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Lowen *LM*-fuzzy topological spaces☆

Hai-Yang Li[a,b,*], Ji-Gen Peng[a]

[a] *Department of Mathematics, Xi'an Jiaotong University, Xi'an 710049, PR China*
[b] *School of Science, Xi'an Polytechnic University, Xi'an 710048, PR China*

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Abstract

The aim of this paper is to define and study Lowen *LM*-fuzzy topological spaces. We discuss the basic properties of Lowen *LM*-fuzzy topological spaces, and introduce the notions of interior Lowen topology and exterior Lowen topology of an *LM*-fuzzy topology and prove that **LLM-FTop** (the category of Lowen *LM*-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in **SLM-FTop** (the category of stratified *LM*-fuzzy topological spaces). Moreover, we also prove that an *LM*-fuzzy topological space is an induced *LM*-fuzzy topological space iff it is a Lowen *LM*-fuzzy topological space and an (IC)*LM*-fuzzy topological space.
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1. Introduction

Since Chang [2] introduced fuzzy set theory into topology, many authors have discussed various aspects of fuzzy topology. However, in a completely different direction, Höhle [6] created the notion of a topology being viewed as an *L*-subset of a powerset (in his case, 2^X). Then Kubiak [8] and Šostak [20] independently extended Höhle's notion to *L*-subsets of L^X. Subsequently, Kubiak and Šostak [9] further extended this notion to *M*-subsets of L^X, thereby creating *LM*-fuzzy topologies and the category **LM-FTop** of *LM*-fuzzy topological spaces and fuzzy continuous mappings (*L* and *M* are distinct complete lattices).

In 2007, Yue [22] studied stratified, induced and weakly induced *LM*-fuzzy topological spaces, and proved that **WILM-FTop** (the category of weakly induced *LM*-fuzzy topological spaces and fuzzy continuous mappings) is a simultaneously reflective and coreflective full subcategory of **LM-FTop**. Recently, Li and Peng [12] have studied the properties of (IC)*LM*-topological spaces, which is the generalization of (weakly) induced *LM*-topological spaces, and introduced the notions of interior (IC)-fication and exterior (IC)-fication of *LM*-fuzzy topologies and showed that**ICLM-FTop** (the category of (IC)*LM*-fuzzy topological spaces and fuzzy continuous mappings) is an

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* Corresponding author at: School of Science, Xi'an Polytechnic University, Xi'an 710048, PR China.
E-mail address: fplihaiyang@126.com (H.-Y. Li).

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isomorphism-closed full proper subcategory of **LM-FTop** and **ICLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **LM-FTop**.

In 1982, Lowen [14] introduced a very important class of fuzzy topological spaces, called fuzzy neighborhood spaces. Lowen's notion has been extended to the L-fuzzy setting (called Lowen space) by Liu and Zhang [16] for L, an arbitrary completely distributive lattice. This kind of L-topological space has received wide attention for categorical properties [13,15,16,21,24]. The aim of the present paper is to define and study Lowen LM-fuzzy topological spaces. We discuss the basic properties of Lowen LM-fuzzy topological spaces, and introduce the notions of interior Lowen topology and exterior Lowen topology of LM-fuzzy topology and prove that **LLM-FTop** (the category of Lowen LM-fuzzy topological spaces and fuzzy continuous mappings) is an isomorphism-closed full proper subcategory of **SLM-FTop** (the category of stratified LM-fuzzy topological spaces and fuzzy continuous mappings) and **LLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **SLM-FTop**. Moreover, we prove that (X, τ) is an induced LM-fuzzy topological space iff (X, τ) is a Lowen LM-fuzzy topological space and an (IC)LM-fuzzy topological space.

For needed categorical notions, please refer to [1,7,17,19].

2. Preliminaries

We now give some definitions and results to be used in this paper. Let L be a complete lattice. An element $r \in L - \{0\}$ is called a coprime element if, for any finite subset $K \subset L$ satisfying $r \leq \bigvee K$ (the supremum of K), there exists a $k \in K$ such that $r \leq k$. An element $s \in L - \{1\}$ is called a prime element if it is a coprime element of L^{op} (the opposite lattice [4] of L). The set of all coprime elements (resp., prime elements) of L will be written as $Copr(L)$ (resp., $Pr(L)$). We say that a is a way-below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$), if for every directed (arbitrary) subset $D \subset L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. From [5], we know that $Copr(L)$ is a join-generating set of L if L is a completely distributive lattice. Hence every element in L is also the supremum of all the coprimes wedge below it. By the definition of completely distributive lattice it is easy to see that a complete lattice L is completely distributive iff the operator $\bigvee : Low(L) \to L$ taking every lower set to its supremum has a left adjoint β, and in the case, $\beta(a) = \{b | b \lhd a\}$ for all $a \in L$. Hence, the wedge below relation has the interpolation property in a completely distributive lattice, that is, $a \lhd b$ implies that there is some $c \in L$ such that $a \lhd c \lhd b$.

The way-below relation on a completely distributive lattice L is called locally multiplicative [23] if for every coprime $a \in Copr(L)$, $a \ll b$ and $a \ll c$ imply $a \ll b \wedge c$ for all $b, c \in L$. If a is a coprime, then $a \ll b$ if and only if $a \lhd b$ (see [5,23]). Hence L is locally multiplicative if for every coprime a, $a \lhd b$ and $a \lhd c$ imply $a \lhd b \wedge c$ for all $b, c \in L$. Clearly, [0,1] and power set lattice 2^X are locally multiplicative.

Throughout this paper, L always stands for a completely distributive lattice with locally multiplicative property and M is a completely distributive lattice. Apparently, the powerset L^X (i.e. the set of all L-subset) with the point-wise order is also a completely distributive lattice with locally multiplicative property, in which the least element and the greatest element of L^X will be written as 0_X and 1_X. Let 1_U denote the characteristic function of $U \in 2^X$, and $[a]$ denote the L-subset taking constant value a, where 2^X is the power set of X and $a \in L$. An L-subset with form $[a] \wedge 1_U$ is called one-step L-subset. Let S denote the set of all the one-step L-subsets of L^X. Each mapping $f : X \longrightarrow Y$ induces a mapping $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-forward powerset operator, cf. [18]), defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) | f(x) = y\} \quad (\forall A \in L^X,\ \forall y \in Y).$$

The right adjoint to $f_L^{\rightarrow}$ (called L-backward powerset operator, cf. [18]) is denoted as $f_L^{\leftarrow}$ and given by

$$f_L^{\leftarrow}(B) = \bigvee\{A \in L^X | f_L^{\rightarrow}(A) \leq B\} = B \circ f \quad (\forall B \in L^Y).$$

It is known that $f_L^{\rightarrow}$ preserves arbitrary unions and that $f_L^{\leftarrow}$ preserves arbitrary unions, arbitrary intersections, and complements when they exist (canonical examples of such morphisms are given in [18]).

