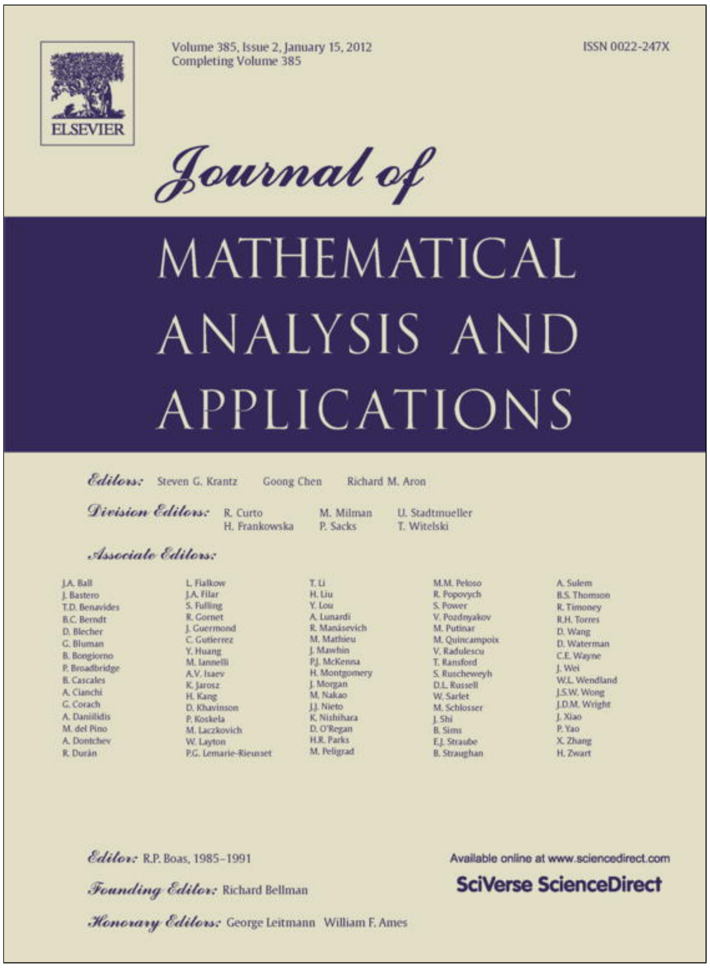
Volume 385, Issue 2, January 15, 2012
Completing Volume 385
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
Division Editors: R. Curto H. Frankowska M. Milman P. Sacks U. Stadtmueller T. Witelski
Associate Editors:
Editor: R.P. Boas, 1985–1991
Founding Editor: Richard Bellman
Honorary Editors: George Leitmann William F. Ames
Available online at www.sciencedirect.com
SciVerse ScienceDirect

J. Math. Anal. Appl. 385 (2012) 786–796

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A novel characteristic of solution operator for the fractional abstract Cauchy problem ☆

Peng Jigen *, Li Kexue

Department of Mathematics, Xi'an Jiaotong University, Xi'an 710049, China

ARTICLE INFO

Article history:
Received 5 January 2011
Available online 8 July 2011
Submitted by M. Iannelli

Keywords:
Fractional abstract Cauchy problem
Fractional derivative
Fractional semigroup
Solution operator

ABSTRACT

Motivated by an equality of the Mittag–Leffler function proved recently by the authors, this paper develops an operator theory for the fractional abstract Cauchy problem (FACP) with order $\alpha \in (0,1)$. The notion of fractional semigroup is introduced. It is proved that a family of bounded linear operator is a solution operator for (FACP) if and only if it is a fractional semigroup. Moreover, the well-posedness of the problem (FACP) is also discussed. It is shown that the problem (FACP) is well-posed if and only if its coefficient operator generates a fractional semigroup.

1. Introduction

Let X be a Banach space. By $C([0,\infty),X)$, resp. $C^k([0,\infty),X)$, we denote the spaces of functions $f:[0,\infty)\to X$, which are continuous, resp. k-times continuously differentiable. Let $\alpha>0$ and $m=[\alpha]$, the smallest integer greater than or equal to α. The α-order Riemann–Liouville fractional derivative of $u\in C([0,\infty),X)$ is defined by

$$ {}_0D_t^{\alpha}u(t)=\frac{d^m}{dt^m}\int_0^t \frac{(t-r)^{m-1-\alpha}}{\Gamma(m-\alpha)}u(r)\,dr, \tag{1.1}$$

where $\Gamma(\cdot)$ stands for the Gamma function. For the sake of convenience, the following notations are commonly used in the literature:

$$ g_\alpha(t)=\begin{cases}\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t>0;\\ 0, & t\leqslant 0,\end{cases} \tag{1.2}$$

and

$$ J_t^{\alpha}u(t)=(g_\alpha * u)(t)=\int_0^t g_\alpha(t-\tau)u(\tau)\,d\tau, \quad u\in C\big([0,\infty),X\big). \tag{1.3}$$

☆ This work was supported by the Natural Science Foundation of China under the contact No. 60970149.
* Corresponding author.
E-mail addresses: jgpeng@mail.xjtu.edu.cn (J. Peng), kexueli@gmail.com (K. Li).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.07.009

Then the α-order derivative operator ${}_0D_t^{\alpha}\cdot$ can be written as

$${}_0D_t^{\alpha}u(t) = \frac{d^m}{dt^m} J_t^{m-\alpha}u(t). \tag{1.4}$$

Let $A : D(A) \subset X \to X$ be a linear operator densely defined in X. Consider the following Cauchy problem for the fractional evolution equation of order α:

$$\text{(FACP)} \quad \begin{cases} {}_0^C D_t^{\alpha}u(t) = Au(t), & t > 0; \\ u(0) = x, \quad u^{(k)}(0) = 0, & k = 1, 2, \ldots, m-1, \end{cases}$$

where ${}_0^C D_t^{\alpha}\cdot$ is the modified Caputo fractional derivative operator

$${}_0^C D_t^{\alpha}u(t) = {}_0D_t^{\alpha}\left(u(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}u^{(k)}(0)\right). \tag{1.5}$$

It is easy to show that for $u \in C^m([0,\infty), X)$, ${}_0^C D_t^{\alpha}u(t)$ can be calculated by

$${}_0^C D_t^{\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-1-\alpha}u^{(m)}(\tau)\,d\tau, \tag{1.6}$$

which is just the original definition of Caputo fractional derivative (see, e.g. [9]).

