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Boundedness, persistence and extinction of a stochastic non-autonomous logistic system with time delays ☆

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ABSTRACT

This paper investigates the stochastic non-autonomous logistic system with time delays. Under two simple assumptions on the environmental noise, it is shown that the stochastic system has a unique global positive solution, and this positive solution is asymptotically bounded. The conditions for extinction, weak persistence of solutions are also obtained by the exponential martingale inequality. Finally, a numerical example is provided to illustrate our results.

1. Introduction

The dynamic analysis of logistic system, in which the members of a single species compete among themselves for a limited amount of food and living space, plays an important role in mathematical biology. The classical non-autonomous logistic equation can be expressed as follows:

$$\dot{x}(t) = x(t)[r(t) - a(t)x(t)], \quad t \geqslant 0, \tag{1}$$

where $x(t)$ is the population density of a single species; $r(t)$, $a(t)$ are continuous bounded functions on $R_+ = [0,+\infty)$; and $r(t)$ is the rate of growth and $r(t)/a(t)$ stands for the carrying capacity at time t, more details see [1]. The initial value of system (1) is $x(0) = x_0 > 0$.

The system (1) has been well concerned and has obtained a lot of interesting results about the stability of positive solutions, see [1–6] and the references therein. However, the natural growth of many populations in the real world is inevitably affected by the random disturbance, which is an important component in an ecosystem [7,8]. Mao et al. [9] has recently revealed an important fact that the environmental noise can suppress a potential population explosion. Thereby, more and more researchers begin to investigate ecological systems with random perturbation subjected to environmental noise.

There are two main aspects that are considered in the literature about the effect of the environmental fluctuations in system (1). One is to assume that the white noise affects $a(t)$ mainly [10–14], i.e., $a(t) \to a(t) + \alpha(t)\dot{B}(t)$. And then by system (1), the environmental perturbed system may be described by the following stochastic differential equation

$$dx(t) = x(t)[r(t) - a(t)x(t)]dt + \alpha(t)x^2(t)dB(t), \quad t \geqslant 0, \tag{2}$$

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where $\alpha(t)$ is a continuous bounded function on R_+ and $\alpha^2(t)$ represents the intensity of the white noise at time t; $\dot{B}(t)$ is the white noise, namely $B(t)$ is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathcal{R}_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). The other is to assume that the noise affects the growth rate $r(t)$ mainly with $r(t) \to r(t) + \alpha(t)\dot{B}(t)$, (see, [14–22]). Then the stochastic form of logistic system will be

$$dx(t) = x(t)[r(t) - a(t)x(t)]dt + \alpha(t)x(t)dB(t), \quad t \geqslant 0. \tag{3}$$

Wang and Liu had investigated the persistence and extinction of systems (2) and (3) in [14], and obtained some significative results. However, like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. And stochastic delay system has received more and more attention and has had lots of nice results [23–27]. Motivated by works mentioned above, in this paper we consider a more general non-autonomous logistic system with time delays,

$$dx(t) = x(t)\left[r(t) - a(t)x(t) - b(t)x(t-\tau) - c(t)\int_{-\infty}^{0} x(t+s)d\mu(s)\right]dt + x(t)[\sigma_1(t) + \sigma_2(t)x(t) + \sigma_3(t)x(t-\tau)]dB(t), \tag{4}$$

where $r(t)$, $a(t)$, $b(t)$, $c(t)$, $\sigma_i(t)$ are continuous bounded functions on R_+; $\tau > 0$ is a time delay; $B(t)$ is a one-dimensional Brownian motion. Let the initial data $\xi(s)$ be positive and belong to the friendly spaces $\mathcal{C}_k$ (see [28]) which defined by

$$\mathcal{C}_k := \left\{\xi \in C((-\infty,0];R_+) : \|\xi\|_{\mathcal{C}_k} = \sup_{-\infty\leqslant s\leqslant 0} e^{ks}|\xi(s)| < \infty,\ k > 0\right\}. \tag{5}$$

Then $\mathcal{C}_k$ is an admissible Banach space. And μ is the probability measure on $(-\infty,0]$ satisfying

$$\mu_k := \int_{-\infty}^{0} e^{-2ks}d\mu(s) < \infty. \tag{6}$$

Now we give some notations that will be used in the following sections.

$$\bar{v} = \sup_{t\in R_+} v(t), \quad \underline{v} = \inf_{t\in R_+} v(t), \quad \langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)ds,$$
$$f_* = \liminf_{t\to\infty} f(t), \quad f^* = \limsup_{t\to\infty} f(t),$$
$$g(t,x,x(t-\tau)) = \sigma_1(t) + \sigma_2(t)x(t) + \sigma_3(t)x(t-\tau).$$

Throughout this paper, we assume that

(H1) $b(t)$, $c(t)$, $\sigma_1(t)$, $\sigma_3(t) > 0$ and $\underline{a}$, $\underline{\sigma_2} > 0$;
(H2) There exist constants $0 < r_2 < 1 < \rho$ such that $\frac{\underline{\sigma}_2^2}{\rho} - \frac{\bar{\sigma}_3^2}{(\rho-1)r_2} > 0$.

