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Fractional resolvents and fractional evolution equations☆

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ABSTRACT

In this work, we present the notion of the fractional resolvent, which can be seen as a generalization of strongly continuous semigroups. We give some of its properties and apply the results to a fractional order abstract evolution equation.

1. Introduction

In recent years, fractional calculus has found many applications in physics, chemistry, engineering and control, etc (see [1,2]). Derivatives and integrals of fractional order are suitable for the description of properties of various real materials. Evolutionary differential equations have been gaining much attention. Berens and Westphal [3] first considered the abstract Cauchy problem for the Riemann–Liouville fractional differentiation operator. Da Prato and Iannelli [4] introduced the concept of resolvent families. Oka [5] introduced integrated solution families. Lizama [6] introduced (a, k)-regularized resolvents. Bajlekova [7] carried out research on fractional evolution equations by using solution operators. In this article we present the fractional resolvent $T_\alpha(t)$ for $0 < \alpha < 1,\ t > 0$ and consider the fractional order abstract evolution equation

$$\begin{cases} D_t^\alpha u(t) = Au(t), \quad t > 0, \\ \lim\limits_{t\to 0+} \Gamma(\alpha) t^{1-\alpha} u(t) = x, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, D_t^α is the Riemann–Liouville fractional derivative operator of order α, $A : D(A) \subset X \to X$ is the infinitesimal generator of an α-order fractional resolvent $T_\alpha(t)$, X is a Banach space with the norm $\|\cdot\|$, and $x \in X$.

The work is organized as follows. In Section 2, we give some preliminaries. In Section 3, we introduce the definition of the fractional resolvent and some of its properties, and study the fractional order abstract Cauchy problem (1).

2. Preliminaries

Assume that $u : [0, \infty) \to X$, where X is a Banach space. Let $0 < \alpha < 1$; following [7], we define the fractional integral of u of order α as

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

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and the Riemann–Liouville fractional derivative of u of order α as

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Let $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\phi_\alpha(t) = 0$ for $t \le 0$; then

$$I_t^\alpha u(t) = (\phi_\alpha * u)(t),$$

$$D_t^\alpha u(t) = \frac{d}{dt}(\phi_{1-\alpha} * u)(t),$$

where the symbol $*$ stands for the convolution operation.

We denote by $C(\mathbb{R}_+; X)$ the space of the X-valued continuous function on $\mathbb{R}_+ = (0, +\infty)$ with the norm $\|u\| = \sup_{t>0} \|u(t)\|$ and by $C^1(\mathbb{R}_+; X)$ the space of the X-valued continuous differentiable function on $\mathbb{R}_+$ with the norm $\|u\| = \sup_{t>0}(\|u(t)\| + \|u'(t)\|)$.

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C}, \tag{2}$$

where $\mathbb{C}$ denotes the set of complex numbers.

Definition 2.1. A function $u \in C(\mathbb{R}_+, X)$ is called a strong solution of (1) if $u \in C(\mathbb{R}_+, D(A))$, $\phi_{1-\alpha} * u \in C^1(\mathbb{R}_+, X)$ and (1) holds.

Definition 2.2. A function $u \in C(\mathbb{R}_+, X)$ is called a mild solution of (1) if $I_t^\alpha u(t) \in D(A)$, and $u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x + A I_t^\alpha u(t),\ t > 0$.

3. The main results

Definition 3.1. For $0 < \alpha < 1$, a function $T_\alpha : \mathbb{R}_+ \to B(X)$ ($B(X)$ denotes the space of all bounded linear operators on X) is called an α-order fractional resolvent if it satisfies the following assumptions:

(P1) $T_\alpha(t)$ is strongly continuous on $\mathbb{R}_+$ and $\lim_{t\to 0+} \Gamma(\alpha) t^{1-\alpha} T_\alpha(t)x = x$ for all $x \in X$.
(P2) $T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s)$ for all $t, s > 0$.
(P3) $T_\alpha(s) I_t^\alpha T_\alpha(t) - I_s^\alpha T_\alpha(s) T_\alpha(t) = \frac{s^{\alpha-1}}{\Gamma(\alpha)} I_t^\alpha T_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I_s^\alpha T_\alpha(s)$ for all $t, s > 0$.

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0+} \frac{t^{1-\alpha} T_\alpha(t)x - \frac{1}{\Gamma(\alpha)} x}{t^\alpha} \text{ exists} \right\}$$

and

$$Ax = \Gamma(2\alpha) \lim_{t\to 0+} \frac{t^{1-\alpha} T_\alpha(t)x - \frac{1}{\Gamma(\alpha)} x}{t^\alpha}, \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the fractional resolvent $T_\alpha(t)$, where $D(A)$ is the domain of A.