Based on the fact that the solvableness of (1.5) is equivalent to that of the following integral equation

$$u(t) = x + \int_0^t g_{\alpha}(t-\tau)Au(\tau)\,d\tau, \quad t \geqslant 0, \tag{1.7}$$

E. Bajlekova [1] introduced the notion of solution operator for (1.5) as follows.

Definition 1. (See [1, Definition 2.3].) A family $\{T_{\alpha}(t)\}_{t\geqslant 0}$ of bounded linear operators of X is called a solution operator for (FACP) if the following three conditions are satisfied:

(a) $T_{\alpha}(t)$ is strongly continuous for $t \geqslant 0$ and $T_{\alpha}(0) = I$ (the identity operator),
(b) $T_{\alpha}(t)D(A) \subset D(A)$ and $AT_{\alpha}(t)x = T_{\alpha}(t)Ax$ for all $x \in D(A)$ and $t \geqslant 0$,
(c) $u(t) = T_{\alpha}(t)x$ is a solution of (1.7) for every $x \in D(A)$.

Remark 1. The notion of solution operator for (FACP) is followed from [11, Definition 1.3], where the notion "resolvent" is defined for some resolvent equation more general than (1.7). Solution operator is also called α-times resolvent family in some references; see, for example, [3,6].

The solution operator for (FACP) has been systematically investigated in the work of E. Bazhlekova and her collaborators (cf. [1]), the obtained results generalize some facts of C_0-semigroups and cosine families. Recently, Chen and Li [3] found a novel characteristic of solution operator, and thereby developed a purely algebraic notion, named α-resolvent operator function.

Definition 2. (See [3, Definition 3.1].) A family $\{S_{\alpha}(t)\}_{t\geqslant 0}$ of bounded linear operators of X is called an α-resolvent operator function if the following conditions are satisfied:

(a) $S_{\alpha}(t)$ is strongly continuous for $t \geqslant 0$ and $S_{\alpha}(0) = I$,
(b) $S_{\alpha}(t)S_{\alpha}(s) = S_{\alpha}(s)S_{\alpha}(t)$ for all $t, s \geqslant 0$, and
(c) there holds for all $t, s \geqslant 0$,

$$S_{\alpha}(s)J_t^{\alpha}S_{\alpha}(t) - J_s^{\alpha}S_{\alpha}(s)S_{\alpha}(t) = J_t^{\alpha}S_{\alpha}(t) - J_s^{\alpha}S_{\alpha}(s), \tag{1.8}$$

where $J_t^{\alpha}\cdot$ is the integral operator defined by (1.3).

It has been proved in [3] that a family $\{S_{\alpha}(t)\}_{t\geqslant 0}$ is an α-resolvent operator function if and only if it is a solution operator for a certain fractional Cauchy problem. This verifies the conjecture, long existing in the literature, that fractional abstract Cauchy problems can be studied using purely algebraic methods, just like using the semigroup property $T(t+s) = T(t)T(s)$ to study first-order abstract Cauchy problems.

The present paper is motivated by the following property of Mittag–Leffler function we recently proved in [8]: for $0<\alpha<1$ and any real a, there holds

$$\int_0^{t+s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{t} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau = \alpha\int_0^t\int_0^s \frac{E_\alpha(ar_1^\alpha)E_\alpha(ar_2^\alpha)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2, \quad t,s\geqslant 0, \tag{1.9}$$

where $E_\alpha(z)$ is the one-parameter Mittag–Leffler function defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad z\in\mathbb{C}. \tag{1.10}$$

Notice that $E_\alpha(at^\alpha)$ solves the particular fractional evolution equation:

$${}_0^C D_t^\alpha u(t)=au(t), \quad t\geqslant 0, \tag{1.11}$$

it is reasonable to infer that an analogue of the equality (1.9) exists for the solution operator for (FACP). The main purpose of this paper is to confirm this idea, and then develop a notion to characterize the solution operator for (FACP). To this end, we restrict ourselves to the case $0<\alpha<1$, due to the obvious fact that the equality (1.9) is unavailable for $\alpha\geqslant 1$.

In the remainder of this section, we recall some properties of the Mittag–Leffler functions and the fractional derivative and integral operators ${}_0^C D_t^\alpha\cdot$ and $J_t^\alpha\cdot$, which will be used frequently in the subsequent sections.

(P1) Let $E_{\alpha,\beta}(z)$ be two-parameter Mittag–Leffler function defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z\in\mathbb{C}, \tag{1.12}$$

where $\alpha,\beta>0$. Then,

$$\frac{dE_\alpha(z)}{dz}=\alpha^{-1}E_{\alpha,\alpha}(z), \quad z\in\mathbb{C}. \tag{1.13}$$

(P2) The Laplace transform of the Mittag–Leffler functions can be derived from the formula

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}\big(at^\alpha\big)\,dt=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-a}, \quad \Re\lambda>a^{1/\alpha},\ a>0, \tag{1.14}$$

where $\Re\lambda$ represents the real part of the complex number λ.

(P3) The set of the integral operators $\{J_t^\alpha\}_{\alpha>0}$ is a semigroup, i.e., $J_t^\alpha J_t^\beta=J_t^{\alpha+\beta}$ for all $\alpha,\beta>0$.

(P4) The derivative operator ${}_0^C D_t^\alpha\cdot$ is a left inverse of the integral operator $J_t^\alpha\cdot$, that is, for $f\in C([0,\infty),X)$,

$${}_0^C D_t^\alpha\big(J_t^\alpha f(t)\big)=f(t), \tag{1.15}$$

but in general not a right inverse:

$$J_t^\alpha\big({}_0^C D_t^\alpha f(t)\big)=f(t)-f(0) \tag{1.16}$$

for those $f\in C([0,\infty),X)$ such that ${}_0^C D_t^\alpha f(t)$ exists.

(P5) The following Laplace transform formulas are available:

$$\widehat{{}_0^C D_t^\alpha f}(\lambda)=\lambda^\alpha\hat f(\lambda)-\lambda^{\alpha-1}f(0), \qquad \widehat{J_t^\alpha f}(\lambda)=\lambda^{-\alpha}\hat f(\lambda), \tag{1.17}$$

where $\hat f(\lambda)$ represents the Laplace transform of f.

