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Existence and properties of stationary solution of dynamical neural field☆

Dequan Jin*, Dong Liang, Jigen Peng

Department of Applied Mathematics, School of Science, Xi'an Jiaotong University, Xi'an, 710049, China

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ABSTRACT

In this paper, we discuss the existence and properties of stationary solution of Amari's dynamical neural field model in $\mathbb{R}^n$. Amari's model has been extensively applied in psychophysics, neurophysiology, machine vision and cognition, where its stationary solution plays an important role. Though many problems in these applications are presented in high dimensional spaces, most investigations on the stationary solutions of neural field are in $\mathbb{R}$ and $\mathbb{R}^2$. Under certain general assumptions on neural field, we obtain some existence conditions for general stationary solution, as well as some typical stationary solutions. Some results on their stabilities are also obtained.

1. Introduction

During 1970s, some approaches that describe large scale activation of cortical neurons as a continuous neural field have been presented [1–6], which are known as dynamical neural field theory later. Among these approaches, the neural field model presented by Amari [1] is an important one. Amari's model describes the statistical average behavior of large scale ensemble of cortical neurons with similar functions or properties, and is mathematically analyzable and biologically reasonable. Therefore, it has been extensively applied in the investigation of psychophysics, neurophysiology, machine vision and cognition, and successfully interprets large amounts of important phenomena and problems [7–14].

Most applications of Amari's model rely on its stationary solutions, so that it is fundamental to study their existence conditions and dynamical properties. Since Amari's model is highly nonlinear system, the investigations of its dynamical properties are almost limited in $\mathbb{R}$ and $\mathbb{R}^2$ [2,4,15–21]. However, the applications of neural field have been extended into high dimensional spaces already, for instance, the feature space arising in pattern recognition [9,11,12,14,13]. Although some important results about the existence of the solution of Amari's model in $\mathbb{R}^n$ have been presented in [22], the existence conditions and stabilities of its stationary solutions have not been considered yet, which significantly constrain the application of neural field.

In this paper, we discuss the existence conditions and properties of stationary solutions of Amari's model in $\mathbb{R}^n$. Some existence conditions for the general stationary solution and the three important types of stationary solutions: ϕ-solution, ∞-solution and ball-solution, are presented, respectively. With some general assumptions, their stabilities are also discussed.

2. Existence conditions for stationary solution

Amari's model is usually given by [1]

$$\tau\frac{\partial u(x,t)}{\partial t}=-u(x,t)+\int_{\Omega}w(x,x')\theta(u(x',t))\mathrm{d}x'+S(x,t)+h \tag{1}$$

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* Corresponding author. Tel.: +86 13772198905.
E-mail addresses: dqjin@yahoo.cn (D. Jin), liangdong@mail.xjtu.edu.cn (D. Liang), jgpeng@mail.xjtu.edu.cn (J. Peng).

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with the initial condition

$$u(x, 0) = u_0(x) \tag{2}$$

where $\|u_0\|_\infty = \sup_{x\in\Omega} |u_0(x)| < \infty$. Ω is called the perceptive field. τ is a positive time constant. h is the resting level of neural field. $S(x, t)$ is an input signal distribution function. The region

$$\{x \in \Omega : u(x, t) > 0\}$$

is called the excited region. $\theta(u)$ is a threshold function that monotonically increases and satisfies $\lim_{u\to-\infty} \theta(u) = 0$ and $\lim_{u\to+\infty} \theta(u) = 1$. The interaction between neurons is introduced by the integration

$$\int_\Omega w(x, x')\theta(u(x', t))\mathrm{d}x'$$

whose strength is determined by the interaction function $w(x, x')$.

Since we mainly concern the stationary solutions of neural field, $S(x, t)$ is supposed to be time-invariant, i.e.,

$$S(x, t) = S(x) \tag{3}$$

for all $t \geq 0$. Let Ω be $\mathbb{R}^n$. Then we have

$$\tau \frac{\partial u(x, t)}{\partial t} = -u(x, t) + \int_{\mathbb{R}^n} w(x, x')\theta(u(x', t))\mathrm{d}x' + S(x) + h. \tag{4}$$

