Lowen $LM$-fuzzy topological spaces

Hai-Yang Li$^{a,b,*}$, Ji-Gen Peng$^a$

$^a$ Department of Mathematics, Xi’an Jiaotong University, Xi’an 710049, PR China
$^b$ School of Science, Xi’an Polytechnic University, Xi’an 710048, PR China

Received in revised form 9 July 2011; accepted 9 July 2011
Available online 21 July 2011

Abstract

The aim of this paper is to define and study Lowen $LM$-fuzzy topological spaces. We discuss the basic properties of Lowen $LM$-fuzzy topological spaces, and introduce the notions of interior Lowen topology and exterior Lowen topology of an $LM$-fuzzy topology and prove that $LLM$-$FTop$ (the category of Lowen $LM$-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in $SLM$-$FTop$ (the category of stratified $LM$-fuzzy topological spaces). Moreover, we also prove that an $LM$-fuzzy topological space is an induced $LM$-fuzzy topological space iff it is a Lowen $LM$-fuzzy topological space and an (IC)$LM$-fuzzy topological space.

© 2011 Elsevier B.V. All rights reserved.

Keywords: Topology; Category; Lowen $LM$-fuzzy topological spaces

1. Introduction


In 2007, Yue [22] studied stratified, induced and weakly induced $LM$-fuzzy topological spaces, and proved that $WILM$-$FTop$ (the category of weakly induced $LM$-fuzzy topological spaces and fuzzy continuous mappings) is a simultaneously reflective and coreflective full subcategory of $LM$-$FTop$. Recently, Li and Peng [12] have studied the properties of (IC)$LM$-topological spaces, which is the generalization of (weakly) induced $LM$-topological spaces, and introduced the notions of interior (IC)-ification and exterior (IC)-ification of $LM$-fuzzy topologies and showed that $ICLM$-$FTop$ (the category of (IC)$LM$-fuzzy topological spaces and fuzzy continuous mappings) is an
isomorphism-closed full proper subcategory of \( \text{LM-FTop} \) and \( \text{ICLM-FTop} \) is a simultaneously bireflective and bicoreflective full subcategory of \( \text{LM-FTop} \).

In 1982, Lowen [14] introduced a very important class of fuzzy topological spaces, called fuzzy neighborhood spaces. Lowen’s notion has been extended to the \( L \)-fuzzy setting (called Lowen space) by Liu and Zhang [16] for \( L \), an arbitrary completely distributive lattice. This kind of \( L \)-topological space has received wide attention for categorical properties [13,15,16,21,24]. The aim of the present paper is to define and study Lowen \( LM \)-fuzzy topological spaces. We discuss the basic properties of Lowen \( LM \)-fuzzy topological spaces, and introduce the notions of interior Lowen topology and exterior Lowen topology of \( LM \)-fuzzy topology and prove that \( \text{LLM-FTop} \) (the category of Lowen \( LM \)-fuzzy topological spaces and fuzzy continuous mappings) is an isomorphism-closed full proper subcategory of \( \text{SLM-FTop} \) (the category of stratified \( LM \)-fuzzy topological spaces and fuzzy continuous mappings) and \( \text{LLM-FTop} \) is a simultaneously bireflective and bicoreflective full subcategory of \( \text{SLM-FTop} \). Moreover, we prove that \((X, τ)\) is an induced \( LM \)-fuzzy topological space iff \((X, τ)\) is a Lowen \( LM \)-fuzzy topological space and an \( (\text{IC})LM \)-fuzzy topological space.

For needed categorical notions, please refer to [1,7,17,19].

2. Preliminaries

We now give some definitions and results to be used in this paper. Let \( L \) be a complete lattice. An element \( r \in L - \{0\} \) is called a coprime element if, for any finite subset \( K \subset L \) satisfying \( r \leq \bigvee K \) (the supremum of \( K \)), there exists a \( k \in K \) such that \( r \leq k \). An element \( s \in L - \{1\} \) is called a prime element if it is a coprime element of \( L^{op} \) (the opposite lattice [4] of \( L \)). The set of all coprime elements (resp., prime elements) of \( L \) will be written as \( \text{Copr}(L) \) (resp., \( \text{Pr}(L) \)). We say that \( a \) is a way-below (wedge below) \( b \), in symbols, \( a \ll b \) (\( a \prec b \)), if for every directed (arbitrary) subset \( D \subset L \), \( \bigvee D \geq b \) implies \( a \leq d \) for some \( d \in D \). From [5], we know that \( \text{Copr}(L) \) is a join-generating set of \( L \) if \( L \) is a completely distributive lattice. Hence every element in \( L \) is also the supremum of all the coprimes wedge below it. By the definition of completely distributive lattice it is easy to see that a complete lattice \( L \) is completely distributive iff the operator \( \bigvee: \text{Low}(L) \rightarrow L \) taking every lower set to its supremum has a left adjoint \( β \), and in the case, \( β(a) = \{b | b \ll a\} \) for all \( a \in L \). Hence, the wedge below relation has the interpolation property in a completely distributive lattice, that is, \( a \ll b \) implies that there is some \( c \in L \) such that \( a \ll c \ll b \).

The way-below relation on a completely distributive lattice \( L \) is called locally multiplicative [23] if for every coprime \( a \in \text{Copr}(L) \), \( a \ll b \) and \( a \ll c \) imply \( a \ll b \wedge c \) for all \( b, c \in L \). If \( a \) is a coprime, then \( a \ll b \) if and only if \( a \prec b \) (see [5,23]). Hence \( L \) is locally multiplicative if for every coprime \( a \), \( a \ll b \) and \( a \ll c \) imply \( a \ll b \wedge c \) for all \( b, c \in L \). Clearly, \([0,1]\) and power set lattice \( 2^X \) are locally multiplicative.