Theorem 3.1. *Let T_α be an α-order fractional resolvent and let A be its infinitesimal generator. Then:*

(a) *$T_\alpha(t)x \in D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$, $t > 0$.*
(b) *For all $x \in X$, $t > 0$,*

$$T_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x + A I_t^\alpha T_\alpha(t)x.$$

(c) *For all $x \in D(A)$, $t > 0$,*

$$T_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x + I_t^\alpha T_\alpha(t)Ax,$$

(d) *A is closed and densely defined.*

Proof. (a) Let $x \in D(A)$; for $t > 0$, $s > 0$, by (P3) of Definition 3.1,

$$\frac{s^{1-\alpha} T_\alpha(s) T_\alpha(t)x - \frac{T_\alpha(t)}{\Gamma(\alpha)} x}{s^\alpha} = \frac{T_\alpha(t) \left(s^{1-\alpha} T_\alpha(s)x - \frac{1}{\Gamma(\alpha)} x \right)}{s^\alpha},$$

and then

$$\Gamma(2\alpha)\lim_{s\to 0+}\frac{s^{1-\alpha}T_\alpha(s)T_\alpha(t)x-\frac{T_\alpha(t)}{\Gamma(\alpha)}x}{s^\alpha}=T_\alpha(t)Ax.$$

That is, $T_\alpha(t)x\in D(A)$ and $AT_\alpha(t)x=T_\alpha(t)Ax$.

(b) For all $x\in X$,

$$\begin{aligned}\left\|\Gamma(2\alpha)\frac{s^{1-\alpha}I_s^\alpha T_\alpha(s)x}{s^\alpha}-x\right\| &= \|\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha T_\alpha(s)x-x\|\\
&=\left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^s s^{1-2\alpha}(s-\tau)^{\alpha-1}T_\alpha(\tau)xd\tau-x\right\|\\
&=\left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^1 s^{1-\alpha}(1-\tau)^{\alpha-1}T_\alpha(s\tau)xd\tau-x\right\|\\
&=\left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 s^{1-\alpha}\Gamma(\alpha)(1-\tau)^{\alpha-1}T_\alpha(s\tau)xd\tau-x\right\|\\
&=\left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}\Gamma(\alpha)(s\tau)^{1-\alpha}T_\alpha(s\tau)xd\tau\right.\\
&\quad\left.-\ \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}xd\tau\right\|\\
&\le \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau\ \sup_{\tau\in[0,1]}\|\Gamma(\alpha)(s\tau)^{1-\alpha}T_\alpha(s\tau)x-x\|\\
&\le \sup_{\tau\in[0,1]}\|\Gamma(\alpha)(s\tau)^{1-\alpha}T_\alpha(s\tau)x-x\|.\end{aligned}\tag{3}$$

Apply (P1) of Definition 3.1 to (3) to obtain

$$\left\|\Gamma(2\alpha)\frac{s^{1-\alpha}I_s^\alpha T_\alpha(s)x}{s^\alpha}-x\right\|\to 0,\quad \text{as } s\to 0+.\tag{4}$$

By (P3) of Definition 3.1 and (4), we have

$$\begin{aligned}AI_t^\alpha T_\alpha(t)x &= \Gamma(2\alpha)\lim_{s\to 0+}\frac{\left(s^{1-\alpha}T_\alpha(s)-\frac{1}{\Gamma(\alpha)}\right)I_t^\alpha T_\alpha(t)x}{s^\alpha}\\
&=\Gamma(2\alpha)\lim_{s\to 0+}\frac{s^{1-\alpha}I_s^\alpha T_\alpha(s)\left(T_\alpha(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x\right)}{s^\alpha}\\
&=T_\alpha(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x,\quad x\in X.\end{aligned}\tag{5}$$

