-\alpha}v(r)dr.$$

From Fubini's theorem, we see that

$$\begin{aligned} S(t) &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t\int_0^r (t-r)^{1-\alpha}T_\alpha(r-s)f(s)dsdr \\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t\int_s^t (t-r)^{1-\alpha}T_\alpha(r-s)f(s)drds \end{aligned}$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t \int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau ds$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{d}{dt} \left(\int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \right) ds$$

$$+ \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} \int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau.$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{d}{dt} \left(\int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \right) ds$$

$$+ \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} T_\alpha((t-s)\tau) f(s) d\tau$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{d}{dt} \left(\int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \right) ds \tag{4.19}$$

By (4.18) and (4.19), we have

$$\begin{aligned} {}^C D_t^\alpha v(t) &= \frac{d}{dt} S(t) \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{d^2}{dt^2} \left(\int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \right) ds \\ &\quad + \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} \frac{d}{dt} \int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \\ &= I_1 + I_2. \end{aligned} \tag{4.20}$$

Since $f \in D(A)$, by property (c), (d) of Proposition 3.7, we get

$$I_1 = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{d^2}{dt^2} \left(\int_0^{t-s} (t-s-\tau)^{1-\alpha} T_\alpha(\tau) f(s) d\tau \right) ds$$

$$= \int_0^t (D_r^\alpha T_\alpha(r) f(s)|_{r=t-s}) \, ds$$

$$= \int_0^t (D_r^\alpha(r) f(s)|_{r=t-s}) ds + \int_0^t (T_\alpha(r) A f(s)|_{r=t-s}) ds$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f(s)ds + \int_0^t T_\alpha(t-s)Af(s)ds$$
$$= J_t^{2-\alpha} f(t) + Av(t). \tag{4.21}$$

By dominated convergence theorem, we have

$$I_2 = \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} \frac{d}{dt} \left((t-s)^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} T_\alpha((t-s)\tau) f(s) d\tau \right)$$
$$= \frac{2-\alpha}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} \tau \left(\frac{T_\alpha((t-s)\tau) f(s)}{(t-s)\tau} \right) d\tau$$
$$+ \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \frac{d}{dt} \int_0^1 (1-\tau)^{1-\alpha} T_\alpha((t-s)\tau) f(\tau) d\tau$$
$$= \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} \frac{d}{dt} T_\alpha((t-s)\tau) f(\tau) d\tau \tag{4.22}$$

By property (c) of Proposition 3.7, for $f \in D(A)$, $T_\alpha(\cdot)f \in C^1([0,T];X)$, this together with (4.22) yield

$$I_2 = \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} \frac{d}{dt} T_\alpha((t-s)\tau) f(\tau) d\tau$$
$$= \frac{1}{\Gamma(2-\alpha)} \lim_{s\to t-0} (t-s)^{2-\alpha} \lim_{s\to t-0} \int_0^1 (1-\tau)^{1-\alpha} \frac{d}{dt} T_\alpha((t-s)\tau) f(\tau) d\tau$$
$$= 0. \tag{4.23}$$

It follows from (4.20), (4.21) and (4.23) that

$$^C D_t^\alpha v(t) = Av(t) + J_t^{2-\alpha} f(t). \tag{4.24}$$

Let $h(t) = J_t^{2-\alpha} f(t)$, then $h(t)$ is continuous on $[0,T]$. In fact,

$$h(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} f(\tau) d\tau$$
$$= \frac{t^{2-\alpha}}{\Gamma(2-\alpha)} \int_0^1 (1-\tau)^{1-\alpha} f(t\tau) d\tau.$$

By dominated convergence theorem, it is clear that $g(t)$ is continuous on $[0,T]$. Since $Av \in C([0,T];X)$ and $g_{2-\alpha} * f = h \in C([0,T];X)$, by (4.24), we obtain $g_{2-\alpha} * v \in C^2([0,T];X)$. Considering that $v \in C([0,T];D(A)) \cap C^1([0,T];X), g_{2-\alpha} * v \in C^2([0,T];X)$ and $v(t)$ satisfies $(FACP_f)$ with $x = 0$, therefore v is a strong solution of $(FACP_f)$ with $x = 0$.

Case (ii): Since $T(t)$ is strongly continuous and $f \in W^{1,1}([0,T];X)$, then v is differentiable and

$$v'(t) = \frac{d}{dt}\int_0^t T_\alpha(s)f(t-s)ds$$
$$= \int_0^t T_\alpha(s)f'(t-s)ds + T_\alpha(t)f(0). \tag{4.25}$$

From Proposition 1.3.4 in [24], it follows that $T_\alpha * f' \in C([0,T];X)$, this together with (4.25) yield

$$v \in C^1([0,T];X). \tag{4.26}$$

By dominated convergence theorem, we have

$$\begin{aligned}(g_{2-\alpha} * f)(0) &= \lim_{s\to 0+}(g_{2-\alpha} * f)(s)\\ &= \lim_{s\to 0+}\frac{1}{\Gamma(2-\alpha)}\int_0^s (s-\tau)^{1-\alpha}f(\tau)d\tau\\ &= \lim_{s\to 0+}\frac{s^{2-\alpha}}{\Gamma(2-\alpha)}\int_0^1 (1-\tau)^{1-\alpha}f(s\tau)d\tau\\ &= 0.\end{aligned} \tag{4.27}$$

Since $g_{2-\alpha} * f \in W^{2,1}([0,T];X)$, from (2.6), (4.27), it follows that

$$\begin{aligned}f(s) &= J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)(0)g_{\alpha-1}(s) + (g_{2-\alpha} * f)'(0)g_\alpha(s)\\ &= J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)'(0)g_\alpha(s).\end{aligned}$$