Throughout this paper, \( L \) always stands for a completely distributive lattice with locally multiplicative property and \( M \) is a completely distributive lattice. Apparently, the power set \( L^X \) (i.e. the set of all \( L \)-subset) with the point-wise order is also a completely distributive lattice with locally multiplicative property, in which the least element and the greatest element of \( L^X \) will be written as \( 0_X \) and \( 1_X \). Let \( 1_Y \) denote the characteristic function of \( U \in 2^X \), and \([a]\) denote the \( L \)-subset taking constant value \( a \), where \( 2^X \) is the power set of \( X \) and \( a \in L \). An \( L \)-subset with form \([a] \wedge U\) is called one-step \( L \)-subset. Let \( S \) denote the set of all the one-step \( L \)-subsets of \( L^X \). Each mapping \( f: X \rightarrow Y \) induces a mapping \( f_{L^Y}^{-}: L^X \rightarrow L^Y \) (called \( L \)-forward powerset operator, cf. [18]), defined by

\[
f_{L^Y}^{-}(A)(y) = \bigvee \{A(x) | f(x) = y\} \quad (\forall A \in L^X, \forall y \in Y).
\]

The right adjoint to \( f_{L^Y}^{-} \) (called \( L \)-backward powerset operator, cf. [18]) is denoted as \( f_{L^Y}^{-} \) and given by

\[
f_{L^Y}^{-}(B) = \bigvee \{A \in L^X | f_{L^Y}^{-}(A) \leq B\} = B \circ f \quad (\forall B \in L^Y).
\]

It is known that \( f_{L^Y}^{-} \) preserves arbitrary unions and that \( f_{L^Y}^{-} \) preserves arbitrary unions, arbitrary intersections, and complements when they exist (canonical examples of such morphisms are given in [18]).

For \( A \in L^X \) and \( p \in L \), the mapping \( \hat{i}_p: L^X \rightarrow 2^X \) is defined by \( \hat{i}_p(A) = \{x | A(x) \ll p\} \), called the strong \( p \)-cut of \( A \). If \( L \) is a locally multiplicative completely distributive lattice, then it is easy to verify the following

**Lemma 2.1** ([Li 10, Yue 22]).

(1) For each \( p \in L \), \( \hat{i}_p: L^X \rightarrow 2^X \) preserves arbitrary suprema and finite meets.
(2) For each \( p \in L \), \( A \in L^X \), \( B \in L^Y \) and any mapping \( f : X \to Y \), we have \( i_{p}(f^{-1}_{L}(A)) = f(i_{p}(A)), \) \( \hat{i}_{p}(f^{-1}_{L}(B)) = f(\hat{i}_{p}(B)). \)

(3) \( A = \bigvee_{p \in L} \{ p \} \wedge 1_{i_{p}(A)} = \bigvee_{p \in \text{Copr}(L)} \{ p \} \wedge 1_{i_{p}(A)}. \)

An \( LM \)-fuzzy topology [9] on a set \( X \) is defined to be a mapping \( \tau : L^X \to M \) satisfying:

\[
(\text{FT1}) \quad \tau(1_X) = \tau(0_X) = 1;
\]

\[
(\text{FT2}) \quad \forall A, B \in L^X, \quad \tau(A \wedge B) \geq \tau(A) \wedge \tau(B);
\]

\[
(\text{FT3}) \quad \tau(\bigvee_{i \in T} A_i) = \bigwedge_{i \in T} \tau(A_i), \quad \text{for every family} \ \{ A_t \mid t \in T \} \subset L^X.
\]

The value \( \tau(A) \) can be interpreted as the degree of openness of \( A \). A fuzzy continuous mapping between \( LM \)-fuzzy topological spaces is a mapping \( f : (X, \tau) \to (Y, \delta) \) such that \( \delta(B) \leq \tau(f^{-1}_{L}(B)) \) for all \( B \in L^Y \). When \( L = \{0, 1\} \), this definition reduced to that of \( M \)-fuzzy topology.

In the following of this section, we give some definitions and results about \( LM \)-fuzzy topology.

**Definition 2.2 (Fang and Yue [3]).** Let \( \tau \) be an \( LM \)-fuzzy topology on \( X \).

(1) \( \mathcal{B} : L^X \to M \) is called a base of \( \tau \) if \( \mathcal{B} \) satisfies the following condition:

\[
\forall A \in L^X, \quad \tau(A) = \bigvee_{\mathcal{B} \mid \mathcal{B}_A = A} \bigwedge_{\lambda \in A} \mathcal{B}(B_{\lambda}),
\]

where the expression \( \bigvee_{\mathcal{B} \mid \mathcal{B}_A = A} \bigwedge_{\lambda \in A} \mathcal{B}(B_{\lambda}) \) will be denoted by \( \mathcal{B}^{(\tau)}(A) \).

(2) \( \phi : L^X \to M \) is called a subbase of \( \tau \) if \( \phi^{(\tau)} : L^X \to M \) is a base of \( \tau \), where

\[
\phi^{(\tau)}(A) = \bigvee_{\phi \mid \phi_{B_{\lambda}} = A} \bigwedge_{\lambda \in J} \phi(B_{\lambda})
\]

for all \( A \in L^X \) with \( \tau \) standing for “finite intersection”.

**Lemma 2.3 (Fang and Yue [3]).** Let \( \tau \) be an \( LM \)-fuzzy topology on \( X \). Then \( \phi : L^X \to M \) is the subbase of \( \tau \) if and only if \( \phi^{(\tau)}(1_X) = 1 \).

**Definition 2.4 (Yue [22]).** Let \( (X, \tau) \) be an \( LM \)-fuzzy topological space. If \( \tau([a]) = 1 \) for all \( a \in L \), then \( (X, \tau) \) is called a stratified \( LM \)-fuzzy topological space.

It is easy to verify that \( (X, \tau) \) is a stratified \( LM \)-fuzzy topological space iff \( \tau([a]) = 1 \) for all \( a \in \text{Copr}(L) \). Let \( \text{SLM-FTop} \) denote the category of stratified \( LM \)-fuzzy topological spaces and fuzzy continuous mappings.

**Lemma 2.5.** Let \( (X, \tau) \) be a stratified \( LM \)-fuzzy topological space. Then \( \tau([a] \wedge 1_U) \leq \tau([b] \wedge 1_U) \) \( (a, b \in L, \ b < a \) and \( U \subset X \)).

**Proof.** Since \( (X, \tau) \) is stratified, we have

\[
\tau([b] \wedge 1_U) = \tau([b] \wedge ([a] \wedge 1_U)) \geq \tau([b] \wedge [a] \wedge 1_U) = \tau([a] \wedge 1_U).
\]

**Definition 2.6 (Yue [22]).** Let \( (X, \tau) \) be an \( LM \)-fuzzy topological space. If \( \tau(A) = \bigwedge_{\tau \in \text{Copr}(L)} \tau(1_{i_{\tau}(A)}) \) holds for all \( A \in L^X \), then \( (X, \tau) \) is called an induced \( LM \)-fuzzy topological space.