In this paper, we suppose that the neural field is homogeneous. Then the interaction function $w(x, x')$ can be written as $w(x - x')$. $w(x)$ is also called interaction kernel. In this paper, simulating mechanism of neurons, $w(x)$ is supposed to be an isotropic function with "Mexican hat" shape such that

$$w(x) \begin{cases} >0, & \text{if } \|x\| < \eta \\ \leq 0, & \text{else} \end{cases} \tag{5}$$

where $\eta > 0$ and

$$\lim_{\|x\|\to\infty} w(x) = 0. \tag{6}$$

With the assumptions that

$$\|w\|_{L(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |w(x)|\mathrm{d}x = M < \infty \tag{7}$$

$$\|w(x_1) - w(x_2)\| < L_w|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n \tag{8}$$

$$|w(x)| < C_\infty, \quad \forall x \in \mathbb{R}^n \tag{9}$$

where M, L_w and C_∞ are positive constants, the solution $u(x, t)$ of neural field system (1) is proved to be bounded [22], i.e.,

$$|u(x, t)| \leq b \tag{10}$$

for any $x \in \mathbb{R}^n$ and $t \geq 0$. b is a constant defined by

$$b = \max\{\|u_0\|_\infty, \widehat{M}\} \tag{11}$$

where

$$\widehat{M} = M + h + S_0. \tag{12}$$

Definition 1. Let $u^*(x)$ be the stationary solution of neural field, then

(1) if $u^*(x) \leq 0$ for all $x \in \mathbb{R}^n$, $u^*(x)$ is called ϕ-solution;
(2) if $u^*(x) > 0$ for all $x \in \mathbb{R}^n$, $u^*(x)$ is called ∞-solution;
(3) if $u^*(x) > 0$ in a ball D with radius $R > 0$ and $u^*(x) \leq 0$ outside D, $u^*(x)$ is called ball-solutions.

In the applications of neural field, the stationary solutions ϕ-solution, ∞-solution and ball-solution are mostly concerned.

By letting $\partial u/\partial t = 0$, all the stationary solutions should satisfy

$$u^*(x) = \int_{\mathbb{R}^n} w(x - x')\theta(u^*(x'))\mathrm{d}x' + S(x) + h. \tag{13}$$

Let $w_{\max}(x) = \max\{0, w(x)\}$ and $w_{\min}(x) = \min\{0, w(x)\}$. Suppose that

$$W_\infty = \int_{\mathbb{R}^n} w(x)\mathrm{d}x < \infty \tag{14}$$

$$W_{\max} = \int_{\mathbb{R}^n} w_{\max}(x)\mathrm{d}x < \infty \tag{15}$$

and

$$W_{\min} = \int_{\mathbb{R}^n} w_{\min}(x)\mathrm{d}x < \infty. \tag{16}$$

When $S(x)$ is bounded by $s_0 \geq 0$ and $S_0 \geq 0$, i.e.,

$$s_0 \leq S(x) \leq S_0 \tag{17}$$

for all $x \in \mathbb{R}^n$, we have the following results.

Proposition 1. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), and $S(x)$ satisfies (17). If*

$$W_{\min} + s_0 + h > 0 \tag{18}$$

then all the stationary solutions of system (4) are ∞-solutions. If there exists an ∞-solution, then

$$W_{\max} + S_0 + h > 0. \tag{19}$$

Proof. Since the threshold function $\theta : \mathbb{R} \to [0, 1]$, when $W_{\min} + s_0 + h > 0$ for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} u^*(x) &= \int_{\mathbb{R}^n} w(x - x')\theta(u^*(x'))\mathrm{d}x' + S(x) + h \\ &\geq \int_{\mathbb{R}^n} w_{\min}(x')\mathrm{d}x' + s_0 + h \\ &= W_{\min} + s_0 + h \\ &> 0. \end{aligned} \tag{20}$$

It follows that all the stationary solutions are ∞-solutions.

When there exists an ∞-solution $u^*(x) > 0$ for all $x \in \mathbb{R}^n$, since

$$\begin{aligned} u^*(x) &= \int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x' + S(x) + h \\ &\leq \int_{\mathbb{R}^n} w_{\max}(x')\mathrm{d}x' + S_0 + h \\ &= W_{\max} + S_0 + h \end{aligned} \tag{21}$$

then

$$W_{\max} + S_0 + h \geq u^*(x) > 0. \tag{22}$$

The proof is complete. □

Proposition 2. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), and $S(x)$ satisfies (17). If*

$$W_{\max} + S_0 + h \leq 0 \tag{23}$$

then all the stationary solutions of system (4) are ϕ-solution. If there exists a ϕ-solution $u^*(x) \leq 0$, then

$$W_{\min} + s_0 + h \leq 0. \tag{24}$$

Proof. Since the threshold function $\theta : \mathbb{R} \to [0, 1]$, when $W_{\max} + S_0 + h < 0$, we have

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x' + S(x) + h \\ &\leq \int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x' + S_0 + h \end{aligned}$$

$$\begin{aligned} &\le \int_{\mathbb{R}^n} w_{\max}(x')\mathrm{d}x' + S_0 + h \\ &= W_{\max} + S_0 + h \\ &< 0. \end{aligned} \tag{25}$$

It follows that all the stationary solutions of system (4) are ϕ-solutions.