2. An equality characteristic of solution operator

The purpose of this section is to prove that the solution operator for (FACP) satisfies an equality similar to (1.9), as conjectured. We first introduce the following notion.

Definition 3. Let $0<\alpha<1$. A one-parameter family $\{T_\alpha(t)\}_{t\geqslant 0}$ of bounded linear operators of X is called strongly continuous fractional semigroup of order α (or α-order fractional semigroup, for short) if it possesses the following two properties:

(i) for every $x \in X$, the mapping $t \mapsto T_\alpha(t)x$ is continuous over $[0, \infty)$;
(ii) $T_\alpha(0) = I$ (the identity operator), and for all $t, s \geqslant 0$,

$$\int_0^{t+s} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} - \int_0^{t} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} - \int_0^{s} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} = \alpha \int_0^t \int_0^s \frac{T_\alpha(\tau_1)T_\alpha(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}}\,d\tau_1\,d\tau_2, \tag{2.1}$$

where the integrals are defined in the strong operator topology.

Remark 2. It should be pointed out that the word "semigroup" is somewhat farfetched because the equality (2.1) does not explicitly exhibit the semigroup property. However, it can be shown that semigroup property is just the limit state of the equality (2.1) as $\alpha \to 1^-$. Indeed, if for each x in some dense set the mapping $t \mapsto T_\alpha(t)x$ is continuously differentiable in $[0, \infty)$, and if the limit $T(t)$ of $T_\alpha(t)$ exists in some sense as $\alpha \to 1^-$, then the limits of two sides of the equality (2.1) multiplied with $1-\alpha$ equal to $T(t+s)$ and $T(t)T(s)$ respectively, that is, the limit $T(t)$ bears the semigroup property: $T(t)T(s) = T(t+s)$, $t, s \geqslant 0$.

For the sake of convenience, we drop the subscript α from $\{T_\alpha(t)\}_{t\geqslant 0}$ throughout the paper.

Proposition 1. *If $\{T(t)\}_{t\geqslant 0}$ is an α-order fractional semigroup, then it is commutative, i.e., $T(t)T(s) = T(s)T(t)$ for all $t, s \geqslant 0$.*

Proof. The symmetry of the left side of (2.1) with respect to t and s yields

$$\int_0^t \int_0^s \frac{T(\tau_1)T(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}}\,d\tau_1\,d\tau_2 = \int_0^s \int_0^t \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2 \tag{2.2}$$

for all $t, s \geqslant 0$.

Given any $a > 0$. Denote by $f_a(t)$ the truncation of $T(t)$ at a, that is, $f_a(t) = T(t)$ for $0 \leqslant t \leqslant a$, and $f_a(t) = 0$ otherwise. Define $R_a : [0, \infty) \times [0, \infty) \to \mathfrak{B}(X)$ (the space of bounded linear operators of X) by

$$R_a(t,s) = \alpha \int_0^t \int_0^s \frac{f_a(\tau_1) f_a(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}}\,d\tau_1\,d\tau_2, \quad t, s \geqslant 0. \tag{2.3}$$

Obviously, $R_a(t,s) = R_a(s,t)$ for all $t, s \geqslant 0$, and for each $t \geqslant 0$, $R_a(t,s)$ is just the convolution of these two functions $\int_0^t (t+s-r)^{-\alpha-1} f_a(r)\,dr$ and $f_a(s)$ of s. Hence, by the convolution property of the Laplace transform, we have

$$\begin{aligned} \widehat{R_a}(t,\lambda) &:= \alpha \int_0^\infty e^{-\lambda s} R_a(t,s)\,ds = \alpha \int_0^\infty e^{-\lambda s} \int_0^t (t+s-r)^{-\alpha-1} f_a(r)\,dr\,ds\,\widehat{f_a}(\lambda) \\ &= \alpha \int_0^\infty e^{-\lambda s} (t+s)^{-\alpha-1} * f_a(t)\,ds\,\widehat{f_a}(\lambda), \end{aligned} \tag{2.4}$$

where $(t+s)^{-\alpha-1} * f_a(t)$ represents the convolution of these two functions $(t+s)^{-\alpha-1}$ and $f_a(t)$ of t, and $\widehat{f_a}(\lambda)$ is the Laplace transform of the function f_a. Further, taking Laplace transform with respect to t, we have

$$\begin{aligned} \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} R_a(t,s)\,ds\,dt &= \alpha \int_0^\infty e^{-\lambda s} \int_0^\infty e^{-\mu t} (t+s)^{-\alpha-1} * f_a(t)\,dt\,ds\,\widehat{f_a}(\lambda) \\ &= \alpha \int_0^\infty e^{-\lambda s} \int_0^\infty e^{-\mu t} (t+s)^{-\alpha-1}\,dt\,ds\,\widehat{f_a}(\mu)\widehat{f_a}(\lambda) \\ &= \alpha \int_0^\infty e^{(\mu-\lambda)s} \int_s^\infty e^{-\mu t} t^{-\alpha-1}\,dt\,ds\,\widehat{f_a}(\mu)\widehat{f_a}(\lambda) \\ &= \frac{\Gamma(1-\alpha)}{\lambda-\mu}\left(\lambda^\alpha - \mu^\alpha\right)\widehat{f_a}(\mu)\widehat{f_a}(\lambda). \end{aligned} \tag{2.5}$$

In the same way, it can be shown that

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_a(s,t)\,ds\,dt=\frac{\Gamma(1-\alpha)}{\lambda-\mu}\big(\lambda^\alpha-\mu^\alpha\big)\widehat{f_a}(\lambda)\widehat{f_a}(\mu). \tag{2.6}$$

Consequently, we get that $\widehat{f_a}(\mu)\widehat{f_a}(\lambda)=\widehat{f_a}(\lambda)\widehat{f_a}(\mu)$. By virtue of the Laplace transform, it follows that $f_a(t)f_a(s)=f_a(s)f_a(t)$ for all $t,s\geqslant 0$. Therefore, the arbitrariness of a implies that $T(s)T(t)=T(t)T(s)$ for $t,s\geqslant 0$, that is, $T(t)$ is commutative. □