For $A \in L^X$ and $p \in L$, the mapping $\hat{\imath}_p : L^X \longrightarrow 2^X$ is defined by $\hat{\imath}_p(A) = \{x | A(x) \lhd p\}$, called the strong p-cut of A. If L is a locally multiplicative completely distributive lattice, then it is easy to verify the following

Lemma 2.1 (*Li [10], Yue [22]*).

(1) *For each* $p \in L$, $\hat{\imath}_p : L^X \longrightarrow 2^X$ *preserves arbitrary suprema and finite meets.*

(2) *For each* $p \in L$, $A \in L^X$, $B \in L^Y$ *and any mapping* $f : X \to Y$, *we have* $\hat{\imath}_p(f_L^{\rightarrow}(A)) = f(\hat{\imath}_p(A))$, $\hat{\imath}_p(f_L^{\leftarrow}(B)) = f(\hat{\imath}_p(B))$.
(3) $A = \bigvee_{p \in L}([p] \wedge 1_{\hat{\imath}_p(A)}) = \bigvee_{p \in Copr(L)}([p] \wedge 1_{\hat{\imath}_p(A)})$.

An *LM*-fuzzy topology [9] on a set *X* is defined to be a mapping $\tau : L^X \longrightarrow M$ satisfying:

(FT1) $\tau(1_X) = \tau(0_X) = 1$;
(FT2) $\forall A, B \in L^X$, $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$;
(FT3) $\tau(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \tau(A_t)$, for every family $\{A_t | t \in T\} \subset L^X$.

The value $\tau(A)$ can be interpreted as the degree of openness of *A*. A fuzzy continuous mapping between *LM*-fuzzy topological spaces is a mapping $f : (X, \tau) \longrightarrow (Y, \delta)$ such that $\delta(B) \leq \tau(f_L^{\leftarrow}(B))$ for all $B \in L^Y$. When $L = \{0, 1\}$, this definition reduced to that of *M*-fuzzying topology.

In the following of this section, we give some definitions and results about *LM*-fuzzy topology.

Definition 2.2 (*Fang and Yue [3]*). Let τ be an *LM*-fuzzy topology on *X*.

(1) $\mathcal{B} : L^X \longrightarrow M$ is called a base of τ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in L^X, \quad \tau(A) = \bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda),$$

where the expression $\bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.
(2) $\phi : L^X \longrightarrow M$ is called a subbase of τ if $\phi^{(\sqcap)} : L^X \longrightarrow M$ is a base of τ, where

$$\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\lambda \in J} B_\lambda = A} \bigwedge_{\lambda \in J} \phi(B_\lambda)$$

for all $A \in L^X$ with $(\sqcap)$ standing for "finite intersection".

Lemma 2.3 (*Fang and Yue [3]*). *Let* τ *be an LM-fuzzy topology on X. Then* $\phi : L^X \longrightarrow M$ *is the subbase of* τ *if and only if* $\phi^{(\sqcup)}(1_X) = 1$.

Definition 2.4 (*Yue [22]*). Let (X, τ) be an *LM*-fuzzy topological space. If $\tau([a]) = 1$ for all $a \in L$, then (X, τ) is called a stratified *LM*-fuzzy topological space.

It is easy to verify that (X, τ) is a stratified *LM*-fuzzy topological space iff $\tau([a]) = 1$ for all $a \in Copr(L)$. Let **SLM-FTop** denote the category of stratified *LM*-fuzzy topological spaces and fuzzy continuous mappings.

Lemma 2.5. *Let* (X, τ) *be a stratified LM-fuzzy topological space. Then* $\tau([a] \wedge 1_U) \leq \tau([b] \wedge 1_U)$ $(a, b \in L, b \lhd a$ *and* $U \subset X)$.

Proof. Since (X, τ) is stratified, we have

$$\tau([b] \wedge 1_U) = \tau([b] \wedge ([a] \wedge 1_U)) \geq \tau([b]) \wedge \tau([a] \wedge 1_U) = \tau([a] \wedge 1_U). \qquad \square$$

Definition 2.6 (*Yue [22]*). Let (X, τ) be an *LM*-fuzzy topological space. If $\tau(A) = \bigwedge_{r \in Copr(L)} \tau(1_{\hat{\imath}_r(A)})$ holds for all $A \in L^X$, then (X, τ) is called an induced *LM*-fuzzy topological space.

Definition 2.7 (*Yue [22]*). Let (X, τ) be an *LM*-fuzzy topological space. If $\tau(A) \leq \bigwedge_{r \in Copr(L)} \tau(1_{\hat{\imath}_r(A)})$ holds for all $A \in L^X$, then (X, τ) is called a weakly induced *LM*-fuzzy topological space. Let **WILM-FTop** denote the category of weakly induced *LM*-fuzzy topological spaces and fuzzy continuous mappings.

Definition 2.8 (*Li and Peng [12]*). Let (X,τ) be an *LM*-fuzzy topological space. If $\tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$ holds for all $A \in L^X$, then (X,τ) is called an (IC)*LM*-fuzzy topological space. Let **ICLM-FTop** denote the category of (IC)*LM*-fuzzy topological spaces and fuzzy continuous mappings.

It is easy to verify that induced and weakly induced *LM*-fuzzy topological spaces are (IC)*LM*-fuzzy topological space.