When there exists a ϕ-solution $u^*(x) = h_\phi \le 0$ for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} u^*(x) &= \int_{\mathbb{R}^n} w(x-x')\theta(u^*(x'))\mathrm{d}x' + S(x) + h \\ &\ge W_{\min} + s_0 + h. \end{aligned} \tag{26}$$

It follows that

$$W_{\min} + s_0 + h \le u^*(x) \le 0. \tag{27}$$

The proof is complete. □

Proposition 3. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), and $S(x)$ satisfies (17). If there exists a ball-solution of system (4), then*

$$W_{\min} + s_0 + h \le 0$$

and

$$W_{\max} + S_0 + h > 0.$$

Proposition 3 can be considered as a corollary of Propositions 1 and 2.

When the threshold function θ is a step function defined by

$$\theta(u) = \begin{cases} 0, & u \le 0 \\ 1, & u > 0 \end{cases} \tag{28}$$

where $u \in \mathbb{R}$, we have the following result.

Corollary 1. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), $S(x)$ satisfies (17), and θ is a step function defined by (28). If*

$$W_{\min} + s_0 + h > 0 \tag{29}$$

then all the stationary solutions of system (4) are ∞-solutions. If there exists an ∞-solution, then

$$W_\infty + S_0 + h > 0. \tag{30}$$

Corollary 2. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), $S(x)$ satisfies (17), and θ is a step function defined by (28). If*

$$W_{\max} + S_0 + h \le 0 \tag{31}$$

then all the stationary solutions of system (4) are ϕ-solution. If there exists a ϕ-solution, then

$$s_0 + h \le 0. \tag{32}$$

Corollary 3. *Suppose that the interaction kernel $w(x)$ satisfies assumptions (15) and (16), $S(x)$ satisfies (17), and θ is a step function defined by (28). If there exists a ball-solution of system (4), then*

$$W_{\min} + s_0 + h \le 0$$

and

$$W_{\max} + S_0 + h > 0.$$

The above conclusions are compatible with the results obtained in $\mathbb{R}$ and $\mathbb{R}^2$ in [1,18].

Suppose that

$$D = \{x \in \Omega : \|x\| \le R\}$$

where $R > 0$. Let

$$W(\|x\|, R) = \int_D w(x - x')\mathrm{d}x'. \tag{33}$$

When $\|x\| = R$, define

$$G(R) = W(R, R) = \int_D w(x - x')\mathrm{d}x'. \tag{34}$$

Similar to the proof of [18, Theorem 1], it is easy to obtain the following result.

Proposition 4. *Suppose that $w(x)$ satisfies assumptions (5) and (6). When $\theta(u)$ is a step function defined by (28) and $S(x) = 0$, there exists a ball-solution of system (4) in $\mathbb{R}^n$, $n = 1, 2, \dots,$ if and only if*

$$G(R) + h = 0. \tag{35}$$

3. Neural field with continuous θ

When $\theta(u)$ is a step function, only activated neurons produce interaction feedback whose strength is totally defined by the interaction kernel $w(x)$. However, it is no longer true when θ is continuous. Neurons with activation under resting level may also produce feedback to other neurons. In this section, we discuss the dynamical properties of neural field when θ is continuous.

3.1. Existence of stationary solution

Let the operator T be defined by

$$Tu(x) = \int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x' + S(x) + h. \tag{36}$$

$BC(\mathbb{R}^n)$ denotes the space of the bounded continuous functions in $\mathbb{R}^n$, which is a Banach space with the norm $\|\cdot\|_\infty$.