**Definition 2.7 (Yue [22]).** Let \( (X, \tau) \) be an \( LM \)-fuzzy topological space. If \( \tau(A) \leq \bigwedge_{\tau \in \text{Copr}(L)} \tau(1_{i_{\tau}(A)}) \) holds for all \( A \in L^X \), then \( (X, \tau) \) is called a weakly induced \( LM \)-fuzzy topological space. Let \( \text{WILM-FTop} \) denote the category of weakly induced \( LM \)-fuzzy topological spaces and fuzzy continuous mappings.
**Definition 2.8 (Li and Peng [12]).** Let \((X, \tau)\) be an LM-fuzzy topological space. If \(\tau(A) \leq \tau(1_{\mathcal{L}(A)})\) holds for all \(A \in \mathcal{L}^X\), then \((X, \tau)\) is called an (IC)LM-fuzzy topological space. Let \(\text{ICLM-FTop}\) denote the category of (IC)LM-fuzzy topological spaces and fuzzy continuous mappings.

It is easy to verify that induced and weakly induced LM-fuzzy topological spaces are (IC)LM-fuzzy topological space.

**Definition 2.9 (Fang and Yue [3]).** Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of LM-fuzzy topological spaces and \(p_t : \prod_{t \in T} X_t \rightarrow X_t\) be the projection. Then the LM-fuzzy topology on \(\prod_{t \in T} X_t\) whose subbase is defined by

\[
\forall A \in L^{\prod_{t \in T} X_t}, \quad \phi(A) = \bigvee_{t \in T} \tau_t(B)
\]

is called the product LM-fuzzy topology of \(\{\tau_t\}_{t \in T}\), denoted by \(\prod_{t \in T} \tau_t\), and \((\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)\) is called the product space of \(\{(X_t, \tau_t)\}_{t \in T}\).

**Definition 2.10 (Yue [22]).** Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of LM-fuzzy topological spaces, different \(X_t\)s be disjoint and \(X = \bigcup_{t \in T} X_t\), and let \(\tau : L^X \rightarrow M\) be defined as follows:

\[
\forall A \in L^X, \quad \tau(A) = \bigwedge_{t \in T} \tau_t(A|X_t).
\]

Then it is easy to verify that \(\tau\) is an LM-fuzzy topology on \(X\), and \(\tau\) is called the sum LM-fuzzy topology of \(\{\tau_t\}_{t \in T}\), denoted by \(\bigoplus_{t \in T} \tau_t\). \((\bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t)\) is called the sum space of \(\{(X_t, \tau_t)\}_{t \in T}\).

**Definition 2.11 (Yue [22]).** Let \((X, \tau)\) be an LM-fuzzy topological space and \(f : X \rightarrow Y\) be a surjective mapping. It is easy to verify that \(\tau/f^+ : L^Y \rightarrow M\) is an LM-fuzzy topology on \(Y\), where \(\tau/f^+\) is defined by

\[
\forall A \in L^Y, \quad \tau/f^+(A) = \tau(f^+(A)).
\]

\(\tau/f^+\) is called the LM-fuzzy quotient topology of \(\tau\) with respect to \(f\), and \((Y, \tau/f^+\) is called the LM-fuzzy quotient space of \((X, \tau)\) with respect to \(f\).

**Definition 2.12 (Rodabaugh [18]).** Let \((X, \tau)\) be an LM-fuzzy topological space and \(Y \subseteq X\). We call \((Y, \tau|_Y)\) a subspace of \((X, \tau)\), where \(\tau|_Y : L^Y \rightarrow M\) is defined by

\[
\forall B \in L^Y, \quad \tau|_Y(B) = \bigvee\{\tau(A)|A \in L^X, A|Y = B\}.
\]

**Lemma 2.13 (Yue [22]).** Let \(f : (X, \tau) \rightarrow (Y, \delta)\) be a mapping and \(\phi\) be a subbase of \(\delta\). If \(\tau(f^{-1}(B)) \geq \phi(B)\) for all \(B \in L^Y\), then \(f\) is fuzzy continuous.

3. Lowen LM-fuzzy topological spaces

In the section, we mainly introduce the definition of a Lowen LM-fuzzy topological space and discuss its basic properties such as the properties that the product space and the sum space of Lowen LM-fuzzy topological spaces are also Lowen LM-fuzzy topological spaces. Moreover, we show that \((X, \tau)\) is an induced LM-fuzzy topological space iff \((X, \tau)\) is a Lowen LM-fuzzy topological space and an (IC)LM-fuzzy topological space.

**Definition 3.1.** An LM-fuzzy topological space \((X, \tau)\) is said to be a Lowen LM-fuzzy topological space iff for all \(A \in L^X\), \(\tau(A) = \bigwedge_{a \in \text{Cop}(L)} \tau(f(a)) \land 1_{\mathcal{L}(A)}\). Let \(\text{LLM-FTop}\) denote the category of Lowen LM-fuzzy topological spaces and fuzzy continuous mappings.
Example 3.2. Let $L = M = 2^X$. Then $L$ is not only a complete Boolean algebra but a locally multiplicative completely distributive lattice. We consider the map $\tau : L^X \to M$ defined by

$$\tau(A) = \left( \bigwedge_{x \in X} A(x) \right) \lor \left( \bigvee_{x \in X} A(x) \to 0 \right).$$

Then $\tau$ is a stratified LM-fuzzy topology on $X$ (See [7, Example 4.2.1(b)]). We will show that $\tau$ is a Lowen LM-fuzzy topology on $X$ in the following.

For any $A \in L^X$ and any $t \in Copr(L)$, if $\hat{\tau}(A) = X$ or $\phi$, then

$$\tau([t] \land 1_{\hat{\tau}(A)}) = 1 \geq \tau(A).$$

If $\hat{\tau}(A) \neq X$ and $\hat{\tau}(A) \neq \phi$, then there exist $x_1, x_2 \in X$ such that $t \ll A(x_1)$ and $t \not\ll A(x_2)$, which imply that $A'(x_1) \leq t'$ and $A(x_2) \leq t'$, and hence

$$\tau(A) = \left( \bigwedge_{x \in X} A(x) \right) \lor \left( \bigvee_{x \in X} A(x) \to 0 \right) \leq t'.$$

Therefore for any $A \in L^X$ and any $t \in Copr(L)$, we have $\tau(A) \leq \tau([t] \land 1_{\hat{\tau}(A)})$, which implies that $\tau$ is the Lowen LM-fuzzy topology on $X$ by Theorem 3.4(2).

**Lemma 3.3.**
1. A Lowen LM-fuzzy topological space $(X, \tau)$ is a stratified LM-fuzzy topological space.
2. An induced LM-fuzzy topological space $(X, \tau)$ is a Lowen LM-fuzzy topological space.