Definition 2.9 (*Fang and Yue [3]*). Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of *LM*-fuzzy topological spaces and $p_t : \prod_{t\in T} X_t \longrightarrow X_t$ be the projection. Then the *LM*-fuzzy topology on $\prod_{t\in T} X_t$ whose subbase is defined by

$$\forall A \in L^{\prod_{t\in T} X_t}, \quad \phi(A) = \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(B)$$

is called the product *LM*-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\prod_{t\in T}\tau_t$, and $(\prod_{t\in T} X_t, \prod_{t\in T}\tau_t)$ is called the product space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.10 (*Yue [22]*). Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of *LM*-fuzzy topological spaces, different X_ts be disjoint and $X = \bigcup_{t\in T} X_t$, and let $\tau : L^X \longrightarrow M$ be defined as follows:

$$\forall A \in L^X, \quad \tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t).$$

Then it is easy to verify that τ is an *LM*-fuzzy topology on *X*, and τ is called the sum *LM*-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\bigoplus_{t\in T}\tau_t$. $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T}\tau_t)$ is called the sum space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.11 (*Yue [22]*). Let (X,τ) be an *LM*-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. It is easy to verify that $\tau/f_L^{\rightarrow} : L^Y \longrightarrow M$ is an *LM*-fuzzy topology on *Y*, where $\tau/f_L^{\rightarrow}$ is defined by

$$\forall A \in L^Y, \quad \tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)).$$

$\tau/f_L^{\rightarrow}$ is called the *LM*-fuzzy quotient topology of τ with respect to *f*, and $(Y, \tau/f_L^{\rightarrow})$ is called the *LM*-fuzzy quotient space of (X,τ) with respect to *f*.

Definition 2.12 (*Rodabaugh [18]*). Let (X,τ) be an *LM*-fuzzy topological space and $Y \subset X$. We call $(Y, \tau|Y)$ a subspace of (X,τ), where $\tau|Y : L^Y \longrightarrow M$ is defined by

$$\forall B \in L^Y, \quad \tau|Y(B) = \bigvee\{\tau(A) | A \in L^X, A|Y = B\}.$$

Lemma 2.13 (*Yue [22]*). *Let $f : (X,\tau) \longrightarrow (Y,\delta)$ be a mapping and ϕ be a subbase of δ. If $\tau(f_L^{\leftarrow}(B)) \geq \phi(B)$ for all $B \in L^Y$, then f is fuzzy continuous.*

3. Lowen *LM*-fuzzy topological spaces

In the section, we mainly introduce the definition of a Lowen *LM*-fuzzy topological space and discuss its basic properties such as the properties that the product space and the sum space of Lowen *LM*-fuzzy topological spaces are also Lowen *LM*-fuzzy topological spaces. Moreover, we show that (X,τ) is an induced *LM*-fuzzy topological space iff (X,τ) is a Lowen *LM*-fuzzy topological space and an (IC)*LM*-fuzzy topological space.

Definition 3.1. An *LM*-fuzzy topological space (X,τ) is said to be a Lowen *LM*-fuzzy topological space iff for all $A \in L^X$, $\tau(A) = \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\iota}_a(A)})$. Let **LLM-FTop** denote the category of Lowen *LM*-fuzzy topological spaces and fuzzy continuous mappings.

Example 3.2. Let $L = M = 2^X$. Then L is not only a complete Boolean algebra but a locally multiplicative completely distributive lattice. We consider the map $\tau : L^X \longrightarrow M$ defined by

$$\tau(A) = \left(\bigwedge_{x\in X} A(x)\right) \vee \left(\left(\bigvee_{x\in X} A(x)\right) \to 0\right).$$

Then τ is a stratified *LM*-fuzzy topology on X (See [7, Example 4.2.1(b)]). We will show that τ is a Lowen *LM*-fuzzy topology on X in the following.

For any $A \in L^X$ and any $t \in Copr(L)$, if $\hat{\imath}_t(A) = X$ or ϕ, then

$$\tau([t] \wedge 1_{\hat{\imath}_t(A)}) = 1 \geq \tau(A).$$

If $\hat{\imath}_t(A) \neq X$ and $\hat{\imath}_t(A) \neq \phi$, then there exist $x_1,\ x_2 \in X$ such that $t \lhd A(x_1)$ and $t \ntriangleleft A(x_2)$, which imply that $A'(x_1) \leq t'$ and $A(x_2) \leq t'$, and hence

$$\tau(A) = \left(\bigwedge_{x\in X} A(x)\right) \vee \left(\left(\bigvee_{x\in X} A(x)\right) \to 0\right) \leq t'.$$

Note

$$\tau([t] \wedge 1_{\hat{\imath}_t(A)}) = \left(\bigwedge_{x\in X} ([t] \wedge 1_{\hat{\imath}_t(A)})(x)\right) \vee \left(\left(\bigvee_{x\in X} ([t] \wedge 1_{\hat{\imath}_t(A)})(x)\right) \to 0\right) = t'.$$

Therefore for any $A \in L^X$ and any $t \in Copr(L)$, we have $\tau(A) \leq \tau([t] \wedge 1_{\hat{\imath}_t(A)})$, which implies that τ is the Lowen *LM*-fuzzy topology on X by Theorem 3.4(2).

Lemma 3.3. (1) *A Lowen LM-fuzzy topological space (X, τ) is a stratified LM-fuzzy topological space.*
(2) *An induced LM-fuzzy topological space (X, τ) is a Lowen LM-fuzzy topological space.*

Proof.

(1) Since (X, τ) is a Lowen *LM*-fuzzy topological space, we have

$$\tau(1_X) \leq \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\imath}_a(1_X)}) = \bigwedge_{a\in Copr(L)} \tau([a]),$$

and hence $\tau([a]) = 1$ ($\forall a \in Copr(L)$), which implies that (X, τ) is a stratified *LM*-fuzzy topological space.

(2) Since (X, τ) is an induced *LM*-fuzzy topological space, for all $A \in L^X$, and $a \in Copr(L)$, we have

$$\tau(1_{\hat{\imath}_a(A)}) = \tau([a]) \wedge \tau(1_{\hat{\imath}_a(A)}) \leq \tau([a] \wedge 1_{\hat{\imath}_a(A)}),$$

which implies that

$$\tau(A) = \bigwedge_{a\in Copr(L)} \tau(1_{\hat{\imath}_a(A)}) \leq \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\imath}_a(A)}).$$

Hence (X, τ) is a Lowen *LM*-fuzzy topological space by Theorem 3.4(2). □

Theorem 3.4. *Let (X, τ) be an LM-fuzzy topological space and S be the set of all the one-step L-subset of L^X. Then the following are equivalent.*

(1) *(X, τ) is a Lowen LM-fuzzy topological space.*
(2) *For all $A \in L^X$ and $a \in Copr(L)$, $\tau(A) \leq \tau([a] \wedge 1_{\hat{\imath}_a(A)})$.*

(3) *There exists a base* $\mathcal{B}$ *of* τ *satisfying* $\mathcal{B}([a]) = 1$ $(\forall a \in L)$ *and*

$$\forall A \in L^X, \quad \tau(A) = \bigvee \left\{ \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda) \middle| \bigvee_{\lambda \in \Lambda} B_\lambda = A \text{ and } B_\lambda \in S \right\}.$$

(4) *There exists a subbase* ϕ *of* τ *satisfying* $\phi([a]) = 1$ $(\forall a \in L)$ *and*

$$\forall A \in L^X, \quad \phi^{(\sqcap)}(A) = \bigvee \left\{ \bigwedge_{\lambda \in J} \phi(B_\lambda) | (\sqcap)_{\lambda \in J} B_\lambda = A \text{ and } B_\lambda \in S \right\}$$

is a base of τ.