Proposition 5. *Suppose that $S(x)$ and θ are continuous. If the interaction kernel $w(x)$ is continuous and satisfies assumptions (7)–(9), then $T : BC(\mathbb{R}^n) \to BC(\mathbb{R}^n)$ is a continuous compact operator.*

Proof. We first prove that $Tu(x)$ is equicontinuous. For any $\epsilon > 0$, since $S(x)$ is continuous, there exists $\delta_0 > 0$ such that for $\|x_1 - x_2\| < \delta_0$,

$$|S(x_1) - S(x_2)| < \epsilon/2.$$

Let $\delta = \min\{\epsilon/(2L_w), \delta_0\}$. When $u \in BC(\mathbb{R}^n)$, for any $x_1, x_2 \in \mathbb{R}^n$ such that $\|x_1 - x_2\| < \delta$, with assumption (8), we have

$$\begin{aligned} |Tu(x_1) - Tu(x_2)| &= \left|\int_{\mathbb{R}^n} (w(x_1 - x') - w(x_2 - x'))\theta(u(x'))\mathrm{d}x' + S(x_1) - S(x_2)\right| \\ &\le \left|\int_{\mathbb{R}^n} (w(x_1 - x') - w(x_2 - x'))\mathrm{d}x'\right| + |S(x_1) - S(x_2)| \\ &\le L_w(x_1 - x_2) + \epsilon/2 \\ &\le \epsilon. \end{aligned} \tag{37}$$

Then $Tu(x)$ is equicontinuous. Next we prove that T is uniformly bounded. Since

$$\begin{aligned} |Tu(x)| &= \left|\int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x' + S(x) + h\right| \\ &\le \left|\int_{\mathbb{R}^n} w(x - x')\theta(u(x'))\mathrm{d}x'\right| + |S(x)| + |h| \\ &\le \int_{\mathbb{R}^n} |w(x')|\mathrm{d}x' + S_0 + h \\ &= M + S_0 + h \\ &= \widehat{M} \end{aligned} \tag{38}$$

then T is uniformly bounded. By Arzela–Ascoli Theorem [23, Chapter 3], T is a compact operator.

Since θ is continuous and bounded, for any $\epsilon > 0$, there exists $\delta = \epsilon/(LM)$ such that for any $u_1, u_2 \in BC(\mathbb{R}^n)$ satisfying $\|u_1 - u_2\|_\infty < \delta$, we have

$$|\theta(u_1) - \theta(u_2)| < \epsilon/M. \tag{39}$$

Since

$$\begin{aligned} |Tu_1(x) - Tu_2(x)| &= \left|\left(\int_{\mathbb{R}^n} w(x-x')\theta(u_1(x'))\mathrm{d}x' + S(x) + h\right) - \left(\int_{\mathbb{R}^n} w(x-x')\theta(u_2(x'))\mathrm{d}x' + S(x) + h\right)\right| \\ &\le \left|\int_{\mathbb{R}^n} w(x')\theta(u_1(x-x'))\mathrm{d}x' - \int_{\mathbb{R}^n} w(x')\theta(u_2(x-x'))\mathrm{d}x'\right| \\ &\le \left|\int_{\mathbb{R}^n} w(x')(\theta(u_1(x-x')) - \theta(u_2(x-x')))\mathrm{d}x'\right| \\ &\le \int_{\mathbb{R}^n} \left|w(x')\right| \cdot \left|(\theta(u_1(x-x')) - \theta(u_2(x-x')))\right|\mathrm{d}x' \\ &\le \frac{\epsilon}{M}\int_{\mathbb{R}^n} |w(x')|\mathrm{d}x' \\ &\le \epsilon \end{aligned} \tag{40}$$

then

$$\|Tu_1 - Tu_2\|_\infty < \epsilon. \tag{41}$$

Therefore, T is a continuous compact operator. The proof is complete. □

Proposition 6. *Suppose that $S(x)$ and θ are continuous. If the interaction kernel $w(x)$ is continuous and satisfies assumptions (7)–(9), then there exists a stationary solution $u^*(x)$ of system (4).*

Proof. Let U be an open region

$$U := \{u \in BC(\mathbb{R}^n) : \|u\|_\infty < b\}$$

where b is defined by (11). Since $u(x,t)$ have been proved to be bounded by b (see [22]), the stationary solution $u(x)$ of system (4) should be bounded by b if it exists. Then the existence of a stationary solution of system (4) is equal to the existence of a solution of the operator equation

$$Tu = u \tag{42}$$

in the Banach space $BC(\mathbb{R}^n)$.