Proposition 2. *If $\{T(t)\}_{t\geqslant 0}$ is a solution operator for (FACP), then it satisfies the equality (2.1) and is therefore an α-order fractional semigroup.*

Proof. Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of equality (2.1), respectively. Obviously, we need to prove $L(t,s)=R(t,s)$ for all $t,s\geqslant 0$. For brevity, we introduce the following notations:

$$H(t,s)=T(s)J_t^\alpha T(t)-J_s^\alpha T(s)T(t),\qquad K(t,s)=J_t^\alpha T(t)-J_s^\alpha T(s),\quad t,s\geqslant 0. \tag{2.7}$$

Moreover, for sufficiently large $a>0$, we denote by $f_a(t)$ the truncation of $T(t)$ at a, and by $R_a(t,s)$, $L_a(t,s)$, $H_a(t,s)$ and $K_a(t,s)$ we denote the quantities resulted from replacing $T(t)$ with $f_a(t)$ in $R(t,s)$, $L(t,s)$, $H(t,s)$ and $K(t,s)$, respectively.

On one hand, it follows from (2.6) that the Laplace transform of $R_a(t,s)$ with respect to t and s is given by

$$\widehat{R_a}(\mu,\lambda)=\frac{\Gamma(1-\alpha)}{\lambda-\mu}\big(\lambda^\alpha-\mu^\alpha\big)\widehat{f_a}(\mu)\widehat{f_a}(\lambda). \tag{2.8}$$

On the other hand, it can be shown that for all $t\geqslant 0$,

$$\begin{aligned}\widehat{L_a}(t,\lambda)&:=\int_0^\infty e^{-\lambda s}\left[\int_t^{t+s}\frac{f_a(\tau)}{(t+s-\tau)^\alpha}\,d\tau\,ds-\int_0^s\frac{f_a(\tau)}{(t+s-\tau)^\alpha}\,d\tau\right]ds\\&=\int_t^\infty f_a(\tau)\int_{\tau-t}^\infty\frac{e^{-\lambda s}}{(t+s-\tau)^\alpha}\,ds\,d\tau-\int_0^\infty f_a(\tau)\int_\tau^\infty\frac{e^{-\lambda s}}{(t+s-\tau)^\alpha}\,ds\,d\tau\\&=\int_t^\infty e^{\lambda(t-\tau)}f_a(\tau)\int_0^\infty e^{-\lambda r}r^{-\alpha}\,dr\,d\tau-e^{\lambda t}\int_t^\infty e^{-\lambda s}s^{-\alpha}\,ds\,\widehat{f_a}(\lambda)\\&=\lambda^{\alpha-1}\Gamma(1-\alpha)\int_t^\infty e^{\lambda(t-\tau)}f_a(\tau)\,d\tau-e^{\lambda t}\int_t^\infty e^{-\lambda s}s^{-\alpha}\,ds\,\widehat{f_a}(\lambda),\end{aligned}$$

with which we further can get the Laplace transform of $\widehat{L_a}(t,\lambda)$ with respect to t as follows:

$$\widehat{L_a}(\mu,\lambda)=\frac{\Gamma(1-\alpha)}{\lambda-\mu}\big(\lambda^{\alpha-1}\widehat{f_a}(\mu)-\mu^{\alpha-1}\widehat{f_a}(\lambda)\big). \tag{2.9}$$

By Laplace transform technique, we shall link $R(t,s)$ to $H(t,s)$, $L(t,s)$ to $K(t,s)$, respectively. For brevity, set

$$P_a(t,s)=\alpha\int_0^t\int_0^s\frac{H_a(r_1,r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2, \tag{2.10}$$

and

$$Q_a(t,s)=\alpha\int_0^t\int_0^s\frac{K_a(r_1,r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2. \tag{2.11}$$

According to [3, Theorem 3.4], $\{T(t)\}_{t\geqslant 0}$ is an α-resolvent operator function, and hence $H(t,s)=K(t,s)$ for all $t,s\geqslant 0$. It therefore follows that for all $t,s\geqslant 0$,

$$\lim_{a\to\infty}P_a(t,s)=\lim_{a\to\infty}Q_a(t,s). \tag{2.12}$$

Now, we calculate the Laplace transforms of $P_a(t,s)$ and $Q_a(t,s)$ with respect to t and s, respectively. Firstly, by the convolution property of the Laplace transform it can be shown that, for all $t\geqslant 0$,

$$\begin{aligned}
\widehat{P_a}(t,\lambda) &= \alpha \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{f_a(r_1) J_{r_2}^\alpha f_a(r_2) - J_{r_1}^\alpha f_a(r_1) f_a(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\, dr_1\, dr_2\, ds \\
&= \alpha \int_0^t \int_0^\infty e^{-\lambda s} \int_0^s \frac{f_a(r_1) J_{r_2}^\alpha f_a(r_2) - J_{r_1}^\alpha f_a(r_1) f_a(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\, dr_1\, dr_2\, ds \\
&= \alpha \int_0^t \widehat{f_a}(\lambda) \widehat{\Pi}(t-r_2,\lambda)\big[J_{r_2}^\alpha f_a(r_2) - \lambda^{-\alpha} f_a(r_2)\big]\, dr_2 \\
&= \alpha \widehat{\Pi}(t,\lambda) * \big[J_t^\alpha f_a(t) - \lambda^{-\alpha} f_a(t)\big] \widehat{f_a}(\lambda),
\end{aligned} \tag{2.13}$$

where $\widehat{\Pi}(t,\lambda)$ stands for the Laplace transform of the function $(t+s)^{-1-\alpha}$ of s. So, we have

$$\widehat{P_a}(\mu,\lambda) = \int_0^\infty e^{-\mu t} \widehat{P_a}(t,\lambda)\, dt = \alpha \widehat{\Pi}(\mu,\lambda)\big[\mu^{-\alpha} - \lambda^{-\alpha}\big] \widehat{f_a}(\mu) \widehat{f_a}(\lambda), \tag{2.14}$$

where $\widehat{\Pi}(\mu,\lambda)$ is the Laplace transform of $\widehat{\Pi}(t,\lambda)$. Secondly, in the same way it can be shown that

$$\widehat{Q_a}(\mu,\lambda) = \alpha \widehat{\Pi}(\mu,\lambda)\big[\lambda^{-1}\mu^{-\alpha} \widehat{f_a}(\mu) - \mu^{-1}\lambda^{-\alpha} \widehat{f_a}(\lambda)\big]. \tag{2.15}$$