Proof. (1) $\Longleftrightarrow$ (2) follows from the definition of *LM*-fuzzy topology.

(2) $\Longrightarrow$ (3): Let $\mathcal{B} : S \longrightarrow M$ defined by $\mathcal{B}(A) = \tau(A)$ $(\forall A \in S)$. Then $\mathcal{B}([a]) = 1$ $(\forall a \in L)$ by Lemma 3.3 (1). We will show $\mathcal{B}$ is a base of τ in the following. By (2) and the definition of *LM*-fuzzy topology, we have

$$\tau(A) \le \bigwedge_{a \in Copr(L)} \tau([a] \wedge 1_{\hat{\imath}_a(A)}) = \bigwedge_{a \in Copr(L)} \mathcal{B}([a] \wedge 1_{\hat{\imath}_a(A)}) \le \bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda) \ (B_\lambda \in S)$$

$$\le \bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \tau(A) = \tau(A),$$

which implies that $\mathcal{B}$ is a base of τ.

(3) $\Longrightarrow$ (4): It is easily concluded by taking $\phi = \mathcal{B}$.

(4) $\Longrightarrow$ (2): For all $a \in Copr(L)$, $A \in L^X$, we have

$$\tau([a] \wedge 1_{\hat{\imath}_a(A)}) = \bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = [a] \wedge 1_{\hat{\imath}_a(A)}} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\sqcap)_{\beta \in \Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi(C_{\lambda\beta}) \ (C_{\lambda\beta} \in S)$$

$$\ge \bigvee_{\bigvee_{\lambda \in \Lambda} D_\lambda = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\sqcap)_{\beta \in \Lambda_\lambda} E_{\lambda\beta} = [a] \wedge 1_{\hat{\imath}_a(D_\lambda)}} \bigwedge_{\beta \in \Lambda_\lambda} \phi(E_{\lambda\beta}) \ (E_{\lambda\beta} \in S)$$

$$\ge \bigvee_{\bigvee_{\lambda \in \Lambda} D_\lambda = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\sqcap)_{\beta \in \Lambda_\lambda} F_{\lambda\beta} = D_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi([a] \wedge 1_{\hat{\imath}_a(F_{\lambda\beta})}) \ (F_{\lambda\beta} \in S)$$

$$\ge \bigvee_{\bigvee_{\lambda \in \Lambda} D_\lambda = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\sqcap)_{\beta \in \Lambda_\lambda} F_{\lambda\beta} = D_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi(F_{\lambda\beta}) \ge \tau(A). \qquad \square$$

Theorem 3.5. *An LM-fuzzy topological space* (X, τ) *is an induced LM-fuzzy topological space iff* (X, τ) *is a Lowen LM-fuzzy topological space and an (IC)LM-fuzzy topological space*.

Proof. It suffices to show the sufficiency. Since (X, τ) is a Lowen *LM*-fuzzy topological space, we have, for all $A \in L^X$,

$$\tau(A) \le \bigwedge_{a \in Copr(L)} \tau([a] \wedge 1_{\hat{\imath}_a(A)}).$$

By the definition of (IC)*LM*-fuzzy topology,

$$\tau([a] \wedge 1_{\hat{\imath}_a(A)}) \le \tau(1_{\hat{\imath}_a(A)}))$$

and hence

$$\tau(A) \le \bigwedge_{a \in Copr(L)} (\tau(1_{\hat{\imath}_a(A)})).$$

By Lemma 3.3(1) and the definition of Lowen *LM*-fuzzy topology, we have

$$\tau(A) \geq \bigwedge_{a\in Copr(L)} \tau([a]\wedge 1_{\hat{\imath}_a(A)}) \geq \bigwedge_{a\in Copr(L)} \tau(1_{\hat{\imath}_a(A)}).$$

Hence

$$\tau(A) = \bigwedge_{a\in Copr(L)} (\tau(1_{\hat{\imath}_a(A)})),$$

which implies that (X, τ) is an induced *LM*-fuzzy topological space. □

Theorem 3.6. *Let (X, τ) be a Lowen LM-fuzzy topological space and $Y \subset X$. Then the subspace $(Y, \tau|Y)$ of (X, τ) is also a Lowen LM-fuzzy topological space.*

Proof. Since (X, τ) is a Lowen *LM*-fuzzy topological space, for all $A \in L^Y$ and $a \in Copr(L)$, we have

$$\tau|Y(A) = \bigvee\{\tau(B)|B \in L^X,\ B|Y = A\} \leq \bigvee\{\tau(B)|B \in L^X,\ [a]\wedge 1_{\hat{\imath}_a(B)}|Y = [a]\wedge 1_{\hat{\imath}_a(A)}\}$$
$$\leq \bigvee\{\tau([a]\wedge 1_{\hat{\imath}_a(B)})|B \in L^X,\ [a]\wedge 1_{\hat{\imath}_a(B)}|Y = [a]\wedge 1_{\hat{\imath}_a(A)}\} \leq \tau|Y([a]\wedge 1_{\hat{\imath}_a(A)})$$

which implies that $(Y, \tau|Y)$ is a Lowen *LM*-fuzzy topological space. □

Theorem 3.7. *Let (X, τ) be a Lowen LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. Then the LM-fuzzy quotient space $(Y, \tau/f_L^{\rightarrow})$ of (X, τ) with respect to f is also a Lowen LM-fuzzy topological space.*

