Applied Mathematics Letters 24 (2011) 2019–2023

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Laplace transform and fractional differential equations

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ARTICLE INFO

Article history:
Received 13 May 2010
Received in revised form 25 May 2011
Accepted 31 May 2011

Keywords:
Laplace transform
Riemann–Liouville fractional derivative
Caputo fractional derivative
Fractional differential equations

ABSTRACT

In this paper, we give a sufficient condition to guarantee the rationality of solving constant coefficient fractional differential equations by the Laplace transform method.

1. Introduction

This paper deals with the rationality of Laplace transform for solving the following fractional differential equation

$$\begin{aligned} &{}_0^C D_t^\alpha x(t) = Ax(t) + f(t), \quad 0 < \alpha < 1,\ t \geq 0, \\ &x(0) = \eta, \end{aligned} \tag{1}$$

where ${}_0^C D_t^\alpha \cdot$ is the Caputo fractional derivative operator, A is a $n \times n$ constant matrix, $f(t)$ is a n-dimensional continuous vector-valued function, which is usually called the forcing term.

Fractional derivatives describe the property of memory and heredity of many materials, and it is the major advantage over integer-order derivatives. Fractional differential equations are differential equations of arbitrary (noninteger) order. Fractional differential equations have attracted considerable attention recently, due to its extensive applications in engineering and science; see the books of Podlubny [1], Miller and Boss [2], Oldham and Spanier [3] and the papers by Friedrich [4], Chen and Moore [5], Ahmad and Sivasundaram [6], Odibat [7].

An effective and convenient method for solving fractional differential equations is needed. Methods in [2,3] for rational order fractional differential equations is not applicable to the case of arbitrary order. Some authors used the series method [3,4], which allows solution of arbitrary order fractional differential equations, but it works only for relatively simple equations. Podlubny [1] introduced a method based on the Laplace transform technique, it is suitable for a large class of initial value problems for fractional differential equations.

However, we find that the existence of Laplace transform is taken for granted in some papers to solve fractional differential equations (see, e.g., [8,9]). In fact, not every function has its Laplace transform, for example, $f(t) = 1/t^2$, $f(t) = e^{t^2}$, do not have the Laplace transform. In this paper, to guarantee the rationality of solving fractional differential equations by the Laplace transform method, we give a sufficient condition, i.e., Theorem 3.1.

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.05.035

The paper has been organized as follows. In Section 2, basic definitions and lemmas related to fractional derivatives and fractional integrals have been summarized. In Section 3, we give the main results.

2. Preliminaries

In this section, we give some definitions and lemmas which are used further in this paper.

Definition 2.1. The Laplace transform is defined by

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt, \tag{2}$$

where $f(t)$ is n-dimensional vector-valued function.

Definition 2.2 ([1]). The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}, \tag{3}$$

where $\alpha > 0,\ \beta > 0$, $\mathbb{C}$ denotes the complex plane.

Definition 2.3 ([10]). The Riemann–Liouville fractional integral of order $0 < \alpha < 1$ of a function $h : (0, \infty) \to R^n$ is defined by

$$_0I_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds. \tag{4}$$

Definition 2.4 ([10]). The Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ of a function $h : (0, \infty) \to R^n$ is defined by

$$_0D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds. \tag{5}$$

Definition 2.5 ([10]). The Caputo fractional derivative of order $0 < \alpha < 1$ of a function $h : (0, \infty) \to R^n$ is defined by

$$^C_0D_t^\alpha h(t) = {}_0D_t^\alpha [h(t) - h(0)]. \tag{6}$$

Definition 2.6. A function f on $0 \le t < \infty$ is said to be exponentially bounded if it satisfies an inequality of the form

$$\|f(t)\| \le Me^{ct} \tag{7}$$

for some real constants $M > 0$ and c, for all sufficiently large t.

Lemma 2.1. *Let C be complex plane, for any $\alpha > 0$, $\beta > 0$ and $A \in C^{n\times n}$,*

$$L\{t^{\beta-1}E_{\alpha,\beta}(At^\alpha)\} = s^{\alpha-\beta}(s^\alpha - A)^{-1} \tag{8}$$

holds for $\Re s > \|A\|^{1/\alpha}$, where $\Re s$ represents the real part of the complex number s.