Since for any $u \in \partial U$, $|Tu(x)| < \widehat{M}$ in $\mathbb{R}^n$, we have $\|Tu\|_\infty \le \widehat{M} \le b$. As a result, we have

$$T(\bar{U}) \subset \bar{U}. \tag{43}$$

By Rothe's Fixed Point Theorem [24], there exists a fixed point $u^* \in \bar{U}$ of operator equation (42). Therefore, there exists a stationary solution $u^*(x) \in \bar{U}$ of neural field (4). The proof is complete. □

When $\theta(u)$ is Lipschitz continuous, i.e., for any $u_1, u_2 \in \mathbb{R}$,

$$|\theta(u_1) - \theta(u_2)| < L|u_1 - u_2| \tag{44}$$

where $L > 0$, we have the following results.

Proposition 7. *Suppose that $S(x)$ is continuous, θ satisfies (44) and the interaction kernel $w(x)$ is continuous and satisfies assumptions (7)–(9). If $LM < 1$, there exists a unique stationary solution $u^*(x)$ of neural field (4).*

Proof. For any $u_1, u_2 \in BC(\mathbb{R}^n)$, since

$$|\theta(u_1(x)) - \theta(u_2(x))| \le L\|u_1 - u_2\|_\infty \tag{45}$$

for all $x \in \mathbb{R}^n$, we have

$$\|\theta(u_1) - \theta(u_2)\|_\infty \le L\|u_1 - u_2\|_\infty. \tag{46}$$

As $M = \int_{\mathbb{R}^n} |w(x)|\mathrm{d}x$, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |Tu_1(x) - Tu_2(x)| &= \left| \left(\int_{\mathbb{R}^n} w(x-x')\theta(u_1(x'))\mathrm{d}x' + S(x) + h \right) - \left(\int_{\mathbb{R}^n} w(x-x')\theta(u_2(x'))\mathrm{d}x' + S(x) + h \right) \right| \\ &= \left| \int_{\mathbb{R}^n} w(x-x')(\theta(u_1(x')) - \theta(u_2(x')))\mathrm{d}x' \right| \\ &\le \int_{\mathbb{R}^n} |w(x-x')| \cdot |\theta(u_1(x')) - \theta(u_2(x'))|\mathrm{d}x' \\ &\le L\|u_1 - u_2\|_\infty \int_{\mathbb{R}^n} |w(x-x')|\mathrm{d}x' \\ &= LM\|u_1 - u_2\|_\infty. \end{aligned} \tag{47}$$

Then

$$\|Tu_1 - Tu_2\|_\infty \le LM\|u_1 - u_2\|_\infty. \tag{48}$$

Since $LM < 1$, T is a strict contraction. By Banach's Fixed Point Theorem [25], there exists a unique fixed $u^* \in BC(\mathbb{R}^n)$ of the operator equation

$$Tu = u.$$

As a result, there exists one and only one stationary solution $u^*(x)$ of dynamical neural field (4). The proof is complete. □

When the threshold function θ is a sigmoid function, we have the following result.

Corollary 4. *When the threshold function θ is a sigmoid function defined by*

$$\theta(u) = \frac{1}{1 + \exp(-u/\alpha^2)} \tag{49}$$

and the interaction kernel $w(x)$ is a DoG function defined by

$$w(x) = A\exp(-\|x\|^2/2\sigma_1^2) - B\exp(-\|x\|^2/2\sigma_2^2) \tag{50}$$

where A, B, α, σ_1, σ_2 are positive constants such that $\sigma_1 < \sigma_2$ and $A > B$, if

$$\left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B < \alpha^2 \tag{51}$$

then there exists a unique stationary solution $u^*(x)$ of (4).

Proof. Since $\theta : \mathbb{R} \to \mathbb{R}$ defined by (49) is a smooth function, its Lipschitz factor L is maximum of its derivative function.

The derivative of $\theta(u)$ is

$$\theta'(u) = \frac{\exp(-u/\alpha^2)}{\alpha^2(1 + \exp(-u/\alpha^2))^2} \tag{52}$$

and the second derivative of $\theta(u)$ is

$$\theta''(u) = \frac{\exp(-u/\alpha^2)(\exp(-u/\alpha^2) - 1)}{\alpha^4(1 + \exp(-u/\alpha^2))^3}. \tag{53}$$

Since $\theta''(u) = 0$ only at $u = 0$, and

$$\theta'(u) \to 0, \quad u \to \infty \tag{54}$$

then

$$\begin{aligned} L &= \theta'(0) \\ &= \frac{1}{2\alpha^2}. \end{aligned} \tag{55}$$