Proof. Since (X, τ) is a Lowen *LM*-fuzzy topological space, for all $A \in L^Y$ and $a \in Copr(L)$, we have

$$\tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)) \leq \tau([a]\wedge 1_{\hat{\imath}_a(f_L^{\leftarrow}(A))}) = \tau(f_L^{\leftarrow}([a]\wedge 1_{\hat{\imath}_a(A)})) = \tau/f_L^{\rightarrow}([a]\wedge 1_{\hat{\imath}_a(A)})),$$

which implies that $(Y, \tau/f_L^{\rightarrow})$ is a Lowen *LM*-fuzzy topological space. □

Theorem 3.8. *Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of Lowen LM-fuzzy topological spaces. Then the product space $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t\in T}$ is also a Lowen LM-fuzzy topological space.*

Proof. Let ϕ be the subbase of τ. Then, for all $A \in L^X$ and $a \in Copr(L)$, we have

$$\phi(A) = \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(B) \leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}([a]\wedge 1_{\hat{\imath}_a(B)})=[a]\wedge 1_{\hat{\imath}_a(A)}} \tau_t(B) \leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}([a]\wedge 1_{\hat{\imath}_a(B)})=[a]\wedge 1_{\hat{\imath}_a(A)}} \tau_t([a]\wedge 1_{\hat{\imath}_a(B)})$$
$$= \phi([a]\wedge 1_{\hat{\imath}_a(A)}).$$

Hence, for all $A \in L^X$ and $a \in Copr(L)$, $\tau(A) \leq \tau([a]\wedge 1_{\hat{\imath}_a(A)})$, which implies that $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ is a Lowen *LM*-fuzzy topological space. □

Theorem 3.9. *Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of Lowen LM-fuzzy topological spaces, different X_t's be disjoint. Then the sum space $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t\in T}$ is a Lowen LM-fuzzy topological space iff for all $t \in T$, (X_t, τ_t) is a Lowen LM-fuzzy topological space.*

Proof. *Necessity*: For all $t \in T$, $A_t \in L^{X_t}$ and $a \in Copr(L)$, we have

$$\tau_t(A_t) = \bigwedge_{t\in T} \tau_t(A*|X_t) = \tau(A*) \leq \tau([a]\wedge 1_{\hat{\imath}_a(A*)}) = \bigwedge_{t\in T} \tau_t([a]\wedge 1_{\hat{\imath}_a(A*)}|X_t) = \tau_t([a]\wedge 1_{\hat{\imath}_a(A)}),$$

where

$$A*(x) = \begin{cases} A(x), & x \in X_t \\ 0, & x \notin X_t \end{cases}$$

Thus for every $t \in T$, (X_t, τ_t) is a Lowen *LM*-fuzzy topological space.

Sufficiency: For every $A \in L^X$ and $a \in Copr(L)$, we have

$$\tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t) \le \bigwedge_{t\in T} \tau_t([a] \wedge 1_{\hat{\iota}_a(A|X_t)}) = \bigwedge_{t\in T} \tau_t([a] \wedge 1_{\hat{\iota}_a(A)}|X_t) = \tau([a] \wedge 1_{\hat{\iota}_a(A)}),$$

which implies that $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} \tau_t)$ is a Lowen *LM*-fuzzy topological space. □

4. Interior and exterior Lowen topology of an *LM*-fuzzy topology

In the section, we mainly study some categorical properties of *LM*-fuzzy topological spaces. Especially, based on the interior and exterior Lowen topology of an *LM*-fuzzy topology, we prove that **LLM-FTop** (the category of Lowen *LM*-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in **SLM-FTop** (the category of stratified *LM*-fuzzy topological spaces). For this we first prove Lemmas 4.1 and 4.2.

Lemma 4.1. *Let (X, τ) be a stratified LM-fuzzy topological space and $I_L(\tau) : L^X \longrightarrow M$ be defined by*

$$I_L(\tau)(A) = \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\iota}_a(A)})\ (\forall A \in L^X).$$

Then $I_L(\tau)$ is the largest Lowen LM-fuzzy topology on X which is contained in τ. We call $I_L(\tau)$ the interior Lowen topology of τ.

Proof. Clearly, $I_L(\tau)(A) \le \tau(A)$ for every $A \in L^X$. By Lemma 2.5 and the definition of I_L, for every $a \in Copr(L)$ and $A \in L^X$, we have

$$I_L(\tau)([a] \wedge 1_{\hat{\iota}_a(A)}) = \bigwedge_{b\lhd a, b\in Copr(L)} \tau([b] \wedge 1_{\hat{\iota}_a(A)}) = \tau([a] \wedge 1_{\hat{\iota}_a(A)}).$$

Hence

$$I_L(\tau)(A) = \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\iota}_a(A)}) = \bigwedge_{a\in Copr(L)} I_L(\tau)([a] \wedge 1_{\hat{\iota}_a(A)}),$$

which implies that $I_L(\tau)$ is the Lowen *LM*-fuzzy topology on *X* which is contained in τ. Set $\delta \le \tau$ and δ be a Lowen *LM*-fuzzy topology on *X*. Then, for all $A \in L^X$,

$$\delta(A) = \bigwedge_{a\in Copr(L)} \delta([a] \wedge 1_{\hat{\iota}_a(A)}) \le \bigwedge_{a\in Copr(L)} \tau([a] \wedge 1_{\hat{\iota}_a(A)}) = I_L(\tau)(A).$$

Therefore, $I_L(\tau)$ is the largest Lowen *LM*-fuzzy topology on *X* which is contained in τ. □

Let (X, τ) be an *LM*-fuzzy topological space and $\phi^\tau : L^X \longrightarrow M$ be defined by

$$\phi^\tau(A) = \begin{cases} \bigvee_{a\lhd r}(\bigvee\{\tau(B)|\hat{\iota}_r(B) = U\}), & A = [a] \wedge 1_U\ (\forall a \in L, r \in Copr(L) \text{ and } U \subset X) \\ \tau(A) & \text{others} \end{cases}$$

It is easy to verify that ϕ^τ is a subbase of one *LM*-fuzzy topology on *X*. We denote this *LM*-fuzzy topology by $E_L(\tau)$ and call it the exterior Lowen topology of τ (see Lemma 4.2).