And the following inequalities will be used frequently in this paper.

$$\frac{u}{\rho} - \frac{v}{\rho-1} \leqslant (u+v)^2 \leqslant \frac{u}{r} + \frac{v}{1-r}, \quad u, v \in R,\ 0 < r < 1 < \rho, \tag{7}$$

$$u^\alpha v^\beta \leqslant \frac{\alpha u^{\alpha+\beta} + \beta v^{\alpha+\beta}}{\alpha+\beta}, \quad u, v \geqslant 0,\ \alpha, \beta > 0. \tag{8}$$

We also need to introduce the definitions of extinction, non-persistence in mean, and weakly persistence [14,29], which will be used in Section 4.

Definition 1. A solution $x = x(t)$ with the initial data $\xi(s) \in \mathcal{C}_k$ of (4) is called a population. A population, x, goes to extinction if $\lim_{t\to+\infty} x(t) = 0$. It is non-persistent in mean if $\langle x\rangle^* = 0$; and it is weakly persistent provided $x^* > 0$.

This paper is organized as follows: in Section 2, we show that system (4) admits a unique solution and the solution will remain in R^+ with probability one. From the biological point of view, the asymptotic bound property is more desired than nonexplosion property, and we consider it in Section 3. Section 4 discusses the extinction, non-persistent in mean and weakly persistence of the solution of system (4). In the last section, an example with its numerical simulation is provided to illustrate our results.

2. Global positive solutions

Theorem 1. *Assume (H1) and (H2) hold. For any initial data $\xi \in \mathcal{C}_k$, there is a unique global positive solution $x(t) = x(t,\xi)$ to system (4).*

Proof. Obviously, the coefficients of (4) are locally Lipschitz continuous, then there is a unique local solution $x(t)$ on $t \in (-\infty, \tau_e]$, where τ_e is the explosion time. Now we show that $x(t)$ is in fact a global solution, i.e., $\tau_e = \infty$. Let $D_k = (1/k, k)$, and define the stopping time $\sigma_k = inf\{-\tau \leqslant t \leqslant \tau_e : x(t) \notin D_k\}$. Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \leqslant \tau_e$. In the following, we will show that $\tau_k \to \infty$ as $k \to \infty$.

Define $V(x) = 4\sqrt{x} - 2\ln x$. It is easy to see that $V(x) \geqslant 4$ on $x > 0$, $V(0) = V(\infty) = \infty$. Applying Itô's formula to $V(x)$, we have

$$dV(x(t)) = V_x dx + \frac{1}{2}V_{xx}(dx)^2 = \mathcal{L}V(x(t), x(t-\tau))dt + 2(\sqrt{x} - 1)g(t, x, x(t-\tau))\mathrm{d}B(t),$$

where

$$\mathcal{L}V(x(t), x(t-\tau)) = 2(\sqrt{x} - 1)\left[r(t) - a(t)x(t) - b(t)x(t-\tau) - c(t)\int_{-\infty}^{0} x(t+s)d\mu(s)\right] + \left(1 - \frac{1}{2}\sqrt{x}\right)g^2(t, x, x(t-\tau)). \quad (9)$$

We may compute that

$$2b(t)(1 - \sqrt{x})x(t-\tau) \leqslant \bar{b}^2(\sqrt{x} - 1)^2 + x^2(t-\tau), \quad (10)$$

$$2c(t)(1 - \sqrt{x})\int_{-\infty}^{0} x(t+s)ds \leqslant \bar{c}^2(\sqrt{x} - 1)^2 + \int_{-\infty}^{0} x^2(t+s)d\mu(s). \quad (11)$$

By (7), we have that

$$g^2(t, x, x(t-\tau) = (\sigma_1(t) + \sigma_2(t)x(t) + \sigma_3(t)x(t-\tau))^2 \leqslant \frac{\bar{\sigma}_3^2}{r_1}x^2(t-\tau) + \frac{(\sigma_1(t) + \sigma_2(t)x(t))^2}{1 - r_1} \quad (12)$$

and

$$g^2(t, x, x(t-\tau) \geqslant \frac{\underline{\sigma}_2^2}{\rho}x^2(t) - \frac{(\sigma_1(t) + \sigma_3(t)x(t-\tau))^2}{\rho - 1} \geqslant \frac{\underline{\sigma}_2^2}{\rho}x^2(t) - \frac{\bar{\sigma}_3^2 x^2(t-\tau)}{(\rho - 1)r_2} - \frac{\sigma_1^2(t)}{(\rho - 1)(1 - r_2)}. \quad (13)$$