It is a routine matter to show that for all $\mu, \lambda > 0$,

$$\widehat{\Pi}(\mu,\lambda) = \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} (t+s)^{-1-\alpha}\, ds\, dt = \frac{\Gamma(1-\alpha)}{\alpha(\lambda-\mu)}\big(\lambda^\alpha - \mu^\alpha\big). \tag{2.16}$$

Therefore, comparing (2.8) with (2.14), and meanwhile comparing (2.9) with (2.15), we get

$$\widehat{P_a}(\mu,\lambda) = \big(\mu^{-\alpha} - \lambda^{-\alpha}\big)\widehat{R_a}(\mu,\lambda), \qquad \widehat{Q_a}(\mu,\lambda) = \big(\mu^{-\alpha} - \lambda^{-\alpha}\big)\widehat{L_a}(\mu,\lambda). \tag{2.17}$$

By virtue of the Laplace transform, it follows that for all $t, s \geqslant 0$,

$$P_a(t,s) = \big(J_t^\alpha - J_s^\alpha\big) R_a(t,s), \qquad Q_a(t,s) = \big(J_t^\alpha - J_s^\alpha\big) L_a(t,s). \tag{2.18}$$

So by (2.12) we have that

$$\big(J_t^\alpha - J_s^\alpha\big) R(t,s) = \big(J_t^\alpha - J_s^\alpha\big) L(t,s), \quad t, s \geqslant 0. \tag{2.19}$$

It follows from (2.19) that $R(t,s) = L(t,s)$ for all $t, s \geqslant 0$. The proof is therefore completed. □

3. Essentiality of equality characteristic for solution operator

It has been proved in Section 2 that the solution operator for (FACP) satisfies the equality (2.1). In this section we further prove that the equality (2.1) is also sufficient for a family of bounded linear operators to become a solution operator for (FACP), that is, a fractional semigroup is also a solution operator. To this end, it is necessary to find a densely defined linear operator A that achieves its values at those $x \in X$ such that the fractional derivative ${}_0^C D_t^\alpha T(t)x$ exists at $t = 0$. Therefore, we introduce the following notion.

Definition 4. Let $\{T(t)\}_{t \geqslant 0}$ be an α-order fractional semigroup of X. Denote by $D(A)$ the set of all $x \in X$ such that the limit

$$\lim_{t \to 0^+} t^{-1} J_t^{1-\alpha}\big(T(t)x - x\big) \tag{3.1}$$

exists. Then, the operator $A : D(A) \to X$ defined by

$$Ax = \lim_{t \to 0^+} t^{-1} J_t^{1-\alpha}\big(T(t)x - x\big) \tag{3.2}$$

is called the generator of $\{T(t)\}_{t \geqslant 0}$.

Obviously, the generator is a linear operator. In the following we prove that an α-order fractional semigroup is just a solution operator for the fractional abstract Cauchy problem governed by its generator.

Proposition 3. *Let $\{T(t)\}_{t\geqslant 0}$ be an α-order fractional semigroup on X and let A be its generator. Let $x \in X$ and define*

$$x_t = J_t^\alpha T(t)x, \quad t \geqslant 0. \tag{3.3}$$

Then, we have that $x_t \in D(A)$ and $T(t)x = Ax_t + x$ for all $t \geqslant 0$.

Proof. Let $x \in X$ and $t > 0$ be fixed. Define the function $H_t(r,s)$ for $r, s \geqslant 0$ by

$$H_t(r,s) = \big(f_t(r) - I\big) J_s^\alpha f_t(s) \tag{3.4}$$

where $f_t(r)$ is the truncation of $T(r)$ at t. Obviously, for sufficiently small $r > 0$,

$$H_t(r,t) = \big(T(r) - I\big) x_t. \tag{3.5}$$

Taking Laplace transform of $H_t(r,s)$ with respect to r and s successively, we can obtain

$$\widehat{H_t}(\mu,\lambda) = \frac{1}{\lambda^\alpha}\widehat{f_t}(\mu)\widehat{f_t}(\lambda)x - \mu^{-1}\frac{1}{\lambda^\alpha}\widehat{f_t}(\lambda)x. \tag{3.6}$$

By (2.9) and (2.8), the above equality can be written as

$$\widehat{H_t}(\mu,\lambda) = \frac{1}{\mu^\alpha}\widehat{f_t}(\mu)\widehat{f_t}(\lambda)x - \frac{1}{\lambda\mu^\alpha}\widehat{f_t}(\mu)x - \frac{\lambda-\mu}{\Gamma(1-\alpha)\lambda^\alpha\mu^\alpha}\big[\widehat{L_t}(\mu,\lambda) - \widehat{R_t}(\mu,\lambda)\big]x.$$

Hence, it follows from (1.15)–(1.17) that for all $r, s \geqslant 0$,

$$H_t(r,s) = \big(f_t(s) - I\big) J_r^\alpha f_t(r)x - \frac{(J_r^\alpha({}_0^C D_s^{1-\alpha}) - ({}_0^C D_r^{1-\alpha})J_s^\alpha)(L_t(r,s) - R_t(r,s))x}{\Gamma(1-\alpha)}. \tag{3.7}$$

Since $f_t(r)$ is the truncation of $T(r)$ at t, then $L_t(r,s) = R_t(r,s)$ for all $r, s \leqslant t$. From (3.7), we have

$$H_t(r,s) = \big(T(s) - I\big) J_r^\alpha T(r)x, \quad \forall r, s \leqslant t, \tag{3.8}$$

particularly, $H_t(r,t) = (T(t) - I)J_r^\alpha T(r)x$ for sufficiently small r. Therefore, by (3.5) we have

$$\lim_{r\to 0^+} r^{-1} J_r^{1-\alpha}\big(T(r) - I\big)x_t = \lim_{r\to 0^+} r^{-1} J_r^{1-\alpha} H_t(r,t) = \lim_{r\to 0^+} r^{-1}\big(T(t) - I\big) J_r^1 T(r)x = T(t)x - x, \tag{3.9}$$

which implies $x_t \in D(A)$ and $Ax_t = T(t)x - x$. The proof is completed. □

Proposition 4. *Let $\{T(t)\}_{t\geqslant 0}$ be an α-order fractional semigroup on X and let A be its generator. Then,*