Lemma 4.2. *Let (X, τ) be an LM-fuzzy topological space. Then $E_L(\tau)$ is the smallest Lowen LM-fuzzy topology on X which contains τ.*

Proof. Firstly, we show that $E_L(\tau)$ is the Lowen *LM*-fuzzy topology on *X*. In fact, for every $A \in L^X$ and $a \in Copr(L)$, we have

$$E_L(\tau)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(C_{\lambda\beta}),$$

and

$$E_L(\tau)([a]\wedge 1_{\hat{\imath}_a(A)})=\bigvee_{\bigvee_{\lambda\in\Lambda}B_\lambda=[a]\wedge 1_{\hat{\imath}_a(A)}}\bigwedge_{\lambda\in\Lambda}\bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda}C_{\lambda\beta}=B_\lambda}\bigwedge_{\beta\in\Lambda_\lambda}\phi^\tau(C_{\lambda\beta})$$
$$\geq\bigvee_{\bigvee_{\lambda\in\Lambda}B_\lambda=A}\bigwedge_{\lambda\in\Lambda}\bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda}C_{\lambda\beta}=B_\lambda}\bigwedge_{\beta\in\Lambda_\lambda}\phi^\tau([a]\wedge 1_{\hat{\imath}_a(C_{\lambda\beta})}).$$

If $C_{\lambda\beta}$ is not a one-step L-subset, then

$$\phi^\tau([a]\wedge 1_{\hat{\imath}_a(C_{\lambda\beta})})=\bigvee_{a\lhd r}\left(\bigvee\{\tau(B)|\hat{\imath}_r(B)=\hat{\imath}_a(C_{\lambda\beta})\}\right)\geq\tau(C_{\lambda\beta})=\phi^\tau(C_{\lambda\beta}).$$

If $C_{\lambda\beta}$ is a one-step L-subset and let $C_{\lambda\beta}=[b]\wedge 1_V$, then

$$\phi^\tau(C_{\lambda\beta})=\bigvee_{b\lhd r}\left(\bigvee\{\tau(B)|\hat{\imath}_r(B)=V\}\right)$$

and, for all $a\lhd b$,

$$\phi^\tau([a]\wedge 1_{\hat{\imath}_a(C_{\lambda\beta})})=\phi^\tau([a]\wedge 1_V)=\bigvee_{a\lhd r}\left(\bigvee\{\tau(B)|\hat{\imath}_r(B)=V\}\right)\geq\phi^\tau(C_{\lambda\beta}).$$

Hence $E_L(\tau)(A)\leq E_L(\tau)([a]\wedge 1_{\hat{\imath}_a(A)})$, which implies that $E_L(\tau)$ is the Lowen *LM*-fuzzy topology on X which contains τ.

Secondly, we show that $E_L(\tau)$ is the smallest Lowen *LM*-fuzzy topology on X which contains τ. Let $\tau\leq\eta$ and η be a Lowen *LM*-fuzzy topology on X. We need to prove that $E_L(\tau)\leq\eta$. It suffices to show that $\phi^\tau(A)\leq\eta(A)$ for all $A\in L^X$. It is clear that $E(\tau)\leq\eta$ when A is not a one-step L-subset. If A is a one-step L-subset and let $A=[b]\wedge 1_U$, then

$$\phi^\tau(A)=\bigvee_{b\lhd r}\left(\bigvee\{\tau(B)|\hat{\imath}_r(B)=U\}\right)\leq\bigvee_{b\lhd r}\left(\bigvee\{\eta(B)|\hat{\imath}_r(B)=U\}\right)\leq\bigvee_{b\lhd r}\left(\bigvee\{\eta([r]\wedge 1_U)|\hat{\imath}_r(B)=U\}\right)$$
$$\leq\bigvee_{b\lhd r}\eta([r]\wedge 1_U)\leq\eta([b]\wedge 1_U)=\eta(A),$$

and thus $E(\tau)\leq\eta$. □

Corollary 4.3. (X,δ) *is a Lowen LM-fuzzy topological space iff any two of* $E_L(\delta)$, $I_L(\delta)$, δ *are equal* (*equivalently, all the three are equal*).

Lemma 4.4.

(1) *Let* (X,δ) *be a stratified LM-fuzzy topological space and* (Y,τ) *be a Lowen LM-fuzzy topological space. Then* $f:(X,\delta)\longrightarrow(Y,\tau)$ *is a fuzzy continuous mapping iff* $f:(X,I_L(\delta))\longrightarrow(Y,I_L(\tau))=(Y,\tau)$ *is a fuzzy continuous mapping.*
(2) *Let* (X,δ) *be an LM-fuzzy topological space and* (Y,τ) *be a Lowen LM-fuzzy topological space. Then* $f:(Y,\tau)\longrightarrow(X,\delta)$ *is a fuzzy continuous mapping iff* $f:(Y,E_L(\tau))=(Y,\tau)\longrightarrow(X,E_L(\delta))$ *is a fuzzy continuous mapping.*

Proof.

(1) The sufficiency is obvious and we need to show the necessity. Suppose that $f:(X,\delta)\longrightarrow(Y,\tau)$ is fuzzy continuous, i.e., $\tau(B)\leq\delta(f_L^\leftarrow(B))$ for every $B\in L^Y$. Since (Y,τ) is a Lowen *LM*-fuzzy topological space, we have

$$\tau(B)=\bigwedge_{a\in Copr(L)}\tau([a]\wedge 1_{\hat{\imath}_a(B)})\leq\bigwedge_{a\in Copr(L)}\delta(f_L^\leftarrow([a]\wedge 1_{\hat{\imath}_a(B)}))$$
$$=\bigwedge_{a\in Copr(L)}\delta([a]\wedge 1_{\hat{\imath}_a(f_L^\leftarrow(B))})=I_L(\delta)(f_L^\leftarrow(B)),$$

which implies that $f:(X,I_L(\delta))\longrightarrow(Y,I(\tau))=(Y,\tau)$ is a fuzzy continuous mapping.