Then together with (8)

$$\begin{aligned} -\frac{1}{2}\sqrt{x}g^2(t, x, x(t-\tau) &\leqslant -\frac{\underline{\sigma}_2^2}{2\rho}x^{2.5}(t) + \frac{\bar{\sigma}_3^2 x^2(t-\tau)\sqrt{x}}{2(\rho - 1)r_2} + \frac{\sigma_1^2(t)\sqrt{x}}{2(\rho - 1)(1 - r_2)} \\ &\leqslant -\frac{\underline{\sigma}_2^2}{2\rho}x^{2.5}(t) + \frac{\sigma_1^2(t)x^{0.5}}{2(\rho - 1)(1 - r_2)} + \frac{\bar{\sigma}_3^2}{2(\rho - 1)r_2}\frac{0.5x^{2.5} + 2x^{2.5}(t-\tau)}{2.5}. \end{aligned} \quad (14)$$

Substituting (10)–(14) into (9) yields

$$\mathcal{L}V(x(t), x(t-\tau)) = F(x) + \left(1 + \frac{\bar{\sigma}_3^2}{r}\right)(x^2(t-\tau) - x^2) + \left(\int_{-\infty}^{0} x^2(t+s)d\mu(s) - x^2\right) + \frac{2\bar{\sigma}_3^2}{5(\rho - 1)r_2}(x^{2.5}(t-\tau) - x^{2.5}), \quad (15)$$

where

$$\begin{aligned} F(x) &= 2r(t)(x^{0.5} - 1) - 2a(t)(x^{1.5} - x) + \frac{\sigma_1^2(t)}{2(\rho - 1)(1 - r_2)}x^{0.5} + (\bar{b}^2 + \bar{c}^2)(x^{0.5} - 1)^2 + \left(2 + \frac{\bar{\sigma}_3^2}{r}\right)x^2 \\ &\quad + \frac{(\sigma_1(t) + \sigma_2(t)x)^2}{1 - r_1} - \left(\frac{\underline{\sigma}_2^2}{\rho} - \frac{\bar{\sigma}_3^2}{2(\rho - 1)r_2}\right)x^{2.5}. \end{aligned} \quad (16)$$

It is straightforward to see that $F(x)$ is bounded when $(H2)$ holds, i.e., $F(x) \leqslant K$, $K \in R_+$. For any $\alpha > 0$, we have

$$\int_0^t x^\alpha(s-\tau)ds \leqslant \int_{-\tau}^{t-\tau} x^\alpha(\theta)d\theta = \int_{-\tau}^{0} x^\alpha(\theta)d\theta + \int_0^t x^\alpha(\theta)d\theta, \quad (17)$$

And for any given positive initial data $\xi \in C_k$, by (6), we obtain that

$$\begin{aligned} \int_0^t \int_{-\infty}^{0} x^2(s+\theta)d\mu(\theta)ds &= \int_0^t \left[\int_{-\infty}^{-s} x^2(s+\theta)d\mu(\theta) + \int_{-s}^{0} x^2(s+\theta)d\mu(\theta)\right]ds \\ &= \int_0^t ds\int_{-\infty}^{-s} e^{2k(s+\theta)}x^2(s+\theta)e^{-2k(s+\theta)}d\mu(\theta) + \int_{-t}^{0} d\mu(\theta)\int_{-\theta}^{t} x^2(s+\theta)ds \\ &\leqslant \|\xi\|_{C_k}^2\int_0^t e^{-2ks}ds\int_{-\infty}^{0} e^{-2k\theta}d\mu(\theta) + \int_{-\infty}^{0} d\mu(\theta)\int_0^t x^2(s)ds \leqslant \|\xi\|_{C_k}\mu_k t + \int_0^t x^2(s)ds. \end{aligned} \quad (18)$$