**Proof.**

1. Since $(X, \tau)$ is a Lowen LM-fuzzy topological space, we have

$$\tau(1_X) \leq \bigwedge_{a \in Copr(L)} \tau([a] \land 1_{\hat{\tau}(A)}) = \bigwedge_{a \in Copr(L)} \tau([a]),$$

and hence $\tau([a]) = 1$ $(\forall a \in Copr(L))$, which implies that $(X, \tau)$ is a stratified LM-fuzzy topological space.

2. Since $(X, \tau)$ is an induced LM-fuzzy topological space, for all $A \in L^X$, and $a \in Copr(L)$, we have

$$\tau(1_{\hat{\tau}(A)}) = \tau([a]) \land \tau(1_{\hat{\tau}(A)}) \leq \tau([a] \land 1_{\hat{\tau}(A)}),$$

which implies that

$$\tau(A) = \bigwedge_{a \in Copr(L)} \tau(1_{\hat{\tau}(A)}) \leq \bigwedge_{a \in Copr(L)} \tau([a] \land 1_{\hat{\tau}(A)}).$$

Hence $(X, \tau)$ is a Lowen LM-fuzzy topological space by Theorem 3.4(2).

**Theorem 3.4.** Let $(X, \tau)$ be an LM-fuzzy topological space and $S$ be the set of all the one-step $L$-subset of $L^X$. Then the following are equivalent.

1. $(X, \tau)$ is a Lowen LM-fuzzy topological space.
2. For all $A \in L^X$ and $a \in Copr(L)$, $\tau(A) \leq \tau([a] \land 1_{\hat{\tau}(A)})$.
(3) There exists a base $B$ of $\tau$ satisfying $\mathcal{B}([a]) = 1$ ($\forall a \in L$) and

$$\forall A \in L^X, \quad \tau(A) = \bigvee \left\{ \bigwedge_{\xi \in A} \mathcal{B}(B_{\xi}) \bigg| \bigvee_{\xi \in A} B_{\xi} = A \text{ and } B_{\xi} \in S \right\}.$$ 

(4) There exists a subbase $\phi$ of $\mathcal{\tau}$ satisfying $\phi([a]) = 1$ ($\forall a \in L$) and

$$\forall A \in L^X, \quad \phi^{(\tau)}(A) = \bigvee \left\{ \bigwedge_{\xi \in J} \phi(B_{\xi})(\tau)_{\xi \in J} B_{\xi} = A \text{ and } B_{\xi} \in S \right\}$$

is a base of $\tau$.

**Proof.** (1) $\iff$ (2) follows from the definition of LM-fuzzy topology.

(2) $\implies$ (3): Let $\mathcal{B} : S \to M$ defined by $\mathcal{B}(A) = \tau(A)$ ($\forall A \in S$). Then $\mathcal{B}([a]) = 1$ ($\forall a \in L$) by Lemma 3.3 (1). We will show $\mathcal{B}$ is a base of $\tau$ in the following. By (2) and the definition of LM-fuzzy topology, we have

$$\tau(A) \leq \bigwedge_{a \in \text{Copr}(L)} \tau([a] \wedge 1_{\text{int}(A)}) = \bigwedge_{a \in \text{Copr}(L)} \mathcal{B}([a] \wedge 1_{\text{int}(A)}) \leq \bigvee_{\bigvee_{\xi \in A} B_{\xi} = A} \bigwedge_{\xi \in A} \mathcal{B}(B_{\xi})(B_{\xi} \in S)$$

$$\leq \bigvee_{\bigvee_{\xi \in A} B_{\xi} = A} \tau(A) = \tau(A),$$

which implies that $\mathcal{B}$ is a base of $\tau$.

(3) $\implies$ (4): It is easily concluded by taking $\phi = \mathcal{B}$.

(4) $\implies$ (2): For all $a \in \text{Copr}(L)$, $A \in L^X$, we have

$$\tau([a] \wedge 1_{\text{int}(A)}) = \bigvee_{\bigvee_{\xi \in A} B_{\xi} = [a] \wedge 1_{\text{int}(A)}} \bigwedge_{\xi \in A} \bigwedge_{\beta \in \text{int}(A)} \bigwedge_{C_{\xi} = B_{\xi} \wedge A_\xi} \phi(C_{\xi, \beta}) (C_{\xi, \beta} \in S)$$

$$\geq \bigvee_{\bigvee_{\xi \in A} D_{\xi} = A} \bigwedge_{\xi \in A} \bigwedge_{\beta \in \text{int}(A)} \bigwedge_{E_{\xi, \beta} = \xi \wedge 1_{\text{int}(A)}} \phi(E_{\xi, \beta}) (E_{\xi, \beta} \in S)$$

$$\geq \bigvee_{\bigvee_{\xi \in A} D_{\xi} = A} \bigwedge_{\xi \in A} \bigwedge_{\beta \in \text{int}(A)} \bigwedge_{F_{\xi, \beta} = D_{\xi} \wedge \beta \in A_\xi} \phi([a] \wedge 1_{\text{int}(A)}) (F_{\xi, \beta} \in S)$$

$$\geq \bigvee_{\bigvee_{\xi \in A} D_{\xi} = A} \bigwedge_{\xi \in A} \bigwedge_{\beta \in \text{int}(A)} \bigwedge_{F_{\xi, \beta} = D_{\xi} \wedge \beta \in A_\xi} \phi(F_{\xi, \beta}) \geq \tau(A). \quad \square$$

**Theorem 3.5.** An LM-fuzzy topological space $(X, \tau)$ is an induced LM-fuzzy topological space iff $(X, \tau)$ is a Lowen LM-fuzzy topological space and an (IC)LM-fuzzy topological space.

**Proof.** It suffices to show the sufficiency. Since $(X, \tau)$ is a Lowen LM-fuzzy topological space, we have, for all $A \in L^X$,

$$\tau(A) \leq \bigwedge_{a \in \text{Copr}(L)} \tau([a] \wedge 1_{\text{int}(A)}).$$

By the definition of (IC)LM-fuzzy topology,

$$\tau([a] \wedge 1_{\text{int}(A)}) \leq \tau(1_{\text{int}(A)})$$

and hence

$$\tau(A) \leq \bigwedge_{a \in \text{Copr}(L)} \tau(1_{\text{int}(A)}).$$
By Lemma 3.3(1) and the definition of Lowen LM-fuzzy topology, we have

\[ \tau(A) \geq \bigwedge_{a \in \text{Copr}(L)} \tau([a] \land 1_{i_d(A)}) \geq \bigwedge_{a \in \text{Copr}(L)} \tau(1_{i_d(A)}). \]

Hence

\[ \tau(A) = \bigwedge_{a \in \text{Copr}(L)} (\tau(1_{i_d(A)})), \]

which implies that \((X, \tau)\) is an induced LM-fuzzy topological space. □

**Theorem 3.6.** Let \((X, \tau)\) be a Lowen LM-fuzzy topological space and \(Y \subset X\). Then the subspace \((Y, \tau|Y)\) of \((X, \tau)\) is also a Lowen LM-fuzzy topological space.