Since

$$\int_{\mathbb{R}^n} |w(x)|\mathrm{d}x = \int_{\mathbb{R}^n} \left| A\exp(-\|x\|^2/2\sigma_1^2) - B\exp(-\|x\|^2/2\sigma_2^2) \right| \mathrm{d}x$$
$$\leq A\int_{\mathbb{R}^n} \exp(-\|x\|^2/2\sigma_1^2)\mathrm{d}x + B\int_{\mathbb{R}^n} \exp(-\|x\|^2/2\sigma_2^2)\mathrm{d}x$$
$$= \left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B \tag{56}$$

we have

$$0 \leq M \leq \left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B. \tag{57}$$

Since

$$LM \leq \frac{\left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B}{\alpha^2} \tag{58}$$

and

$$\left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B < \alpha^2 \tag{59}$$

we have

$$LM < 1. \tag{60}$$

According to Proposition 7, there exists a unique stationary solution $u^*(x)$ of (4). The proof is complete. □

The above inequality

$$\left(\sqrt{2\pi}\sigma_1\right)^n A + \left(\sqrt{2\pi}\sigma_2\right)^n B < \alpha^2 \tag{61}$$

can be written as

$$2\pi\sigma_1^2 A + 2\pi\sigma_2^2 B < \alpha^2 \tag{62}$$

in $\mathbb{R}^2$, which is easy to be verified.

3.2. Existence and stability of ϕ, ∞ and ball-solution

Suppose that $S(x) = 0$. Then we have

$$\tau\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{\mathbb{R}^n} w(x-x')\theta(u(x',t))\mathrm{d}x' + h. \tag{63}$$

When $w(x)$ satisfies assumption (14), let

$$F(u) = -u + \theta(u)W_\infty + h \tag{64}$$

where $u \in \mathbb{R}$. Since θ is bounded and $W_\infty < \infty$, we have

$$\lim_{u\to-\infty} F(u) > 0$$

and

$$\lim_{u\to+\infty} F(u) < 0.$$

It is easy to obtain the following results.

Proposition 8. *Suppose that θ is continuous and $w(x)$ satisfies assumption (14), then we have the following.*

(1) *There exists a ϕ-solution $u(x) = h_\phi < 0$ of (63) if and only if*

$$F(h_\phi) = 0.$$

(2) *There exists an ∞-solution $u(x) = h_\infty > 0$ of (63) if and only if*

$$F(h_\infty) = 0.$$

(3) *There exists at least one ϕ-solution or ∞-solution of system (63).*

Proposition 9. *Suppose that θ is differentiable and $w(x)$ satisfies assumption (14),*

(1) *a ϕ-solution $u(x) = h_\phi < 0$ of (63) is stable if*

$$F'(h_\phi) < 0;$$

(2) *an ∞-solution $u(x) = h_\infty > 0$ of (63) is stable if*

$$F'(h_\infty) < 0.$$

Proposition 10. *Suppose that θ is differential and $w(x)$ satisfies assumption (14). If $LW_\infty < 1$, where L is the Lipschitz factor of θ, there exists one and only one stable ϕ-solution or ∞-solution.*

Proof. Since $L = \max_{u\in\mathbb{R}} \theta'(u)$ when θ is differential, we have

$$F'(u) = -1 + \theta'(u)W_\infty \le -1 + LW_\infty < 0 \tag{65}$$

then $F(u)$ monotonously decreases. By Proposition 8, it has one and only one solution. As $F'(u) < 0$, this solution is stable. The proof is complete. □

Notice that in Proposition 10, θ is assumed to be differentiable, which is stronger than the condition in Proposition 7.

When $LW_\infty > 1$, it is difficult to discuss the nature of stationary solution. So we assume that $\theta'(u)$ is uniformly continuous and monotonously increases in $(-\infty, 0)$ and monotonously decreases in $(0, \infty)$, which are not constraint assumptions. With Barbalat's lemma [26, Lemma 8.2], we obtain that

$$\lim_{u\to\infty} \theta'(u) = \lim_{u\to-\infty} \theta'(u) = 0.$$

Since $LW_\infty > 1$, we have

$$\begin{cases} F'(u) > 0, & u \in (y_1, y_2) \\ F'(u) = 0, & u \in \{y_1, y_2\} \\ F'(u) < 0, & u \in (-\infty, y_1) \cup (y_2, \infty) \end{cases}$$

where $y_1 < 0 < y_2$.