Denote by $\mathcal{M}^2([-\tau, \tau_e], R)$ the space of all real-valued measurable $\mathcal{F}_t$-adapted process $\{\zeta(t)\}_{t \leqslant \tau_e}$. Followed by the boundedness of $\sigma_i(t)$, $i = 1, 2, 3$ on $[0, +\infty)$, we have $(\sqrt{x(t)} - 1)g(t, x(t), x(t-\tau)) \in \mathcal{M}^2([-\tau, \tau_e], R)$ for the solutions $x(t)$ of (4) (see, [30]). Then we have

$$E\left[\int_0^t (\sqrt{x(s)}-1)g(s,x(s),x(s-\tau))dB(s)=0,\right.$$

and by (16)–(18) we have

$$EV(x(t)) = EV(\xi(0)) + E\int_0^t \mathcal{L}V(x(t),x(t-\tau))ds \leqslant EV(\xi(0)) + Kt + E\int_0^t\left[\int_{-\infty}^0 x^2(s+\theta)d\mu(\theta) - x^2(s)\right]ds$$
$$+\left(1+\frac{\bar{\sigma}_3^2}{r}\right)E\int_0^t (x^2(s-\tau(s))-x^2)ds + \frac{2\bar{\sigma}_3^2}{5(\rho-1)r_2}E\int_0^t (x^{2.5}(s-\tau(s))-x^{2.5})ds \leqslant Const + (K+\check{b}\|\xi\|_{C_k}\mu_k)t,$$

where $Const = EV(\xi(0)) + \left(1+\frac{\bar{\sigma}_3^2}{r}\right)E\int_{-\tau}^0 x^2(s)ds + \frac{2\bar{\sigma}_3^2}{5(\rho-1)r_2}E\int_{-\tau}^0 x^{2.5}(s)ds$. Let $t = \tau_k \wedge T$, we obtain that

$$EV(x(\tau_k\wedge T)) \leqslant Const + (K+\check{b}\|\xi\|_{C_r}\mu_r)T.$$

By the definition of τ_k, we have $x(\tau_k) = k$ or $x(\tau_k) = 1/k$, and

$$\mathbb{P}(\tau_k \leqslant T)[V(k)\wedge V(1/k)] \leqslant \mathbb{P}(\tau_k\leqslant T)V(x(\tau_k\wedge T)) \leqslant EV(x(t)) \leqslant Const + (K+\check{b}\|\xi\|_{C_r}\mu_r)T,$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}(\tau_k\leqslant T) \leqslant \lim_{k\to\infty}\frac{Const + (K+\check{b}\|\xi\|_{C_r}\mu_r)T}{[V(k)\wedge V(1/k)]} = 0.$$

Then we have $\mathbb{P}(\tau_e \leqslant \infty) = 0$ as required since $T > 0$ is arbitrary. □

3. Asymptotic bound property

In this section, we will discuss asymptotic bounded property for the unique global positive solution of Eq. (4) under assumptions $(H1)$ and $(H2)$.

Theorem 2. *If (H1) and (H2) hold, for any $p\in(0,1)$, there exists a positive constant $K = K(p)$ independent of the initial data $\xi\in\mathcal{C}_r$ such that the solution $x(t)$ of system (4) has the property*

$$\limsup_{t\to\infty} Ex^p \leqslant K.$$

Proof. By $(H2)$, one have that there exists some $\varepsilon\in(0,k)$ such that

$$\frac{\underline{\sigma}_2^2}{\rho} - \frac{\bar{\sigma}_3^2(p+2e^{\varepsilon\tau})}{r_2(\rho-1)(p+2)} > 0,\quad p\in(0,1). \tag{19}$$

Define $V(x) = x^p$. For the ε given above, applying Itô's formula to $e^{\varepsilon t}V(x)$ and taking expectation yield

$$e^{\varepsilon t}EV(x(t)) = EV(\xi(0)) + E\int_0^t e^{\varepsilon s}[\mathcal{L}V(x(t),x(t-\tau)) + \varepsilon V(x(t))]ds,$$

where

$$\mathcal{L}V(x,x(t-\tau)) = px^p\left[r(t) - a(t)x - b(t)x(t-\tau) - c(t)\int_{-\infty}^0 x(t+s)d\mu(s) - \frac{1-p}{2}g^2(t,x,x(t-\tau))\right].$$

By $(H1)$, (8) and (13), we get

$$\begin{aligned}\mathcal{L}V(x(t),x(t-\tau)) + \varepsilon V(x) &\leqslant (pr(t)+\varepsilon)x^p - \frac{p(1-p)}{2}x^p g^2(t,x,x(t-\tau))\\
&\leqslant (pr(t)+\varepsilon)x^p - \frac{p(1-p)\underline{\sigma}_2^2}{2\rho}x^{p+2} + \frac{p(p-1)\bar{\sigma}_3^2}{2r_2(\rho-1)}\left(\frac{px^{p+2}+2x^{p+2}(t-\tau)}{p+2}\right)\\
&\quad + \frac{p(1-p)\sigma_1^2(t)x^p}{2(\rho-1)(1-r_2)}\\
&= H(x) + \frac{p(1-p)\bar{\sigma}_3^2}{r_2(\rho-1)(p+2)}(x^{p+2}(t-\tau) - e^{\varepsilon\tau}x^{p+2}(t)),\end{aligned} \tag{20}$$

where

$$H(x) = \left(pr(t) + \frac{p(1-p)\sigma_1^2(t)}{2(\rho-1)(1-r_2)} + \varepsilon\right)x^p - \frac{p(1-p)}{2}\left[\frac{\underline{\sigma}_2^2}{\rho} - \frac{\bar{\sigma}_3^2(p+2e^{\varepsilon\tau})}{r_2(\rho-1)(p+2)}\right]x^{p+2}.$$