Proof. For $\Re s > \|A\|^{1/\alpha}$, we have $\sum_{k=0}^\infty A^k s^{-(k+1)\alpha} = (s^\alpha - A)^{-1}$. Then

$$\begin{aligned} L[t^{\beta-1}E_{\alpha,\beta}(At^\alpha)] &= L\left[t^{\beta-1}\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha k+\beta)}\right] \\ &= \sum_{k=0}^\infty \frac{A^k L[t^{\alpha k+\beta-1}]}{\Gamma(\alpha k+\beta)} \\ &= s^{\alpha-\beta}\sum_{k=0}^\infty A^k s^{-(k+1)\alpha} \\ &= s^{\alpha-\beta}(s^\alpha - A)^{-1}. \quad \square \end{aligned}$$

Lemma 2.2 *([11]). Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s \tag{9}$$

on this interval. Then

$$u(t) \leq a(t) + \theta\int_0^t E'_\beta(\theta(t-s))a(s)\mathrm{d}s, \quad 0 \leq t < T, \tag{10}$$

where

$$\theta = (b\Gamma(\beta))^{1/\beta}, \qquad E_\beta(z) = \sum_{n=0}^{\infty} z^{n\beta}/\Gamma(n\beta+1), \qquad E'_\beta(z) = \frac{\mathrm{d}}{\mathrm{d}z}E_\beta(z),$$

$$E'_\beta(z) \simeq z^{\beta-1}/\Gamma(\beta) \quad \text{as } z \to 0+, \qquad E'_\beta(z) \simeq \frac{1}{\beta}\mathrm{e}^z \quad \text{as } z \to +\infty,$$

(and $E_\beta(z) \simeq \frac{1}{\beta}\mathrm{e}^z$ as $z \to +\infty$). If $a(t) \equiv a$, constant, then $u(t) \leq aE_\beta(\theta t)$.

3. Main results

In this section, we discuss the reliability of the Laplace transform method for solving (1).

Theorem 3.1. *Assume (1) has a unique continuous solution $x(t)$, if $f(t)$ is continuous on $[0,\infty)$ and exponentially bounded, then $x(t)$ and its Caputo derivative ${}_0^C D_t^\alpha x(t)$ are both exponentially bounded, thus their Laplace transforms exist.*

Proof. Since $f(t)$ is exponentially bounded, there exist positive constants M, σ and enough large T such that $\|f(t)\| \leq M\mathrm{e}^{\sigma t}$ for all $t \geq T$.

It is easy to see that Eq. (1) is equivalent to the following integral equation

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[Ax(\tau)+f(\tau)]\mathrm{d}\tau, \quad 0 \leq t < \infty. \tag{11}$$

For $t \geq T$, (11) can be rewritten as

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}[Ax(\tau)+f(\tau)]\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}[Ax(\tau)+f(\tau)]\mathrm{d}\tau$$

In view of assumptions of Theorem 3.1, the solution $x(t)$ ($x(0) = \eta$) is unique and continuous on $[0,\infty)$, then $Ax(t)+f(t)$ is bounded on $[0,T]$, i.e., there exists a constant $K > 0$ such that $\|Ax(t)+f(t)\| \leq K$. We have

$$\|x(t)\| \leq \|\eta\| + \frac{K}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|A\|\,\|x(\tau)\|\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|f(\tau)\|\mathrm{d}\tau$$

Multiply this inequality by $\mathrm{e}^{-\sigma t}$ and note that $\mathrm{e}^{-\sigma t} \leq \mathrm{e}^{-\sigma T}$, $\mathrm{e}^{-\sigma t} \leq \mathrm{e}^{-\sigma\tau}$, $\|f(t)\| \leq M\mathrm{e}^{\sigma t}$ $(t \geq T)$ to obtain