Proposition 11. *Suppose θ is differential and $w(x)$ satisfies assumption (14). If $LW_\infty > 1$,*

(1) *if $F(y_1) > 0$, there exists one and only one stable ∞-solution;*
(2) *if $F(y_1) < 0$ and $F(y_2) < 0$, there exists one and only one stable ϕ-solution;*
(3) *if $F(y_1) < 0$ and $F(y_2) > 0$, there exists one stable ϕ-solution, one stable ∞-solution, and one unstable ϕ-solution or ∞-solution.*

Proof. It is easy to see that y_1 is a local minimum point and y_2 is a local maximum point of $F(u)$. $F(u)$ monotonously decreases in $(-\infty, y_1) \cup (y_2, \infty)$ and monotonously increases on $[y_1, y_2]$.

(1) If $F(y_1) > 0$, $F(u) > 0$ in $(-\infty, y_2)$. Since $F(u)$ monotonously decreases in $(-\infty, y_1) \cup (y_2, \infty)$ and monotonously increases on $[y_1, y_2]$, we can see that the equation $F(u) = 0$ has only one positive solution $u^* > 0$. Since $F'(u^*) < 0$, this solution is stable. Therefore, there exists one and only one stable ∞-solution.

(2) Since $\lim_{u\to-\infty} > 0$ and $F(y_1) < 0$, there exists a solution $u^* < y_1$. As $F'(u^*) < 0$, u^* is stable. Since $F(y_2) < 0$ and $F(u)$ monotonously decreases in $(-\infty, y_1) \cup (y_2, \infty)$ and monotonously increases on $[y_1, y_2]$, there is no other solutions of $F(u) = 0$. Therefore, there exists one and only one stable ϕ-solution.

(3) $F(y_1) < 0$ and $F(y_2) > 0$; there exist three solutions u_1^*, u_2^* and u_3^* such that $u_1^* < y_1 < u_2^* < y_2 < u_3^*$. It is easy to show that $u_1^* < 0$ and $u_3^* > 0$ are stable solutions and u_2^* is not stable. Therefore, there exist one stable ϕ-solution, one stable ∞-solution, and one unstable ϕ-solution or ∞-solution.

The proof is complete. □

To discuss the existence condition and stability of ball-solution, we further suppose that θ is a sigmoid function defined by (49). If there exists a ball-solution with radius R_0, the equation

$$\int_\Omega w(x - x')\theta(u(x', t))\mathrm{d}x' + h = 0 \tag{66}$$

for $\|x\| = R_0$ should be satisfied. Let

$$G_\theta(R(u)) = \int_\Omega w(x - x')\theta(u(x', t))\mathrm{d}x' \tag{67}$$

where $\|x\| = R(u)$.

Suppose that a neural activation ball with radius R satisfies $u(x', t) = 0$ only on $\{x \in \mathbb{R}^n : \|x\| = R\}$. Let

$$E(R, \alpha) = G_\theta(R) - G(R). \quad (68)$$

Assume

$$|E(R, \alpha)| < b_0 \quad (69)$$

for $R \in [0, R_b]$, where $R_b > 0$ is large. It follows that

$$G_\theta(R) > G(R) - b_0 \quad (70)$$

and

$$G_\theta(R) < G(R) + b_0 \quad (71)$$

for any $R \in [0, R_b]$. Suppose that $G(R)$ is convex in $[0, \infty)$ and $\lim_{R\to\infty} G(R) < 0$.

Proposition 12. *Suppose that θ is continuous and $E(R, \alpha)$ satisfies assumption (69). R_1 and R_2 are constants that satisfy $0 < R_1 < R_2 < R_b$. Then a ball-solution with radius $R \in (R_1, R_2)$ can be sustained if*

$$G(R_1) - b_0 + h \geq 0 \quad (72)$$

and

$$G(R_2) + b_0 + h \leq 0. \quad (73)$$

Proof. For any neural activation ball with radius R_1 in neural field, we have

$$G_\theta(R_1) > G(R_1) - b_0. \quad (74)$$

As

$$G(R_1) - b_0 + h \geq 0 \quad (75)$$

we have

$$G_\theta(R_1) + h > G(R_1) - b_0 + h \geq 0. \quad (76)$$

It follows that the neurons on $\partial B(R_1)$ would be activated. Then the radius of the neural activation ball will increase.

For any neural activation ball with radius $R = R_2$ in neural field, we have

$$G_\theta(R_2) < G(R_2) + b_0. \quad (77)$$

As

$$G(R_2) + b_0 + h \leq 0 \quad (78)$$

we have

$$G_\theta(R_2) + h < G(R_2) + b_0 + h \leq 0. \quad (79)$$

It follows that the neurons on $\partial B(R_2)$ would be suppressed. Then the radius of neural activation ball will decrease.