So, (19) implies that $H(x)$ is bounded, i.e., $H(x) \leqslant K_1$, $K_1 \in R_+$. And we can also compute that

$$\int_0^t e^{\varepsilon s} x^{p+2}(s-\tau)ds = \int_{-\tau}^{t-\tau} e^{\varepsilon(\theta+\tau)} x^{p+2}(\theta)d\theta \leqslant e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon s}x^{p+2}(s)ds + e^{\varepsilon\tau}\int_0^t e^{\varepsilon s}x^{p+2}(s)ds.$$

That is

$$\int_0^t e^{\varepsilon s}(x^{p+2}(s-\tau) - e^{\varepsilon\tau}x^{p+2}(s))ds \leqslant e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon s}x^{p+2}(s)ds.$$

So we have

$$\begin{aligned} e^{\varepsilon t}EV(x(t)) &= EV(\xi(0)) + E\int_0^t e^{\varepsilon s}[\mathcal{L}V(x(t),x(t-\tau)) + \varepsilon V(x(t))]ds \\ &\leqslant EV(\xi(0)) + E\int_0^t e^{\varepsilon s}H(x)ds + \frac{p(1-p)\bar{\sigma}_3^2}{r_2(\rho-1)(p+2)}E\int_0^t e^{\varepsilon s}(x^{p+2}(t-\tau) - e^{\varepsilon\tau}x^{p+2}(t))\Big]ds \\ &\leqslant EV(\xi(0)) + \varepsilon^{-1}K_1e^{\varepsilon t} + I, \end{aligned}$$

where $I = \frac{p(1-p)\bar{\sigma}_3^2}{r_2(\rho-1)(p+2)}E\int_{-\tau}^0 e^{\varepsilon s}x^{p+2}(s)ds$. Then one can obtain that

$$\limsup_{t\to\infty} Ex^p(t) = \limsup_{t\to\infty} EV(x(t)) \leqslant \varepsilon^{-1}K_1$$

The proof is completed. □

4. Extinction and persistence in time average

Now, we will discuss extinction and persistence of the unique global positive solution of Eq. (4) under assumptions ($H1$) and ($H2$).

Theorem 3. *If (H1), (H2) hold, then*

(i) $\langle r\rangle^* < 0$ *species x(t) will go to extinction almost surely (a.s.);*
(ii) $\langle r\rangle^* = 0$ *species x(t) will will be non-persistent in the mean a.s.*

Proof. For case (i): $\langle r\rangle^* < 0$. Applying Itô's formula to $\ln x(t)$ leads to

$$d\ln x = \frac{dx}{x} - \frac{(dx)^2}{2x^2} = \left[r(t) - a(t)x - b(t)x(t-\tau) - c(t)\int_{-\infty}^0 x(t+s)d\mu(s) - \frac{1}{2}g^2(t,x,x(t-\tau))\right]dt + g(t,x,x(t-\tau))dB(t),$$

Then we have

$$\ln x(t) = \ln x(0) + \int_0^t \left[r(s) - a(s)x - b(s)x(s-\tau) - c(s)\int_{-\infty}^0 x(s+\theta)d\mu(\theta) - \frac{1}{2}g^2(s,x,x(s-\tau))\right]ds + M(t), \tag{21}$$

where $M(t) = \int_0^t g(t,x,x(t-\tau))dB(s)$ is a real-valued continuous local martingale vanishing at $t = 0$. For every integer $n \geqslant 1$, by exponential martingale inequality, we have

$$P\left\{\sup_{0\leqslant t\leqslant n}\left[M(t) - \frac{1}{2}\int_0^t g^2(t,x,x(t-\tau))ds\right] > 2\ln n\right\} \leqslant \frac{1}{n^2}.$$