$$\begin{aligned}
\|x(t)\|\mathrm{e}^{-\sigma t} &\leq \|\eta\|\mathrm{e}^{-\sigma t} + \frac{K\mathrm{e}^{-\sigma t}}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}\mathrm{d}\tau + \frac{\mathrm{e}^{-\sigma t}}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|A\|\,\|x(\tau)\|\mathrm{d}\tau \\
&\quad + \frac{\mathrm{e}^{-\sigma t}}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|f(\tau)\|\mathrm{d}\tau \\
&\leq \|\eta\|\mathrm{e}^{-\sigma T} + \frac{K\mathrm{e}^{-\sigma T}}{\alpha\Gamma(\alpha)}(t^\alpha - (t-T)^\alpha) + \frac{\|A\|}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|x(\tau)\|\mathrm{e}^{-\sigma\tau}\mathrm{d}\tau \\
&\quad + \frac{\mathrm{e}^{-\sigma t}}{\Gamma(\alpha)}\int_T^t (t-\tau)^{\alpha-1}\|f(\tau)\|\mathrm{d}\tau \\
&\leq \|\eta\|\mathrm{e}^{-\sigma T} + \frac{K\mathrm{e}^{-\sigma T}}{\alpha\Gamma(\alpha)}(t^\alpha - (t-T)^\alpha) + \frac{\|A\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|x(\tau)\|\mathrm{e}^{-\sigma\tau}\mathrm{d}\tau \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\mathrm{e}^{\sigma(\tau-t)}\mathrm{d}\tau \\
&\leq \|\eta\|\mathrm{e}^{-\sigma T} + \frac{KT^\alpha\mathrm{e}^{-\sigma T}}{\alpha\Gamma(\alpha)} + \frac{\|A\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|x(\tau)\|\mathrm{e}^{-\sigma\tau}\mathrm{d}\tau + \frac{M}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}\mathrm{e}^{-\sigma s}\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&\le \|\eta\|e^{-\sigma T}+\frac{KT^{\alpha}e^{-\sigma T}}{\alpha\Gamma(\alpha)}+\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau+\frac{M}{\Gamma(\alpha)}\int_0^{\infty}s^{\alpha-1}e^{-\sigma s}ds\\&\le \|\eta\|e^{-\sigma T}+\frac{KT^{\alpha}e^{-\sigma T}}{\alpha\Gamma(\alpha)}+\frac{M}{\sigma^{\alpha}}+\frac{\|A\|}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau,\quad t\ge T.\end{aligned}$$

Denote

$$a=\|\eta\|e^{-\sigma T}+\frac{KT^{\alpha}e^{-\sigma T}}{\alpha\Gamma(\alpha)}+\frac{M}{\sigma^{\alpha}},\qquad b=\frac{\|A\|}{\Gamma(\alpha)},\qquad r(t)=\|x(t)\|e^{-\sigma t},$$

we get

$$r(t)\le a+b\int_0^t(t-\tau)^{\alpha-1}r(\tau)d\tau,\quad t\ge T. \tag{12}$$

By Lemma 2.2,

$$r(t)\le aE_{\alpha}(\theta t)=a\sum_{n=0}^{\infty}\frac{(b\Gamma(\alpha))^n t^{n\alpha}}{\Gamma(n\alpha+1)},\quad t\ge T.$$

Recall that the Mittag-Leffler function is defined as

$$F_{\alpha}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)},\quad \alpha>0,$$

then

$$r(t)\le aF_{\alpha}(b\Gamma(\alpha)t^{\alpha}),\quad t\ge T. \tag{13}$$

It is known that (see [10], p. 21) the Mittag-Leffler type function $F_{\alpha}(\omega t^{\alpha})$ satisfies the following inequality

$$F_{\alpha}(\omega t^{\alpha})\le Ce^{\omega^{1/\alpha}t},\quad t\ge 0,\ \omega\ge 0,\ 0<\alpha<2, \tag{14}$$

where C is a positive constant.

With (13) and (14), we have

$$r(t)\le aCe^{(b\Gamma(\alpha))^{1/\alpha}t},\quad t\ge T.$$

then

$$\|x(t)\|\le aCe^{[(b\Gamma(\alpha))^{1/\alpha}+\sigma]t},\quad t\ge T.$$

From Eq. (1), we obtain

$$\begin{aligned}\|{}_0^C D_t^{\alpha}x(t)\| &\le \|A\|\,\|x(t)\|+\|f(t)\|\\&\le a\|A\|Ce^{[(b\Gamma(\alpha))^{1/\alpha}+\sigma]t}+Me^{\sigma t}\\&\le (a\|A\|C+M)e^{[(b\Gamma(\alpha))^{1/\alpha}+\sigma]t},\quad t\ge T.\quad\square\end{aligned}$$

Taking Laplace transform with respect to t in both sides of (1), we obtain

$$\hat{x}(s)=s^{\alpha-1}(s^{\alpha}-A)^{-1}\eta+(s^{\alpha}-A)^{-1}\hat{f}(s), \tag{15}$$

where $\hat{x}(s)$ and $\hat{f}(s)$ denotes the Laplace transform of $x(t)$ and $f(t)$, respectively. The inverse Laplace transform using (8) yields

$$x(t)=E_{\alpha,1}(t)\eta+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(\tau)d\tau. \tag{16}$$

Remark 3.1. The Laplace transform method is suitable for constant coefficient fractional differential equations, but it demands for forcing terms, so not every constant coefficient fractional differential equation can be solved by the Laplace transform method.

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