**Proof.** Since \((X, \tau)\) is a Lowen LM-fuzzy topological space, for all \(A \in L^Y\) and \(a \in \text{Copr}(L)\), we have

\[ \tau|Y(A) = \bigvee \{\tau(B)|B \in L^X, B|Y = A\} \leq \bigvee \{\tau(B)|B \in L^X, [a] \land 1_{i_d(B)}|Y = [a] \land 1_{i_d(A)}\} \]

which implies that \((Y, \tau|Y)\) is a Lowen LM-fuzzy topological space. □

**Theorem 3.7.** Let \((X, \tau)\) be a Lowen LM-fuzzy topological space and \(f: X \rightarrow Y\) be a surjective mapping. Then the LM-fuzzy quotient space \((Y, \tau/f_{L^y}^-)\) of \((X, \tau)\) with respect to \(f\) is also a Lowen LM-fuzzy topological space.

**Proof.** Since \((X, \tau)\) is a Lowen LM-fuzzy topological space, for all \(A \in L^Y\) and \(a \in \text{Copr}(L)\), we have

\[ \tau/f_{L^y}^-(A) = \tau(f_{L^y}^-(A)) \leq \tau([a] \land 1_{i_d(f_{L^y}^-(A))}) = \tau(f_{L^y}^-([a] \land 1_{i_d(A)})) = \tau/f_{L^y}^-(([a] \land 1_{i_d(A)})), \]

which implies that \((Y, \tau/f_{L^y}^-)\) is a Lowen LM-fuzzy topological space. □

**Theorem 3.8.** Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of Lowen LM-fuzzy topological spaces. Then the product space \((\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)\) of \(\{(X_t, \tau_t)\}_{t \in T}\) is also a Lowen LM-fuzzy topological space.

**Proof.** Let \(\phi\) be the subbase of \(\tau\). Then, for all \(A \in L^X\) and \(a \in \text{Copr}(L)\), we have

\[ \phi(A) = \bigvee_{t \in T} (\tau_t(B) \leq \bigvee_{t \in T} (\tau_t([a] \land 1_{i_d(B)})) \leq \bigvee_{t \in T} \tau_t([a] \land 1_{i_d(A)})) \]

Hence, for all \(A \in L^X\) and \(a \in \text{Copr}(L)\), \(\tau(A) \leq \tau([a] \land 1_{i_d(A)})\), which implies that \((\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)\) is a Lowen LM-fuzzy topological space. □

**Theorem 3.9.** Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of Lowen LM-fuzzy topological spaces, different \(X_t\)’s be disjoint. Then the sum space \((\bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t)\) of \(\{(X_t, \tau_t)\}_{t \in T}\) is a Lowen LM-fuzzy topological space iff for all \(t \in T\), \((X_t, \tau_t)\) is a Lowen LM-fuzzy topological space.

**Proof.** Necessity: For all \(t \in T\), \(A_t \in L^{X_t}\) and \(a \in \text{Copr}(L)\), we have

\[ \tau_t(A_t) = \bigwedge_{t \in T} \tau_t(A_t) = \tau(A_t) \leq \tau([a] \land 1_{i_d(A_t)}) = \bigwedge_{t \in T} \tau_t([a] \land 1_{i_d(A_t)}), \]

where

\[ A\#(x) = \begin{cases} A(x), & x \in X_t \\ 0, & x \notin X_t \end{cases} \]

Thus for every \(t \in T\), \((X_t, \tau_t)\) is a Lowen LM-fuzzy topological space.
Sufficiency: For every $A \in L^X$ and $a \in \text{Copr}(L)$, we have
\[
\tau(A) = \bigwedge_{t \in T} \tau_t(A|X_t) \leq \bigwedge_{t \in T} \tau_t([a] \land 1_{l_a(A)|X_t}) = \bigwedge_{t \in T} \tau_t([a] \land 1_{l_a(A)}|X_t) = \tau([a] \land 1_{l_a(A)}),
\]
which implies that $(\bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t)$ is a Lowen LM-fuzzy topological space. □

4. Interior and exterior Lowen topology of an LM-fuzzy topology

In the section, we mainly study some categorical properties of LM-fuzzy topological spaces. Especially, based on the interior and exterior Lowen topology of an LM-fuzzy topology, we prove that LLM-FTop (the category of Lowen LM-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in SLM-FTop (the category of stratified LM-fuzzy topological spaces). For this we first prove Lemmas 4.1 and 4.2.

Lemma 4.1. Let $(X, \tau)$ be a stratified LM-fuzzy topological space and $I_L(\tau) : L^X \rightarrow M$ be defined by
\[
I_L(\tau)(A) = \bigwedge_{a \in \text{Copr}(L)} \tau([a] \land 1_{l_a(A)}) \quad (\forall A \in L^X).
\]
Then $I_L(\tau)$ is the largest Lowen LM-fuzzy topology on $X$ which is contained in $\tau$. We call $I_L(\tau)$ the interior Lowen topology of $\tau$.

Proof. Clearly, $I_L(\tau)(A) \leq \tau(A)$ for every $A \in L^X$. By Lemma 2.5 and the definition of $I_L$, for every $a \in \text{Copr}(L)$ and $A \in L^X$, we have
\[
I_L(\tau)([a] \land 1_{l_a(A)}) = \bigwedge_{b \leq a, b \in \text{Copr}(L)} \tau([b] \land 1_{l_b(A)}) = \tau([a] \land 1_{l_a(A)}).
\]
Hence
\[
I_L(\tau)(A) = \bigwedge_{a \in \text{Copr}(L)} \tau([a] \land 1_{l_a(A)}) = \bigwedge_{a \in \text{Copr}(L)} I_L(\tau)([a] \land 1_{l_a(A)}),
\]
which implies that $I_L(\tau)$ is the Lowen LM-fuzzy topology on $X$ which is contained in $\tau$. Set $\delta \leq \tau$ and $\delta$ be a Lowen LM-fuzzy topology on $X$. Then, for all $A \in L^X$,
\[
\delta(A) = \bigwedge_{a \in \text{Copr}(L)} \delta([a] \land 1_{l_a(A)}) \leq \bigwedge_{a \in \text{Copr}(L)} \tau([a] \land 1_{l_a(A)}) = I_L(\tau)(A).
\]
Therefore, $I_L(\tau)$ is the largest Lowen LM-fuzzy topology on $X$ which is contained in $\tau$. □