It is clear that $\sum_{n=1}^\infty n^{-2}$ converges, the well-known Borel–Cantelli lemma yields that there exists an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $k_0(\omega)$, when $n \geqslant k_0(\omega)$ and $n-1 \leqslant t \leqslant n$,

$$M(t) \leqslant 2\ln n + \frac{1}{2}\int_0^t g^2(t,x,x(t-\tau))ds. \tag{22}$$

From (21), (22) we obtain

$$\ln x(t) \leqslant \ln x(0) + \int_0^t r(s)ds - \int_0^t a(s)x(s)ds + 2\ln n \leqslant \ln x(0) + \int_0^t r(s)ds + 2\ln n \tag{23}$$

for $n \geqslant n_0(\omega)$ and $n-1 \leqslant t \leqslant n$. That is

$$t^{-1}(\ln x(t) - \ln x(0)) < t^{-1}\int_0^t r(s)ds + \frac{2\ln n}{n-1},$$

which indicates $[t^{-1}\ln x(t)]^* \leqslant \langle r\rangle^* < 0$. And one can obtain that $\lim_{t\to\infty} x(t) = 0$, which means that species $x(t)$ will go to extinction almost surely (a.s.). Case (ii): $\langle r\rangle^* = 0$. This implies that for $\forall\ \epsilon > 0$, $\exists\ T_1$ such that $\int_0^t r(s)ds \leqslant \epsilon t/2$ for $t \geqslant T_1$. Then it follows from (23) that

$$\ln x(t) \leqslant \ln x(0) + \epsilon t/2 - \int_0^t a(s)x(s)ds + 2\ln n, \tag{24}$$

for $n-1 \leqslant t \leqslant n$, $n \geqslant \max\{k_0(\omega), T_1+1\}$. Note that for sufficiently large $t \in (n-1, n)$, we have $2(\ln n)/(n-1) \leqslant \epsilon t/2$. Then

$$\ln x(t) \leqslant \ln x(0) + \epsilon t - \underline{a}\int_0^t x(s)ds.$$

Let $h(t) = \int_0^t x(s)ds$, then we have deduced that

$$e^{\underline{a}h(t)}dh(t) \leqslant x(0)e^{\epsilon t}dt.$$

Integrating this inequality from $T > T_1$ to t results in

$$e^{\underline{a}h(t)} \leqslant e^{\underline{a}h(T)} + x(0)\underline{a}\epsilon^{-1}e^{\epsilon t} - x(0)\underline{a}\epsilon^{-1}e^{\epsilon T}.$$

It follows that

$$h(t) \leqslant \underline{a}^{-1}\ln(x(0)\underline{a}\epsilon^{-1}e^{\epsilon t} + e^{\underline{a}h(T)} - x(0)\underline{a}\epsilon^{-1}e^{\epsilon T}),$$

which indicates

$$[t^{-1}h(t)]^* \leqslant \underline{a}^{-1}\limsup_{t\to\infty}\ln(x(0)\underline{a}\epsilon^{-1}e^{\epsilon t} + e^{\underline{a}h(T)} - x(0)\underline{a}\epsilon^{-1}e^{\epsilon T})/t \leqslant \epsilon/\underline{a},$$

Since ϵ is arbitrary and $x(t) > 0 (t > 0)$, one can obtain that $\langle x\rangle^* = 0$. So, species $x(t)$ will be non-persistent in mean a.s. □

Theorem 4. *If (H1), (H2) hold and $\langle r\rangle^* > 0$, then species x(t) will be weakly persistent a.s.*

Proof. Applying Itô's formula to $e^{\epsilon t}\ln x(t)$, $\epsilon \in (0, 2k)$ leads to

$$\begin{aligned} d(e^{\epsilon t}\ln x(t)) &= e^{\epsilon t}\ln x(t)dt + e^{\epsilon t}d\ln x(t) \\ &= e^{\epsilon t}\left[\ln x(t) + r(t) - a(t)x - b(t)x(t-\tau)) - c(t)\int_0^\infty x(t+s)d\mu(s) - \frac{1}{2}g^2(t,x,x(t-\tau))\right]dt \\ &+ e^{\epsilon t}g(t,x,x(t-\tau))dB(t), \end{aligned}$$

Then we have

$$e^{\epsilon t}\ln x(t) = \ln x(0) + \int_0^t e^{\epsilon t}\left[\ln x(s) + r(s) - a(s)x - b(s)x(s-\tau) - c(s)\int_{-\infty}^0 x(s+\theta)d\mu(\theta) - \frac{1}{2}g^2(t,x,x(t-\tau))\right]ds + N(t), \tag{25}$$

where $N(t) = \int_0^t e^{\epsilon t}g(t,x,x(t-\tau))dB(s)$ is a real-valued continuous local martingale vanishing at $t = 0$. For every integer $k \geqslant 1$, $p \in (0,1)$ and $\delta > 1$, by exponential martingale inequality, we have

$$P\left\{\sup_{0\leqslant t\leqslant k}\left[N(t) - \frac{p}{2e^{\epsilon k}}\int_0^t e^{2\epsilon s}g^2(s,x,x(s-\tau))ds\right] > \frac{\delta e^{\epsilon k}\ln k}{p}\right\} \leqslant \frac{1}{k^\delta}.$$