(2) The sufficiency is obvious and we need to show the necessity. It suffices to show for all $A = [a] \wedge 1_U$ ($\forall a \in L$ and $U \subset X$), $\phi^{\delta}(A) \leq \tau(f_L^{\leftarrow}(A))$ by the definition of $E_L(\delta)$ and Lemma 2.13. Suppose that $f : (Y, \tau) \longrightarrow (X, \delta)$ is a fuzzy continuous mapping, i.e., $\delta(A) \leq \tau(f_L^{\leftarrow}(A))$ for all $A \in L^X$. Since (Y, τ) is a Lowen *LM*-fuzzy topological space, we have

$$\tau(A) = \bigwedge_{a \in Copr(L)} \tau([a] \wedge 1_{\hat{\iota}_a(A)}),$$

and hence

$$\begin{aligned}\phi^{\delta}(A) &= \bigvee_{a \lhd r} \left(\bigvee\{\delta(B) | \hat{\iota}_r(B) = U\}\right) \leq \bigvee_{a \lhd r} \left(\bigvee\{f_L^{\leftarrow}(B) | \hat{\iota}_r(B) = U\}\right) \\ &= \bigvee_{a \lhd r} \tau([r] \wedge 1_{f^{-1}(U)}) \leq \tau([a] \wedge 1_{f^{-1}(U)}) = \tau(f_L^{\leftarrow}(A)),\end{aligned}$$

which implies that $f : (Y, E_L(\tau)) = (Y, \tau) \longrightarrow (X, E_L(\delta))$ is a fuzzy continuous mapping. □

Let **i** : **LLM-FTop** $\longrightarrow$ **SLM-FTop** (**LM-FTop**) be the inclusion functor. By proof of Lemma 4.4, we may show the following

Theorem 4.5. (1) $\mathbf{I_L}$: **SLM-FTop** $\longrightarrow$ **LLM-FTop** *is functor and* $\mathbf{I_L} \dashv \mathbf{i}$.
(2) $\mathbf{E_L}$: **LM-FTop** $\longrightarrow$ **LLM-FTop** *is functor and* $\mathbf{i} \dashv \mathbf{E_L}$.

Corollary 4.6. **LLM-FTop** *is an isomorphism-closed full proper subcategory of* **SLM-FTop** *which is simultaneously bireflective and bicoreflective in* **SLM-FTop**, *and given a stratified LM-fuzzy topological space* (X, δ), *its reflection and coreflection are given by* $id_X : (X, \delta) \longrightarrow (X, I_L(\delta))$ *and* $id_X : (X, E_L(\delta)) \longrightarrow (X, \delta)$, *respectively*, *where* $id_X : X \longrightarrow X$ *is the identity mapping*.

As every right adjoint preserves limits and every left adjoint preserves colimits, we have the following Corollaries.

Corollary 4.7.

(1) *Let* $\{(X_i, \tau_i)\}_{i \in I}$ *be a family of LM-fuzzy topological spaces*. *Then*

$$E_L\left(\prod_{i \in I} X_i, \prod_{i \in I} \delta_i\right) = \left(\prod_{i \in I} X_i, \prod_{i \in I} E_L(\delta_i)\right).$$

(2) *Let* $\{(X_i, \tau_i)\}_{i \in I}$ *be a family of stratified LM-fuzzy topological spaces*, *different* $X_t s$ *be disjoint*. *Then*

$$I_L\left(\bigoplus_{i \in I} X_i, \bigoplus_{i \in I} \delta_i\right) = \left(\bigoplus_{i \in I} X_i, \bigoplus_{i \in I} I_L(\delta_i)\right).$$

Corollary 4.8. (1) *Let* (X, δ) *be an LM-fuzzy topological space and* $Y \subset X$. *Then* $E_L(\delta|Y) = E_L(\delta)|Y$.
(2) *Let* (X, δ) *be a stratified LM-fuzzy topological space and* $f : X \longrightarrow Y$ *be a surjective mapping*. $(Y, \delta/f_L^{\rightarrow})$ *is the LM-fuzzy quotient space of* (X, δ) *with respect to f*. *Then* $I_L(\delta/f_L^{\rightarrow}) = I_L(\delta)/f_L^{\rightarrow}$.

Theorem 4.9. (1) *Let* (X, τ) *be a stratified LM-fuzzy topological space and* $U \subset X$. *Then* $I_L(\tau)|U \leq I_L(\tau|U)$.
(2) *Let* (X, δ) *be an LM-fuzzy topological space and* $(Y, \delta/f_L^{\rightarrow})$ *be the LM-fuzzy quotient space of* (X, δ) *with respect to f*. *Then* $E_L(\delta/f_L^{\rightarrow}) \leq E_L(\delta)/f_L^{\rightarrow}$.

Proof.

(1) We have $I_L(\tau)|U \leq I_L(\tau|U)$ by Theorem 3.6 and the definition of $I_L(\delta)$.
(2) We have $E_L(\delta/f_L^{\rightarrow}) \leq E_L(\delta)/f_L^{\rightarrow}$ by Theorem 3.7 and the definition of $E_L(\delta)$. □

Remark 4.10. The following two counterexamples show that the above inequalities in Theorem 4.9 cannot be replaced by equalities.

(1) Let $X = L = M = [0, 1]$, $U = [0, 0.5]$ and (X, τ) be a stratified *LM*-fuzzy topological space, where $\tau([a] \wedge 1_{[0.25,0.5]}) = \tau(1_{[0.25,0.5]} \vee ([0.5] \wedge 1_{[0,0.25]})) = 1$ $(\forall a \in L)$ and for others $A \in L^X$, $\tau(A) = 0$.
It is easy to verify that $I_L(\tau) : L^X \longrightarrow M$ is defined by $I_L(\tau)([a] \wedge 1_{[0.25,0.5]}) = I_L(\tau)([a]) = 1$ $(\forall a \in L)$ and for others $A \in L^X$, $I_L(\tau)(A) = 0$. On the one hand, $I_L(\tau)|U(1_{[0.25,0.5]} \vee ([0.5] \wedge 1_{[0,0.25]})) = 0$. On the other hand, $I_L(\tau|U)(1_{[0.25,0.5]} \vee ([0.5] \wedge 1_{[0,0.25]})) = 1$. Therefore, $I_L(\tau)|U < I_L(\tau|U)$.
(2) Let $X = [-1, 1]$, $Y = L = M = [0, 1]$, $\delta : L^X \longrightarrow M$ is defined by $\delta(0_X) = \delta(1_X) = \delta([0.5] \wedge 1_{[0.5,1]}) = \delta(1_{[-1,-0.5]}) = 1$.
It is easy to verify that $\delta/f_L^{\rightarrow} : L^Y \longrightarrow M$ is defined by $\delta/f_L^{\rightarrow}(0_Y) = \delta/f_L^{\rightarrow}(1_Y) = 1$ and for others $B \in L^Y$, $\delta/f_L^{\rightarrow}(B) = 0$. Hence $E_L(\delta/f_L^{\rightarrow})([0.5] \wedge 1_{[0.5,1]}) = 0$. On the other hand, $E_L(\delta)/f_L^{\rightarrow}([0.5] \wedge 1_{[0.5,1]}) = E_L(\delta)(f_L^{\leftarrow}([0.5] \wedge 1_{[0.5,1]})) = E_L(\delta)([0.5] \wedge 1_{[0.5,1] \cup [-1,-0.5]})) = 1$. Therefore, $E(\delta/f_L^{\rightarrow}) < E(\delta)/f_L^{\rightarrow}$.