Since $u(x, t)$ is continuous on t, a neural activation ball with radius in (R_1, R_2) can neither vanish under R_1 nor spread over R_2. As a result, the neural activation ball can be sustained by the neural field.

The proof is complete. □

Since θ is a sigmoid function, neurons with non-positive activation is also able to produce positive feedback. Then the radius of the excited region in a ball-solution depends on not only the radius of the excited region but also the activation value of initial activation distribution $u(x, 0)$. Different $u(x, 0)$ with the same radius of excited region may lead to different ball-solutions. This is very different from the situation discussed in [1,18], where θ is defined as a step function.

The bound b_0 of $E_u(R, \alpha)$ is important. Since $E_u(R, \alpha)$ depends on the activation $u(x, t)$, it is difficult to determine b_0 directly. This is why we assume $u(x, t) = 0$ only on the boundary of the ball. With this assumption, b_0 can be controlled by selecting sufficiently small α, so that it is possible to obtain a b_0 that is much smaller than h. In this case, the conditions (72) and (73) can be reduced to

$$G(R_1) + h > 0 \quad (80)$$

and

$$G(R_2) + h < 0. \quad (81)$$

Since $G(R)$ is supposed to be continuous, if R_1 and R_2 exist, they should be found in a small neighborhood of a solution R^* of

$$G(R) + h = 0. \quad (82)$$

As a result, the radius of the orbicular excited region that can be sustained by a neural field should be around R^*.

4. Discussion

In this paper, the existence conditions and stabilities of the stationary solutions of neural field are discussed in $\mathbb{R}^n$. Some important results are obtained, which provide an elementary theoretical foundation for the application of Amari's neural field in high dimensional spaces.

When the threshold function θ and the input distribution $S(x)$ is continuous, with assumptions (7)–(9) on the interaction kernel $w(x)$, Proposition 6 guarantees that there exists at least one stationary solution such that there may coexist two or more solutions. Under the stronger condition that θ is Lipschitz continuous and $LM < 1$, where L is the Lipschitz factor of θ and M is defined by (7), Proposition 7 guarantees that there exists one and only one solution.

In discussing the properties of the stationary solutions of neural field, the interaction kernel $w(x)$ is supposed to satisfy some assumptions. These assumptions are general in the investigations of neural field. They are not constraint, since most of the popular types of $w(x)$ satisfy them.

The known results on the stationary solutions of neural field in $\mathbb{R}^2$ are usually obtained by supposing that the threshold function θ is a step function [18,21], where the neurons under resting level would not produce feedback. Differently, we discuss the case that θ is a sigmoid function, where neurons with non-positive activation may also produce positive feedback to others. It makes the discussion more difficult and no results on this are given. With some assumptions on neural activation $u(x, t)$, we obtain some results on evaluating the radius of excited region in the ball-solution of neural field in $\mathbb{R}^n$, which is important in the actual applications.

In most applications, neural fields are usually designed to have a continuous threshold function θ, and those which are able to sustain local excited regions with proper size are preferred. The excited regions usually represent the reaction of neurons to the input stimulus, or indicates the patterns in the input stimulus. For instance, when describing visual working memory, a neural field that is able to sustain a large excited region indicates good association ability in memory, while those that are able to sustain a smaller excited region indicate better precision in memory [8,11,12]. Another example is in object recognition, where an excited region in the memory indicates the memory of a class of actual objects [9,27]. Therefore, the size of local excited region that can be sustained is very important for a neural field. By calculating the radius R^*, we can evaluate such capacity of a neural field. In this way, R^* can be considered as a measure describing the memory ability of neural field which is meaningful in actual applications.

Some sufficient and necessary existence conditions for ϕ-solution and ∞-solution of neural field are also presented in the above discussions. In many applications of neural field, monostable ϕ-solution and ∞-solution are considered to be ill-posed. In these cases, the necessary conditions for their existence should be avoided. For some applications, monostable ϕ-solution and ∞-solution are also useful. For example, when we want to design a neural field illustrating the fadeaway of short-term memory, a monostable ϕ-solution is preferred; when a neural field is employed to construct a hierarchical clustering approach, as presented in [27], an ∞-solution becomes useful. Therefore, the discussions on the existence of ϕ-solution and ∞-solution are also meaningful.

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