It is clear that $\sum_{k=1}^\infty k^{-\delta}$ converges, the well-known Borel–Cantelli lemma yields that there exists an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $k_0(\omega)$, when $k \geqslant k_0(\omega)$ and $k-1 \leqslant t \leqslant k$,

$$N(t) \leqslant \frac{p}{2}\int_0^t e^{\epsilon s}g^2(s,x,x(s-\tau))ds + \frac{\delta e^{\epsilon(t+1)}\ln(t+1)}{p}. \tag{26}$$

From (25), (26) we obtain

$$e^{\epsilon t}\ln x(t) \leqslant \ln x(0) + \int_0^t e^{\epsilon s}H(x)ds + \frac{\delta e^{\epsilon(t+1)}\ln(t+1)}{p},$$

where

$$\begin{aligned} H(x) &= \ln x(s) + r(s) - a(s)x - b(s)x(s-\tau(s)) - c(s)\int_{-\infty}^0 x(s+\theta)d\mu(\theta) - \frac{1-p}{2}g^2(t,x,x(t-\tau)) \leqslant \ln x(s) + r(s) - a(s)x(s) \\ &\leqslant K_2 < \infty, \end{aligned}$$

for $b(t), c(t) > 0$, $p \in (0,1)$. Then we obtain

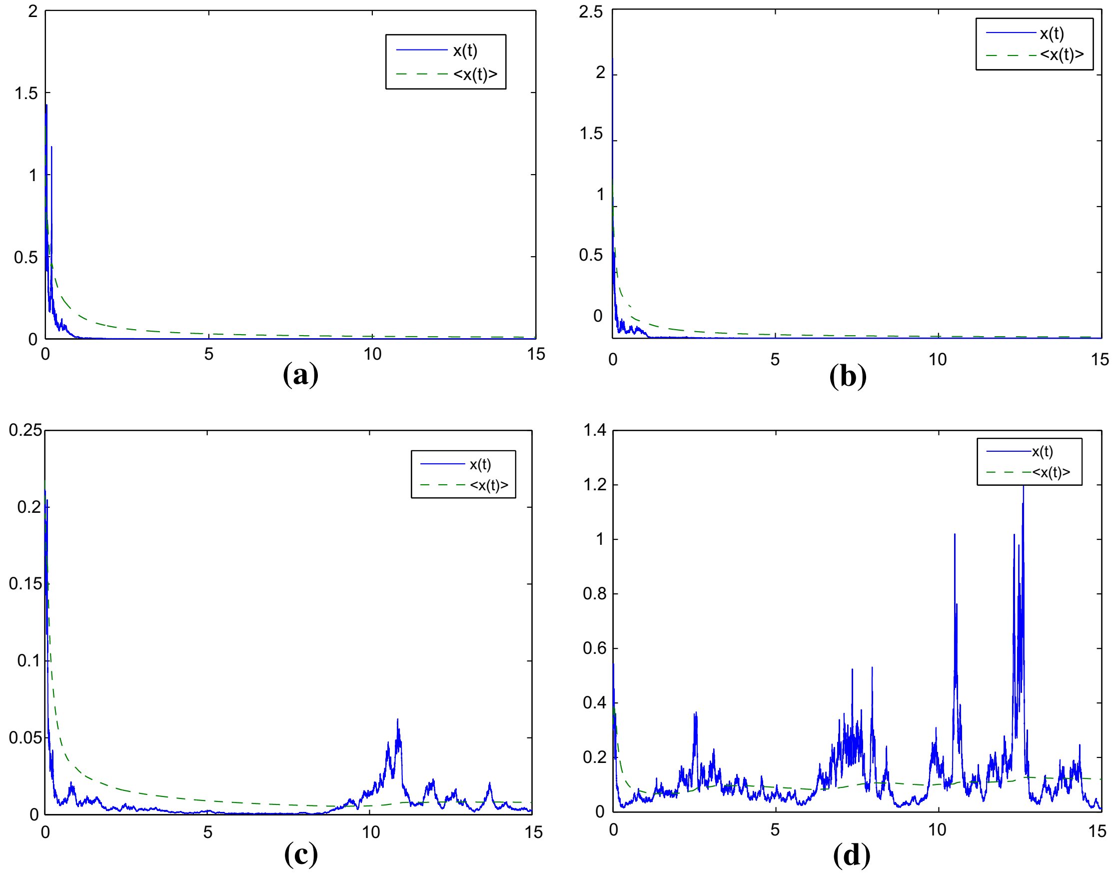

Fig. 1. Solutions of system (29) for $a(t) = 0.2 + 0.01\cos 2t$, $b(t) = c(t) = 0.3 + 0.05\sin 2t$, $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 3 + \sin t$, $x(s) = \xi(s) = e^{-0.5s}$ as $s < 0$, $\tau = 0.3$, step size $\Delta = 0.003$. (a) Choose $r(t) = -1 + 0.5\sin 2t$, (b) choose $r(t) = 0.5\sin 2t$, (c) choose $r(t) = 0.02 + 0.1\sin t$ and (d) choose $r(t) = 0.5 + 0.2\sin 2t$, which correspond to $\langle r\rangle^* > 0$, $\langle r\rangle^* = 0$, $\langle r\rangle^* > 0$ and $\langle r\rangle_* > 0$, respectively.

$$\ln x(t) \leqslant e^{-\epsilon t}\ln x(0) + \epsilon^{-1}K_2(1 - e^{-\epsilon t}) + p^{-1}\delta e^{\epsilon}\ln(t+1).$$

for $k - 1 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega$. This implies

$$\limsup_{t\to\infty} \ln x(t)/t \leqslant 0. \tag{27}$$

Now, we will prove that $x^* > 0$ for $\langle r\rangle^* > 0$. If this assert is not true, then there exists a positive initial data $\xi \in \mathcal{C}_r$ such that $x^* = 0$, which means that $\lim_{t\to\infty} x(t) = 0$. By (21), we have

$$\ln x(t)/t = \ln x(0)/t + \langle r(s)\rangle - \langle a(s)x\rangle - \langle b(s)x(s-\tau)\rangle - \langle c(s)\int_{-\infty}^{0} x(s+\theta)d\mu(\theta)\rangle - \frac{1}{2}\langle g^2(s,x,x(s-\tau))\rangle + M(t)/t. \tag{28}$$