(a) *$T(t)(D(A)) \subset D(A)$, and $T(t)Ax = AT(t)x$ for $x \in D(A)$;*
(b) *for all $x \in D(A)$,*

$$T(t)x = x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} T(\tau)Ax\, d\tau; \tag{3.10}$$

(c) *A is equivalently defined by*

$$Ax = \Gamma(1+\alpha)\lim_{t\to 0^+}\frac{T(t)x - x}{t^\alpha} \tag{3.11}$$

and $D(A)$ consists of those $x \in X$ for which this limit exists;
(d) *A is closed and densely defined.*

Proof. (a) and (b) are immediate from Propositions 1 and 3. We will prove (c) and (d).

(c) Denote by D the set of those $x \in X$ such that the limit $\lim_{t\to 0^+} t^{-\alpha}(T(t)x - x)$ exists. Let $x \in D(A)$. Then, by (b) we have

$$\frac{T(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} T(t\tau)Ax\, d\tau. \tag{3.12}$$

Obviously, the limit of the left side as $t \to 0^+$ exists and

$$\lim_{t\to 0^+}\frac{T(t)x - x}{t^\alpha} = \frac{1}{\alpha\Gamma(\alpha)}Ax = \frac{Ax}{\Gamma(1+\alpha)}, \tag{3.13}$$

which implies that $x \in D$ and then $D(A) \subset D$ by the arbitrariness of $x \in D(A)$. Therefore it remains to show that $D \subset D(A)$. Let $x \in D$, since the limit $\lim_{t\to 0^+} t^{-\alpha}(T(t)x - x)$ exists, we have

$$\lim_{t\to 0^+} \frac{1}{t} J_t^{1-\alpha}\big(T(t)x - x\big) = \lim_{t\to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{-\alpha} \tau^{\alpha} \frac{T(t\tau)x - x}{(t\tau)^{\alpha}}\, d\tau = \frac{B(1-\alpha, 1+\alpha)}{\Gamma(\alpha)} \lim_{t\to 0^+} t^{-\alpha}\big(T(t)x - x\big)$$
$$= \Gamma(1+\alpha) \lim_{t\to 0^+} t^{-\alpha}\big(T(t)x - x\big) \tag{3.14}$$

where $B(a,b)$ is the Beta function. Hence, $x \in D(A)$ and

$$Ax = \Gamma(1+\alpha) \lim_{t\to 0^+} t^{-\alpha}\big(T(t)x - x\big).$$

(d) Assume that $D(A) \ni x_n \to x$ and $Ax_n \to y$. Then, by the equality (3.10) it can be shown that, for all $t > 0$,

$$T(t)x - x = \lim_{n\to\infty} T(t)x_n - x_n = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T(\tau) A x_n\, d\tau = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T(\tau) y\, d\tau$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha} \int_0^1 (1-\tau)^{\alpha-1} T(t\tau) y\, d\tau. \tag{3.15}$$

Obviously, the limit $\lim_{t\to 0^+} t^{-\alpha}(T(t)x - x)$ exists and equals to $\frac{1}{\Gamma(1+\alpha)} y$. Hence, $x \in D(A)$ and $y = Ax$ by (c). This implies that A is closed. Since for every $x \in X$, $x_t = \int_0^t (t-\tau)^{\alpha-1} T(\tau)x\, d\tau \in D(A)$ and $\alpha t^{-\alpha} x_t \to x$ as $t \to 0^+$, then A is densely defined. □

It follows from the above proposition that a fractional semigroup is necessarily associated with the solution operator for a fractional abstract Cauchy problem governed by its generator. This together with Proposition 2 imply that the equality (2.1) is essential for a family of bounded linear operators to become a solution operator for a certain fractional abstract Cauchy problem.

Proposition 5. *Let $\{T(t)\}_{t\geqslant 0}$ is an α-order fractional semigroup of X and let A be its generator. If $\{T(t)\}_{t\geqslant 0}$ is exponentially bounded, i.e., $\|T(t)\| \leqslant Me^{wt}$, $t \geqslant 0$ for some $M, w > 0$, then for all $\lambda \in \mathbb{C}$ with $\Re\lambda > w$, λ^{α} belongs to the resolvent set $\rho(A)$ of A, and the corresponding resolvent operator is given by*

$$R\big(\lambda^{\alpha}, A\big)x = \lambda^{1-\alpha} \int_0^{\infty} e^{-\lambda t} T(t)x\, dt, \quad x \in X, \tag{3.16}$$

where $\Re\lambda$ stands for the real part of λ.

Proof. It follows from Proposition 3 that for every $x \in X$,

$$T(t)x = x + A J_t^{\alpha} T(t)x, \quad t \geqslant 0. \tag{3.17}$$

Let $x \in X$. Since $T(t)x$ is exponentially bounded, its Laplace transform, denoted by $\widehat{T}(\lambda, x)$, exists, and

$$\widehat{T}(\lambda, x) = \lambda^{-1}x + \lambda^{-\alpha} A \widehat{T}(\lambda, x), \quad \Re\lambda > w. \tag{3.18}$$

This implies that $\widehat{T}(\lambda, x) \in D(A)$ and $\lambda^{\alpha} I - A$ is surjective whenever $\Re\lambda > w$. Moreover, it is easy to show that $\lambda^{\alpha} I - A$ is injective whenever $\Re\lambda > w$. Indeed, if $\mu^{\alpha} x = Ax$ for some $x \in D(A)$ and for some $\mu \in \mathbb{C}$ satisfying $\Re\mu > w$, then from (b) of Proposition 4 it follows that

$$T(t)x = x + \frac{\mu^{\alpha}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T(\tau)x\, d\tau, \quad t \geqslant 0. \tag{3.19}$$

Taking Laplace transform to both sides of (3.19), we obtain

$$\widehat{T}(\lambda, x) = \lambda^{-1}x + \mu^{\alpha}\lambda^{-\alpha} \widehat{T}(\mu, x), \quad \Re\lambda > w. \tag{3.20}$$

Letting $\lambda = \mu$, we have $x = 0$. Therefore the proof is completed. □

The following proposition is immediate from Proposition 5.