Then

$$\begin{aligned}v(t) &= \int_0^t T_\alpha(t-s)f(s)ds\\ &= \int_0^t T_\alpha(t-s)\left(J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)'(0)g_\alpha(s)\right)ds\\ &= \int_0^t T_\alpha(t-s)J_s^\alpha D_s^\alpha f(s)ds + \int_0^t g_\alpha(s)T(t-s)(g_{2-\alpha} * f)'(0)ds\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t\int_\tau^t (s-\tau)^{\alpha-1}T_\alpha(t-s)D_\tau^\alpha f(\tau)dsd\tau + J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^{t-\tau}(t-\tau-r)^{\alpha-1}T_\alpha(r)D_\tau^\alpha f(\tau)drd\tau + J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0).\end{aligned}$$

Hence the closedness of A and property (b) of proposition 3.7 imply that $v(t) \in D(A)$ and

$$\begin{aligned}
Av(t) &= \frac{1}{\Gamma(\alpha)} A \int_0^t \int_0^{t-\tau} (t-\tau-r)^{\alpha-1} T_\alpha(r) D_\tau^\alpha f(\tau) dr d\tau \\
&\quad + A J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t A \int_0^{t-\tau} (t-\tau-r)^{\alpha-1} T_\alpha(r) D_\tau^\alpha f(\tau) dr d\tau \\
&\quad + T_\alpha(t)(g_{2-\alpha} * f)'(0) - t(g_{2-\alpha} * f)'(0) \\
&= \int_0^t (T_\alpha(t-\tau) D_\tau^\alpha f(\tau) - (t-\tau) D_\tau^\alpha f(\tau)) d\tau \\
&\quad + T_\alpha(t)(g_{2-\alpha} * f)'(0) - t(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau) d\tau - J_t^2 D_t^\alpha f(t) \\
&\quad + T_\alpha(t)(g_{2-\alpha} * f)'(0) - t(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau) d\tau - J_t^{2-\alpha} f(t) + t(g_{2-\alpha} * f)'(0) \\
&\quad + T_\alpha(t)(g_{2-\alpha} * f)'(0) - t(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau) d\tau - J_t^{2-\alpha} f(t) + T_\alpha(t)(g_{2-\alpha} * f)'(0). \qquad (4.28)
\end{aligned}$$

By Definition 2.3, we have

$$\begin{aligned}
D_t^\alpha v(t) &= D_t^\alpha (v(t) - v(0) - t v'(0)) \\
&= D_t^\alpha v(t) \\
&= D_t^\alpha \int_0^t T_\alpha(\tau) f(t-\tau) d\tau \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t \int_0^r (t-r)^{1-\alpha} T_\alpha(s) f(r-s) ds dr \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \left(\frac{d}{dt} \int_0^t \int_0^r (t-r)^{1-\alpha} T_\alpha(s) f(r-s) ds dr \right). \qquad (4.29)
\end{aligned}$$

Let

$$w(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t \int_0^r (t-r)^{1-\alpha} T_\alpha(s) f(r-s) ds dr.$$

Using Fubini's theorem, we obtain

$$
\begin{aligned}
w(t) &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t\int_s^t (t-r)^{1-\alpha}T_\alpha(s)f(r-s)drds \\
&= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t T_\alpha(s)\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T_\alpha(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds \\
&\quad + \frac{1}{\Gamma(2-\alpha)}T_\alpha(s)\lim_{s\to t-0}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)dr \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T_\alpha(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds \\
&\quad + \frac{1}{\Gamma(2-\alpha)}T_\alpha(s)\lim_{s\to t-0}(t-s)^{2-\alpha}\int_0^1(1-r)^{1-\alpha}f((t-s)r)dr \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T_\alpha(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds. \qquad (4.30)
\end{aligned}
$$

From (4.29) and (4.30), we see that

$$
\begin{aligned}
{}^C D_t^\alpha v(t) &= \frac{d}{dt}w(t) \\
&= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t T(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T(s)\frac{d^2}{dt^2}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)drds \\
&\quad + \frac{1}{\Gamma(2-\alpha)}T(t)\lim_{s\to t-0}\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)dr \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T(t-\tau)\frac{d^2}{d\tau^2}\int_0^{\tau}(\tau-r)^{1-\alpha}f(r)drd\tau
\end{aligned}
$$

$$+\frac{1}{\Gamma(2-\alpha)}T(t)\lim_{s\to 0+}\frac{d}{ds}\int_0^s (s-r)^{1-\alpha}f(r)dr$$

$$=\int_0^t T(t-\tau)D_\tau^\alpha f(\tau)d\tau + T_\alpha(t)(g_{2-\alpha}*f)'(0). \tag{4.31}$$

Therefore, by (4.28) and (4.31), we have

$$^C D_t^\alpha v(t) = Av(t) + J_t^{2-\alpha} f(t).$$

Since $Av \in C([0,T];X)$, $g_{2-\alpha}*f \in C([0,T];X)$, then $g_{2-\alpha}*v \in C^2([0,T];X)$. Considering that $v \in C([0,T];D(A)) \cap C^1([0,T];X)$, $g_{2-\alpha}*v \in C^2([0,T];X)$ and v satisfies $(FACP_f)$ with $x = 0$, therefore v is a strong solution of $(FACP_f)$ with $x = 0$. The proof is completed.

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L. Kexue (✉) and P. Jigen
Department of Mathematics
Xi'an Jiaotong University
Xi'an 710049
China
e-mail: kexueli@gmail.com;
jgpeng@mail.xjtu.edu.cn

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