Moreover, we have

Theorem 4.11. *Let $\{(X_t, \delta_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different X_t's be disjoint. Then $\bigoplus_{t\in T} E_L(\delta_t) = E_L(\bigoplus_{t\in T} \delta_t)$.*

Proof. Clearly, $E_L(\bigoplus_{t\in T} \delta_t) \le \bigoplus_{t\in T} E_L(\delta_t)$ by Theorem 3.9 and the definition of $E_L(\delta)$. Conversely, let $\lambda \in Copr(L)$ and $\lambda \lhd \bigoplus_{t\in T} E_L(\delta_t)(A)$ $(\forall A \in L^X)$, i.e.,

$$\lambda \lhd \bigoplus_{t\in T} E_L(\delta_t)(A) = \bigwedge_{t\in T} E_L(\delta_t)(A|X_t) = \bigwedge_{t\in T} \bigvee_{\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t} \bigwedge_{\lambda\in\Lambda^t} \bigvee_{(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda} = D^t_\lambda} \bigwedge_{\beta\in\Lambda^t_\lambda} \phi^{\delta_t}(E^t_{\beta\lambda}).$$

Then for all $t \in T$, there exists $\{D^t_\lambda\}_{\lambda\in\Lambda^t} \subset L^{X_t}$ such that

(i) $\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t$;
(ii) For each $\lambda \in \Lambda^t$, there exists $\{E^t_{\lambda\beta}\}_{\beta\in\Lambda^t_\lambda} \subset L^{X_t}$ such that $(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda} = D^t_\lambda$;
(iii) For each $\beta \in \Lambda^t_\lambda$, we have $\lambda \le \phi^{\delta_t}(E^t_{\beta\lambda})$.

Let $(D^t_\lambda)* \in L^X$, $(E^t_{\beta\lambda})* \in L^X$ be defined as follows:

$$(D^t_\lambda)*(x) = \begin{cases} D^t_\lambda(x), & x \in X_t \\ 0, & x \notin X_t \end{cases}$$

$$(E^t_{\beta\lambda})*(x) = \begin{cases} E^t_{\beta\lambda}(x), & x \in X_t \\ 0, & x \notin X_t \end{cases}$$

Then we have

$$\bigvee_{t\in T} \bigvee_{\lambda\in\Lambda^t} (D^t_\lambda)* = A, \quad (\sqcap)_{\beta\in\Lambda^t_\lambda} (E^t_{\beta\lambda})* = (D^t_\lambda)* \quad \text{and} \quad \phi^{\delta_t}(E^t_{\beta\lambda}) = \phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})*).$$

Hence $\lambda \le \phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})*)$. Note that

$$E_L\left(\bigoplus_{t\in T} \delta_t\right)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\beta\lambda} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^{\bigoplus_{t\in T}\delta_t}(C_{\beta\lambda}).$$

We have $\lambda \le E_L(\bigoplus_{t\in T} \delta_t)(A)$, and hence $\bigoplus_{t\in T} E_L(\delta_t) \le E_L(\delta)$. □

Remark 4.12. Let $\{(X_t, \delta_t)\}_{t\in T}$ be a family of *LM*-fuzzy topological spaces. Then $I_L(\prod_{t\in T} X_t, \prod_{t\in T} \delta_t) \ge (\prod_{t\in T} X_t, \prod_{t\in T} I_L(\delta_t))$, and the inequality cannot be replaced by equality.

Proof. It is easy to verify that $I_L\left(\prod_{t\in T} X_t, \prod_{t\in T}\delta_t\right) \geq \left(\prod_{t\in T} X_t, \prod_{t\in T} I_L(\delta_t)\right)$ by Theorem 3.8 and the definition of $I_L(\delta)$. The following example shows that the inequality cannot be replaced by equality.

Let $X = L = M = [0,1]$ and (X,τ) be a stratified *LM*-fuzzy topological space, where $\tau([a]\wedge 1_{[0.25,0.5]}) = \tau(1_{[0.25,0.5]}\vee([0.5]\wedge 1_{[0,0.25]})) = 1$ $(\forall a\in L)$ and for others $A\in L^X$, $\tau(A)=0$. It is easy to verify that $I_L(\tau): L^X \longrightarrow M$ is defined by $I_L(\tau)([a]\wedge 1_{[0.25,0.5]}) = I_L(\tau)([a]) = 1$ $(\forall a\in L)$ and for others $A\in L^X$, $I_L(\tau)(A)=0$. On the one hand $I_L(\tau)\times I_L(\tau)([0.5]\wedge 1_{[0,0.25]\times X\cup X\times[0,0.25]}) = 0$. On the other hand, $\tau\times\tau([0.5]\wedge 1_{[0,0.25]\times X\cup X\times[0,0.25]}) = 1$ and $I_L(\tau\times\tau)([0.5]\wedge 1_{[0,0.25]\times X\cup X\times[0,0.25]}) = 1$. Therefore $I_L(\tau)\times I_L(\tau) < I_L(\tau\times\tau)$. □

5. Conclusion

In the paper, we firstly introduce the definition of a Lowen *LM*-fuzzy topological space and discuss its basic properties such as the properties that the product space and the sum space of Lowen *LM*-fuzzy topological spaces are also Lowen *LM*-fuzzy topological spaces. Secondly, we study some categorical properties of Lowen *LM*-fuzzy topological spaces. For example, based on the interior and exterior Lowen topology of an *LM*-fuzzy topology, we show that **LLM-FTop** (the category of Lowen *LM*-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in **SLM-FTop** (the category of stratified *LM*-fuzzy topological spaces). Moreover, we also show that (X,τ) is an induced *LM*-fuzzy topological space iff (X,τ) is a Lowen *LM*-fuzzy topological space and an (IC)*LM*-fuzzy topological space. In the future, there are still some categorical properties of Lowen *LM*-fuzzy topological spaces which are worth studying. For example, is **LLM-FTop** Cartesian closed?

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