Proposition 6. *An exponentially bounded α-order fractional semigroup is uniquely determined by its generator. That is, if exponentially bounded α-order fractional semigroups $\{T(t)\}_{t\geqslant 0}$ and $\{S(t)\}_{t\geqslant 0}$ share the same generator A, then they are identical, i.e., $T(t)=S(t)$ for all $t\geqslant 0$.*

4. Well-posedness of fractional abstract Cauchy problem

Based on the propositions established in Section 3, we shall investigate the well-posedness of (FACP) in this section. In addition to the discussion on uniqueness of solution to (FACP), it is concluded that a fractional abstract Cauchy problem is well-posed if and only if its coefficient operator A generates a fractional semigroup. To this end, we first introduce the following notions; see, e.g. [1].

Definition 5. A function $u\in C([0,\infty),X)$ is called a strong solution of (1.5) if $u(t)\in D(A)$ for all $t\geqslant 0$, and the mapping

$$t\mapsto \int_0^t (t-r)^{-\alpha}\big(u(r)-u(0)\big)\,dr \tag{4.1}$$

is continuously differentiable such that (1.5) holds on $[0,\infty)$.

Definition 6. The problem (FACP) is said to be well-posed if for any $x\in D(A)$ there exists a unique strong solution $u(t,x)$, and $D(A)\ni x_n\to 0$ implies that $u(t,x_n)\to 0$ as $n\to\infty$ in X, uniformly on any compact subinterval of $[0,\infty)$.

Lemma 1. *Let $\lambda\in\mathbb{C}$ and $f(t)$ be a continuous function on $[0,\infty)$. If the continuous function $v(t)$ solves the inhomogeneous linear equation*:

$${}_0^C D_t^\alpha v(t)=\lambda v(t)+f(t),\quad t\geqslant 0, \tag{4.2}$$

then it is given by

$$v(t)=E_\alpha\big(\lambda t^\alpha\big)v(0)+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\tau)^\alpha\big)f(\tau)\,d\tau,\quad t\geqslant 0. \tag{4.3}$$

Proof. For the sake of convenience, let $g(t)=t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$. Then, by the property (1.13) of the Mittag–Leffler function,

$$\frac{d}{dt}\big(E_\alpha\big(\lambda t^\alpha\big)-1\big)=\lambda t^{\alpha-1}E_{\alpha,\alpha}\big(\lambda t^\alpha\big)=\lambda g(t),\quad t\geqslant 0, \tag{4.4}$$

which will be used frequently in the subsequent proof. Since $v(t)$ solves (4.2), we have

$$g(t)*\big({}_0^C D_t^\alpha v(t)\big)=\lambda g(t)*v(t)+g(t)*f(t),\quad t\geqslant 0, \tag{4.5}$$

where $*$ represents the convolution operation. On one hand, it follows from (4.4) that

$$\begin{aligned}\lambda g(t)*v(t)&=\lambda\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\tau)^\alpha\big)v(\tau)\,d\tau=\int_0^t\frac{d}{dt}\big(E_\alpha\big(\lambda(t-\tau)^\alpha\big)-1\big)v(\tau)\,d\tau\\&=\frac{d}{dt}\int_0^t\big(E_\alpha\big(\lambda(t-\tau)^\alpha\big)-1\big)v(\tau)\,d\tau=\frac{d}{dt}\int_0^t\big(E_\alpha\big(\lambda(t-\tau)^\alpha\big)\big)v(\tau)\,d\tau-v(t).\end{aligned} \tag{4.6}$$

On the other hand, using the integration by parts, we can get

$$\begin{aligned}g(t)*\big({}_0^C D_t^\alpha v(t)\big)&=\int_0^t g(t-\tau)\big({}_0^C D_\alpha^\alpha v(\tau)\big)\,d\tau=\frac{d}{\lambda\,dt}\int_0^t\big(E_\alpha\big(\lambda(t-\tau)^\alpha\big)-1\big){}_0^C D_\alpha^\alpha v(\tau)\,d\tau\\&=\frac{d}{\lambda\,dt}\int_0^t\frac{d}{d\tau}\big(E_\alpha\big(\lambda(t-\tau)^\alpha\big)\big)\int_0^\tau\frac{(\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\big(v(r)-v(0)\big)\,dr\,d\tau\\&=\frac{d}{\lambda\,dt}\int_0^t\int_r^t\frac{(\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{d\tau}E_\alpha\big(\lambda(t-\tau)^\alpha\big)\big(v(r)-v(0)\big)\,d\tau\,dr\end{aligned}$$

$$= \frac{d}{\lambda\, dt}\int_0^t\int_0^{t-r}\frac{(t-r-p)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{dp}E_\alpha\big(\lambda p^\alpha\big)\big(v(r)-v(0)\big)\,dp\,dr$$

$$= \frac{d}{\lambda dt}\int_0^t {}_0^C D_{t-r}^\alpha E_\alpha\big(\lambda(t-r)^\alpha\big)\big(v(r)-v(0)\big)\,dr = \frac{d}{dt}\int_0^t E_\alpha\big(\lambda(t-r)^\alpha\big)\big(v(r)-v(0)\big)\,dr$$

$$= \frac{d}{dt}\int_0^t E_\alpha\big(\lambda(t-r)^\alpha\big)v(r)\,dr - E_\alpha\big(\lambda t^\alpha\big)v(0). \tag{4.7}$$

Putting the equalities (4.6) and (4.7) into Eq. (4.5), we have

$$v(t) = E_\alpha\big(\lambda t^\alpha\big)v(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\tau)^\alpha\big)f(\tau)\,d\tau, \quad t \geqslant 0. \tag{4.8}$$

The proof is therefore completed. □

Proposition 7. *Assume that A is densely defined. If the resolvent set $\rho(A)$ of A is nonempty and $(\lambda_0^\alpha, \infty) \subset \rho(A)$ for some sufficiently large λ_0 and*

$$\limsup_{\lambda\to+\infty}\lambda^{-1}\log\big(\lambda^{\alpha-1}\big\|R\big(\lambda^\alpha, A\big)\big\|\big) = 0, \tag{4.9}$$

then (FACP) has at most one solution for each $x \in D(A)$.