Let $(X, \tau)$ be an LM-fuzzy topological space and $\phi^\tau : L^X \rightarrow M$ be defined by
\[
\phi^\tau(A) = \begin{cases} \bigvee_{a \in \text{Copr}} (\bigwedge \{ \tau_r(B) \mid \tau_r(B) = U \}) & A = [a] \land 1_U \quad (\forall a \in L, r \in \text{Copr}(L) \text{ and } U \subset X) \\ \tau(A) & \text{others} \end{cases}
\]
It is easy to verify that $\phi^\tau$ is a subbase of one LM-fuzzy topology on $X$. We denote this LM-fuzzy topology by $E_L(\tau)$ and call it the exterior Lowen topology of $\tau$ (see Lemma 4.2).

Lemma 4.2. Let $(X, \tau)$ be an LM-fuzzy topological space. Then $E_L(\tau)$ is the smallest Lowen LM-fuzzy topology on $X$ which contains $\tau$.

Proof. Firstly, we show that $E_L(\tau)$ is the Lowen LM-fuzzy topology on $X$. In fact, for every $A \in L^X$ and $a \in \text{Copr}(L)$, we have
\[
E_L(\tau)(A) = \bigvee_{\beta \in A} \bigwedge_{\beta \in A} \bigvee_{\beta \in A} \phi^\tau(C_{\beta}),
\]
and
\[ E_L(\tau)([a] \land 1_{i_a(A)}) = \bigvee_{i \in A} \bigwedge_{B_i \subseteq \tau} \bigwedge_{i \in A} \bigwedge_{b \in \tau_i(A)} \phi^*(\tau_i B) \]
\[ \geq \bigvee_{i \in A} \bigwedge_{B_i \subseteq \tau} \bigwedge_{i \in A} \bigwedge_{b \in \tau_i(A)} \phi^*(\tau_i B) \land 1_{i_a(C_{i_B})}. \]

If \( C_{i_B} \) is not a one-step \( L \)-subset, then
\[ \phi^*(\tau_i B) \land 1_{i_a(C_{i_B})} = \bigvee_{a \leq r} \left( \bigvee_{\tau_i(B)} \lceil r \rceil(B) = \lceil a(C_{i_B}) \rceil \right) \geq \tau(C_{i_B}) = \phi^*(\tau_i B). \]

If \( C_{i_B} \) is a one-step \( L \)-subset and let \( C_{i_B} = [b] \land 1_V \), then
\[ \phi^*(C_{i_B}) = \bigvee_{b \leq r} \left( \bigvee_{\tau_i(B)} \lceil r \rceil(B) = V^1 \right) \]
and, for all \( a < b \),
\[ \phi^*(\tau_i B) \land 1_{i_a(C_{i_B})} = \phi^*(\tau_i B) \land 1_V = \bigvee_{a \leq r} \left( \bigvee_{\tau_i(B)} \lceil r \rceil(B) = V^1 \right) \geq \phi^*(C_{i_B}). \]

Hence \( E_L(\tau)(A) \leq E_L(\tau)([a] \land 1_{i_a(A)}) \), which implies that \( E_L(\tau) \) is the Lowen \( LM \)-fuzzy topology on \( X \) which contains \( \tau \).

Secondly, we show that \( E_L(\tau) \) is the smallest Lowen \( LM \)-fuzzy topology on \( X \) which contains \( \tau \). Let \( \tau \leq \eta \) and \( \eta \) be a Lowen \( LM \)-fuzzy topology on \( X \). We need to prove that \( E_L(\tau) \leq \eta \). It suffices to show that \( \phi^*(A) \leq \eta(A) \) for all \( A \in L^X \). It is clear that \( E(\tau) \leq \eta \) when \( A \) is not a one-step \( L \)-subset. If \( A \) is a one-step \( L \)-subset and let \( A = [b] \land 1_U \), then
\[ \phi^*(A) = \bigvee_{b \leq r} \left( \bigvee_{\tau_i(B)} \lceil r \rceil(B) = U^1 \right) \leq \bigvee_{b \leq r} \left( \bigvee_{\eta(B)} \lceil r \rceil(B) = U^1 \right) \leq \bigvee_{b \leq r} \left( \bigvee_{\eta([r] \land 1_U)} \lceil r \rceil(B) = U^1 \right) \]
\[ \leq \bigvee_{b \leq r} \eta([r] \land 1_U) \leq \eta([b] \land 1_U) = \eta(A), \]
and thus \( E(\tau) \leq \eta \). \( \square \)

**Corollary 4.3.** \( (X, \delta) \) is a Lowen \( LM \)-fuzzy topological space iff any two of \( E_L(\delta), I_L(\delta), \delta \) are equal (equivalently, all the three are equal).

**Lemma 4.4.**

1. Let \( (X, \delta) \) be a stratified \( LM \)-fuzzy topological space and \( (Y, \tau) \) be a Lowen \( LM \)-fuzzy topological space. Then \( f : (X, \delta) \rightarrow (Y, \tau) \) is a fuzzy continuous mapping iff \( f : (X, I_L(\delta)) \rightarrow (Y, I_L(\tau)) = (Y, \tau) \) is a fuzzy continuous mapping.

2. Let \( (X, \delta) \) be an \( LM \)-fuzzy topological space and \( (Y, \tau) \) be a Lowen \( LM \)-fuzzy topological space. Then \( f : (Y, \tau) \rightarrow (X, \delta) \) is a fuzzy continuous mapping iff \( f : (Y, E_L(\tau)) = (Y, \tau) \rightarrow (X, E_L(\delta)) \) is a fuzzy continuous mapping.