(c) For $x\in D(A)$, the limit

$$\lim_{s\to 0+}\frac{s^{1-\alpha}T_\alpha(s)x-\frac{1}{\Gamma(\alpha)}x}{s^\alpha}$$

exists; then the function

$$f(s)=\frac{s^{1-\alpha}T_\alpha(s)x-\frac{1}{\Gamma(\alpha)}x}{s^\alpha}$$

is bounded for sufficiently small $s>0$. By (5), (P2) of Definition 3.1 and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}T_\alpha(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x &= AI_t^\alpha T_\alpha(t)x\\
&=\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\lim_{s\to 0+}\frac{s^{1-\alpha}T_\alpha(s)-\frac{1}{\Gamma(\alpha)}}{s^\alpha}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)xd\tau\\
&=\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\lim_{s\to 0+}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)\frac{s^{1-\alpha}T_\alpha(s)x-\frac{1}{\Gamma(\alpha)}x}{s^\alpha}d\tau\end{aligned}$$

$$= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau) \lim_{s\to 0+} \frac{s^{1-\alpha}T_\alpha(s)x - \frac{1}{\Gamma(\alpha)}x}{s^\alpha} d\tau$$
$$= I_t^\alpha T_\alpha(t)Ax. \quad (6)$$

(d) Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. By (6) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x &= \lim_{n\to\infty}\left(T(t)x_n - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x_n\right) \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)Ax_n d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)y d\tau \\ &= I_t^\alpha T_\alpha(t)y. \end{aligned} \quad (7)$$

Using (7) and (4), we have

$$\begin{aligned} Ax &= \Gamma(2\alpha) \lim_{t\to 0+} \frac{t^{1-\alpha}T_\alpha(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha} \\ &= \Gamma(2\alpha) \lim_{t\to 0+} t^{1-2\alpha}\left(T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x\right) \\ &= \Gamma(2\alpha) \lim_{t\to 0+} t^{1-2\alpha} I_t^\alpha T_\alpha(t)y \\ &= y. \end{aligned} \quad (8)$$

The closeness of A is proved.

For every $x \in X$, set $x_t = I_t^\alpha T_\alpha(t)x$; from part (b) we know that $x_t \in D(A)$, and $\Gamma(\alpha)t^{1-2\alpha}x_t \to x$ as $t \to 0+$. Thus $\overline{D(A)} = X$. □

Theorem 3.2. (I) *$T_\alpha(t)x$ is a strong solution of (1) for every $x \in D(A)$.*
(II) *$T_\alpha(t)x$ is a mild solution of (1) for every $x \in X$.*

Proof. (I) For each $x \in D(A)$, by part (a) of Theorem 3.1, $T_\alpha(t)x \in C(R_+, D(A))$. Using part (c) of Theorem 3.1, we have

$$T_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + I_t^\alpha T_\alpha(t)Ax,$$

and then

$$\begin{aligned} D_t^\alpha T_\alpha(t)x &= D_t^\alpha\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}x + I_t^\alpha T_\alpha(t)Ax\right) \\ &= D_t^\alpha \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + D_t^\alpha I_t^\alpha T_\alpha(t)Ax \\ &= T_\alpha(t)Ax \\ &= AT_\alpha(t)x. \end{aligned} \quad (9)$$

Hence $D_t^\alpha T_\alpha(t)x \in C(R_+; X)$, i.e. $(\phi_{1-\alpha} * T_\alpha)(t)x \in C^1(R_+; X)$. This completes the proof, so (I) holds.
(II) It follows from Definition 2.2 and part (b) of Theorem 3.1 that (II) holds. □

Example 3.1. Consider the fractional evolution equation for $u = u(x, t)$

$$D_t^\alpha u = k^2 u_{xx}, \quad 0 < x < 1, t > 0, 0 < \alpha < 1, k \in \mathbb{R}, \quad (10)$$

with conditions $u(0, t) = u(1, t) = 0$, and $\lim_{t\to 0+} \Gamma(\alpha)t^{1-\alpha}u(x, t) = f(x)$.

Let $X = L^2(0, 1)$, and $W^{2,2}(0, 1) = \{\varphi | \exists h \in L^2(0, 1) : \varphi(t) = ct + t * h(t), t \in (0, 1)\}$; note that $c = \varphi'(0)$, $h(t) = \varphi''(t)$. Let $A = k^2 \frac{\partial^2}{\partial x^2}$, and $D(A) = \{v | v \in W^{2,2}(0, 1), v(0) = v(1) = 0\}$. It is clear that A has eigenvalues $\lambda_n = -k^2n^2\pi^2$ and eigenfunctions $v(x) = \sin n\pi x$, $n \in \mathbb{N}$. If $f(x) = \sum_{n=1}^\infty c_n \sin n\pi x$, then the solution to (10) is $u(x, t) = \sum_{n=1}^\infty t^{\alpha-1}E_{\alpha,\alpha}(-k^2n^2\pi^2t^\alpha)c_n \sin n\pi x$, and from Definition 3.1, it follows that A is the infinitesimal generator of an α-order fractional resolvent $T_\alpha(t)$, $T_\alpha(t)f(x) = u(x, t)$.

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