Proof. It suffices to prove that $u = 0$ is the unique solution to (FACP) with $x = 0$. We suppose that $u(t)$ is a solution with $x = 0$. It is easy to show that

$${}_0^C D_t^\alpha\big(R\big(\lambda^\alpha, A\big)u(t)\big) = \lambda^\alpha R\big(\lambda^\alpha, A\big)u(t) - u(t), \quad t \geqslant 0,\ \lambda > \lambda_0, \tag{4.10}$$

that is, for all $\lambda > \lambda_0$, $v_\lambda(t) = R(\lambda^\alpha, A)u(t)$ solves the equation ${}_0^C D_t^\alpha v(t) = \lambda^\alpha v(t) - u(t)$. Hence, it follows from Lemma 1 that

$$R\big(\lambda^\alpha, A\big)u(t) = -\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda^\alpha(t-\tau)^\alpha\big)u(\tau)\,d\tau, \quad t \geqslant 0,\ \lambda > \lambda_0. \tag{4.11}$$

Apply (4.9) to (4.11) to see that for any fixed $\delta > 0$,

$$M(t) := \sup_{\lambda>\lambda_0} e^{-\delta\lambda}\left\|\int_0^t (\lambda\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda^\alpha\tau^\alpha\big)u(t-\tau)\,d\tau\right\| < \infty, \quad \forall t \geqslant 0. \tag{4.12}$$

According to [9, Theorem 1.3], the function $E_{\alpha,\alpha}(z)$ can be written as

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha}z^{(1-\alpha)/\alpha}\exp\big(z^{1/\alpha}\big) + R(z), \tag{4.13}$$

where $R(z) = O(z^{-2})$ as $z \to +\infty$, which implies that

$$\big\|R(z)\big\| \leqslant M_1\big|z^{-2}\big|, \quad \forall z > Z_1, \tag{4.14}$$

for some constant M_1 and some sufficiently large Z_1. So, for any $t > 0$ and sufficiently large λ,

$$\left\|\int_0^t (\lambda\tau)^{\alpha-1}R\big((\lambda\tau)^\alpha\big)u(t-\tau)\,d\tau\right\|$$

$$= \frac{1}{\lambda}\left\|\int_0^{\lambda t} r^{\alpha-1}R\big(r^\alpha\big)u(t-r/\lambda)\,dr\right\| \leqslant \frac{1}{\lambda}\left\|\int_0^{Z_1} r^{\alpha-1}R\big(r^\alpha\big)u(t-r/\lambda)\,dr\right\| + \frac{1}{\lambda}\left\|\int_{Z_1}^{\lambda t} r^{\alpha-1}R\big(r^\alpha\big)u(t-r/\lambda)\,dr\right\|$$

$$\leqslant \frac{1}{\lambda}\left\|\int_0^{Z_1} r^{\alpha-1}R\big(r^\alpha\big)u(t-r/\lambda)\,dr\right\| + \frac{1}{\lambda\alpha}M_1\big(Z_1^{-\alpha} - (\lambda t)^{-\alpha}\big)\sup_{0\leqslant r\leqslant t}\big\|u(r)\big\|. \tag{4.15}$$

Thus, by inequality (4.12) and the formula (4.13), it can be shown that for any given $\delta > 0$ and for all $t \geqslant 0$,

$$\limsup_{\lambda \to +\infty} \left\| \int_{-\delta}^{t-\delta} e^{\lambda \tau} u(t - \tau - \delta)\, d\tau \right\| < +\infty. \tag{4.16}$$

By [10, Lemma 1.1], it follows that $u(t) = 0$ for all $t \geqslant 0$. □

Similarly to [11, Proposition 1.1], we can derive the following proposition from Propositions 4 and 5.

Proposition 8. *The fractional abstract Cauchy problem* (*FACP*) *is well-posed if and only if A generates an α-order fractional semigroup $\{T(t)\}_{t \geqslant 0}$ and in this case $u(t) = T(t)x$ is the unique solution for every $x \in D(A)$.*

Remark 3. It should be noted that the equality (2.1) is unavailable for $\alpha \geqslant 1$, so we restricted ourselves to the case $0 < \alpha < 1$ in this paper. In the recent paper [5], we introduced the notion of fractional solution operator, and considered the well-posedness of a certain fractional abstract Cauchy problem of order $1 < \alpha < 2$ by the fractional solution operator method.

In the rest of the paper, we would like to point out that fractional linear systems have received increasing interest in the last few decades due to their extensive applications in physics, chemistry, materials and engineering (see, e.g. [2,4,7,12] and references therein). However, many aspects of fractional linear systems theory remain to be further clarified. In this paper we develop the notion of fractional semigroup to characterize solution operator for fractional abstract linear systems. We expect to see more work to be presented.

References

[1] E. Bazhlekova, Fractional Evolution Equations in Banach Spaces, University Press Facilities, Eindhoven University of Technology, 2001.
[2] M.M. El-Borai, K. El-Said El-Nadia, E.G. El-Akabawy, On some fractional evolution equations, Comput. Math. Appl. 59 (2010) 1352–1355.
[3] C. Chen, M. Li, On fractional resolvent operator functions, Semigroup Forum 80 (2010) 121–142.
[4] A.A. Kilbas, T. Pierantozzi, J.J. Trujillo, L. Vazquez, On the solution of fractional evolution equations, J. Phys. A 37 (2004) 3271–3283.
[5] L. Kexue, P. Jigen, Fractional abstract Cauchy problems, Integral Equations Operator Theory 70 (3) (2011) 333–361.
[6] F.-B. Li, M. Li, Q. Zheng, Fractional evolution equations governed by coercive differential operators, Abstr. Appl. Anal. (2009), Article ID 438690, 14 pp.
[7] Z.M. Odibat, Analytic study on linear systems of fractional differential equations, Comput. Math. Appl. 59 (2010) 1171–1183.
[8] J. Peng, K. Li, A note on the Mittag–Leffler function, J. Math. Anal. Appl. 370 (2010) 635–638.
[9] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[11] J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser, Basel, Berlin, 1993.
[12] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer-Verlag, The Netherlands, 2007.