**Proof.**

1. The sufficiency is obvious and we need to show the necessity. Suppose that \( f : (X, \delta) \rightarrow (Y, \tau) \) is fuzzy continuous, i.e., \( \tau(B) \leq \delta(f_L^{-1}(B)) \) for every \( B \in L^Y \). Since \( (Y, \tau) \) is a Lowen \( LM \)-fuzzy topological space, we have
\[ \tau(B) = \bigwedge_{a \in Copr(L)} \tau([a] \land 1_{i_a(B)}) \leq \bigwedge_{a \in Copr(L)} \delta(f_L^{-1}([a] \land 1_{i_a(B)})) \]
\[ = \bigwedge_{a \in Copr(L)} \delta([a] \land 1_{i_a(f_L^{-1}(B))}) = I_L(\delta)(f_L^{-1}(B)), \]
which implies that \( f : (X, I_L(\delta)) \rightarrow (Y, I(\tau)) = (Y, \tau) \) is a fuzzy continuous mapping.
(2) The sufficiency is obvious and we need to show the necessity. It suffices to show for all \( A = [a] \land 1_U \) (\( \forall a \in L \) and \( U \subset X \)), \( \phi^*(A) \leq \tau(f^-_L(A)) \) by the definition of \( E_L(\delta) \) and Lemma 2.13. Suppose that \( f : (Y, \tau) \rightarrow (X, \delta) \) is a fuzzy continuous mapping, i.e., \( \delta(A) \leq \tau(f^-_L(A)) \) for all \( A \in L^X \). Since \( (Y, \tau) \) is a Lowen LM-fuzzy topological space, we have

\[
\tau(A) = \bigwedge_{a \in \text{Copr}(L)} \tau([a] \land 1_{i_d(A)}),
\]

and hence

\[
\phi^*(A) = \bigvee_{a < r} \left( \bigvee \{ \tau([B]) \mid r(B) = U \} \right) \leq \bigvee_{a < r} \left( \bigvee \{ f^-_L([B]) \mid r(B) = U \} \right) = \bigvee_{a < r} \tau([a] \land 1_{f^{-1}(U)}) = \tau(f^-_L(A)),
\]

which implies that \( f : (Y, E_L(\tau)) = (Y, \tau) \rightarrow (X, E_L(\delta)) \) is a fuzzy continuous mapping. \( \square \)

Let \( i : \text{LLM-FTop} \rightarrow \text{SLM-FTop} \) be the inclusion functor. By proof of Lemma 4.4, we may show the following

**Theorem 4.5.**

(1) \( I_L : \text{SLM-FTop} \rightarrow \text{LLM-FTop} \) is functor and \( I_L^{-1} \) is.

(2) \( E_L : \text{LM-FTop} \rightarrow \text{LLM-FTop} \) is functor and \( i \circ E_L \).

**Corollary 4.6.** \( \text{LLM-FTop} \) is an isomorphism-closed full proper subcategory of \( \text{SLM-FTop} \) which is simultaneously bireflective and bicoreflective in \( \text{SLM-FTop} \), and given a stratified LM-fuzzy topological space \( (X, \delta) \), its reflection and coreflection are given by \( \text{id}_X : (X, \delta) \rightarrow (X, I_L(\delta)) \) and \( \text{id}_X : (X, E_L(\delta)) \rightarrow (X, \delta) \), respectively, where \( \text{id}_X : X \rightarrow X \) is the identity mapping.

As every right adjoint preserves limits and every left adjoint preserves colimits, we have the following Corollaries.

**Corollary 4.7.**

(1) Let \( \{(X_i, \tau_i)\}_{i \in I} \) be a family of LM-fuzzy topological spaces. Then

\[
E_L \left( \bigotimes_{i \in I} X_i, \bigotimes_{i \in I} \delta_i \right) = \left( \bigotimes_{i \in I} X_i, \bigotimes_{i \in I} E_L(\delta_i) \right).
\]

(2) Let \( \{(X_i, \tau_i)\}_{i \in I} \) be a family of stratified LM-fuzzy topological spaces, different \( X_i \)'s be disjoint. Then

\[
i_L \left( \bigoplus_{i \in I} X_i, \bigoplus_{i \in I} \delta_i \right) = \left( \bigoplus_{i \in I} X_i, \bigoplus_{i \in I} i_L(\delta_i) \right).
\]

**Corollary 4.8.**

(1) Let \( (X, \delta) \) be an LM-fuzzy topological space and \( Y \subset X \). Then \( E_L(\delta|Y) = E_L(\delta)|Y \).

(2) Let \( (X, \delta) \) be a stratified LM-fuzzy topological space and \( f : X \rightarrow Y \) be a surjective mapping. \( (Y, \delta|f_L^-) \) is the LM-fuzzy quotient space of \( (X, \delta) \) with respect to \( f \). Then \( i_L(\delta|f_L^-) = i_L(\delta)/f_L^- \).

**Theorem 4.9.**

(1) Let \( (X, \tau) \) be a stratified LM-fuzzy topological space and \( U \subset X \). Then \( i_L(\tau)\mid U \leq i_L(\tau|U) \).

(2) Let \( (X, \delta) \) be an LM-fuzzy topological space and \( (Y, \delta|f_L^-) \) be the LM-fuzzy quotient space of \( (X, \delta) \) with respect to \( f \). Then \( E_L(\delta|f_L^-) \leq E_L(\delta)/f_L^- \).

**Proof.**

(1) We have \( i_L(\tau|U) \leq i_L(\tau|U) \) by Theorem 3.6 and the definition of \( i_L(\delta) \).

(2) We have \( E_L(\delta|f_L^-) \leq E_L(\delta)/f_L^- \) by Theorem 3.7 and the definition of \( E_L(\delta) \). \( \square \)
Remark 4.10. The following two counterexamples show that the above inequalities in Theorem 4.9 cannot be replaced by equalities.

(1) Let \( X = L = M = [0, 1], U = [0, 0.5] \) and \((X, \tau)\) be a stratified LM-fuzzy topological space, where \( \tau([a] \land 1_{[0,25,0.5]]) = \tau([1_{0,25,0.5}] \lor (0.5] \land 1_{[0,25,0.5]}) = 1 \) (\( \forall a \in L \)) and for others \( A \in L^X, \tau(A) = 0 \).

It is easy to verify that \( I_L(\tau) : L^X \rightarrow M \) is defined by \( I_L(\tau)([a] \land 1_{[0,25,0.5]}) = I_L(\tau)([a]) = 1 \) (\( \forall a \in L \)) and for others \( A \in L^X, I_L(\tau)(A) = 0 \).

On the other hand, \( I_L(\tau)(U(1_{[0,25,0.5]}) \lor (0.5] \land 1_{[0,25,0.5]}) = 0 \). On the other hand, \( I_L(\tau)(U) = 1 \).

Therefore, \( I_L(\tau)(U) < I_L(\tau)(U) \).

(2) Let \( X = [-1, 1], Y = L = M = [0, 1], \delta : L^X \rightarrow M \) is defined by \( \delta(0_X) = \delta(1_X) = \delta([0.5] \land 1_{[0,5,11]}) = \delta(1_{[-1,-0.5]}) \).

It is easy to verify that \( \delta/f_L^- : L^Y \rightarrow M \) is defined by \( \delta/f_L^-(0_Y) = \delta/f_L^-(1_Y) = 1 \) and for others \( B \in L^Y, \delta/f_L^-(B) = 0 \).

Hence \( E_L(\delta/f_L^-)([0.5] \land 1_{[0,5,11]}) = 0 \). On the other hand, \( E_L(\delta/f_L^-)([0.5] \land 1_{[0,5,11]}) = E_L(\delta)([0.5] \land 1_{[0,5,11]}) = 1 \). Therefore, \( E(\delta/f_L^-) < E(\delta)/f_L^- \).

Moreover, we have

Theorem 4.11. Let \( \{(X_t, \delta_t)\}_{t \in T} \) be a family of LM-fuzzy topological spaces, different \( X_t \)'s be disjoint. Then \( \bigoplus_{t \in T} E_L(\delta_t) = E_L(\bigoplus_{t \in T} \delta_t) \).

Proof. Clearly, \( E_L(\bigoplus_{t \in T} \delta_t) \leq \bigoplus_{t \in T} E_L(\delta_t) \) by Theorem 3.9 and the definition of \( E_L(\delta) \). Conversely, let \( \lambda \in Copr(L) \) and \( \lambda \leq \bigoplus_{t \in T} E_L(\delta_t)(A) (\forall A \in L^X) \), i.e.,

\[
\lambda \leq \bigoplus_{t \in T} E_L(\delta_t)(A) = \bigvee_{t \in T} E_L(\delta_t)(A) = \bigwedge_{t \in T} \bigvee_{\lambda \in A} D_{\lambda}^t = A_{X_t} \bigwedge_{\lambda \in A} E_{\lambda}^t,
\]

Then for all \( t \in T \), there exists \( \{D_{\lambda}^t\}_{\lambda \in A_t} \subset L^X_t \) such that

(i) \( \bigvee_{\lambda \in A_t} D_{\lambda}^t = A_{X_t} \),

(ii) For each \( \lambda \in A_t \), there exists \( \{E_{\lambda}^t\}_{\lambda \in A_t^t} \subset L^X_t \) such that \( \bigcap_{\lambda \in A_t^t} E_{\lambda}^t = D_{\lambda}^t \),

(iii) For each \( \lambda \in A_t^t \), we have \( \lambda \leq \bigoplus_{t \in T} E_L(\delta_t)(A) \).

Let \( \{D_{\lambda}^t\}_{t \in T, \lambda \in A_t} \subset L^X_t \) be defined as follows:

\[
(D_{\lambda}^t)(x) = \begin{cases} D_{\lambda}^t(x), & x \in X_t \\ 0, & x \notin X_t \end{cases}
\]

\[
(E_{\lambda}^t)(x) = \begin{cases} E_{\lambda}^t(x), & x \in X_t \\ 0, & x \notin X_t \end{cases}
\]

Then we have

\[
\bigvee_{t \in T} \bigwedge_{\lambda \in A_t^t} (D_{\lambda}^t) = A_t, \quad \bigcap_{\lambda \in A_t^t} (E_{\lambda}^t) = (D_{\lambda}^t) \quad \text{and} \quad \phi(\delta_t)(E_{\lambda}^t) = \phi(\bigoplus_{t \in T} \delta_t)((E_{\lambda}^t)).
\]

Hence \( \lambda \leq \phi(\bigoplus_{t \in T} \delta_t)((E_{\lambda}^t)) \).

Remark 4.12. Let \( \{(X_t, \delta_t)\}_{t \in T} \) be a family of LM-fuzzy topological spaces. Then \( I_L(\prod_{t \in T} X_t, \prod_{t \in T} \delta_t) \geq (\prod_{t \in T} X_t, \prod_{t \in T} I_L(\delta_t)) \), and the inequality cannot be replaced by equality.
Proof. It is easy to verify that $I_L (\prod_{i \in T} X_i, 1 \prod_{i \in T} I_L (\delta_i)) \geq (\prod_{i \in T} X_i, 1 \prod_{i \in T} I_L (\delta_i))$ by Theorem 3.8 and the definition of $I_L (\delta)$. The following example shows that the inequality cannot be replaced by equality.

Let $X = L = M = [0, 1]$ and $(X, \tau)$ be a stratified $LM$-fuzzy topological space, where $\tau([a] \land I_{[0.25, 0.5}}) = \tau(I_{[0.5]} \land I_{[0.25, 0.5]}) = 1$ ($\forall a \in L$) and for others $A \in L^X, \tau(A) = 0$. It is easy to verify that $I_L (\tau) : L^X \rightarrow M$ is defined by $I_L (\tau)([a] \land I_{[0.25, 0.5}}) = I_L (\tau)(a) = 1$ ($\forall a \in L$) and for others $A \in L^X, I_L (\tau)(A) = 0$. On the one hand $I_L (\tau) \times I_L (\tau)([0.5] \land [0.25, 0.5]) = 0$. On the other hand, $\tau \times \tau([0.5] \land [0.25, 0.5]) = 1$ and $I_L (\tau \times \tau)([0.5] \land [0.25, 0.5]) = 1$. Therefore $I_L (\tau) \times I_L (\tau) < I_L (\tau \times \tau)$. □

5. Conclusion

In the paper, we firstly introduce the definition of a Lowen $LM$-fuzzy topological space and discuss its basic properties such as the properties that the product space and the sum space of Lowen $LM$-fuzzy topological spaces are also Lowen $LM$-fuzzy topological spaces. Secondly, we study some categorical properties of Lowen $LM$-fuzzy topological spaces. For example, based on the interior and exterior Lowen topology of an $LM$-fuzzy topology, we show that $LLM-FTop$ (the category of Lowen $LM$-fuzzy topological spaces) is isomorphism-closed and simultaneously bireflective and bicoreflective in $SLM-FTop$ (the category of stratified $LM$-fuzzy topological spaces). Moreover, we also show that $(X, \tau)$ is an induced $LM$-fuzzy topological space iff $(X, \tau)$ is a Lowen $LM$-fuzzy topological space and an (IC)$LM$-fuzzy topological space. In the future, there are still some categorical properties of Lowen $LM$-fuzzy topological spaces which are worth studying. For example, is $LLM-FTop$ Cartesian closed?

Acknowledgments

The authors wish to thank Prof. S.E. Rodabaugh and the anonymous referees for their valuable comments and helpful suggestions.

References