Then

$$[\ln x(t)/t]^* = \langle r\rangle^* > 0,$$

which contradicts (27). So, species $x(t)$ will be weakly persistent a.s. □

Furthermore, for the case $\langle r\rangle_* > 0$, the delays make it difficult to investigates the persistence of the species $x(t)$, and we will consider this problem in the future work. However, the numerical simulations show that species $x(t)$ will be persistent for $\langle r\rangle_* > 0$ a.s. in Section 5.

5. Numerical simulations

In this section, we explore system behavior using numerical solutions of the stochastic non-autonomous system (4). For convenience, let the probability measure $\mu(\theta)$ be e^{θ} on $(-\infty, 0]$. So the stochastic non-autonomous system (4) will be written as

$$dx(t) = x(t)\left[r(t) - a(t)x(t) - b(t)x(t-\tau) - c(t)e^t\int_{-\infty}^{0} e^s\xi(s)ds - c(t)e^t\int_0^t e^s x(s)ds\right]dt + x(t)[\sigma_1(t) + \sigma_2(t)x(t) + \sigma_3(t)x(t-\tau)]dB(t), \quad (29)$$

where $a(t) = 0.2 + 0.01\cos 2t$, $b(t) = c(t) = 0.3 + 0.05\sin 2t$, $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 3 + \sin t$ and $x(t) = \xi(t)$ as $t < 0$. Obviously, $(H1)$, $(H2)$ hold. There exists a unique global positive solution $x(t) = x(t,\xi)$ to system (29) for any initial data $\xi \in \mathcal{C}_k$. Take $\xi(s) = e^{-0.5s}$, $\tau = 0.3$ and step size $\Delta = 0.003$. We employ the Euler scheme to discretize this equation, where the integral term is approximated by using the composite θ-rule as a quadrature [31], we may obtain the discrete approximate solution with respect to (29)

$$x_{k+1} = x_k + x_k\Delta\left[r(k\Delta) - a(k\Delta)x_k - b(k\Delta)x_{k-1000} - c(k\Delta)e^{k\Delta}\int_{-\infty}^{0} e^{0.5s}ds - \Delta c(k\Delta)e^{k\Delta}\sum_{j=0}^{k}\omega_j^{(k)}e^{j\Delta}x_j\right] + x_k[1 + (3 + \sin(k\Delta))x_k + x_{k-1000}]\Delta B_k,$$

where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$, $k = 0, 1, 2, \ldots$, the general composite θ-rule has weights

$$\left\{\omega_0^{(k)}, \omega_1^{(k)}, \ldots, \omega_k^{(k)}\right\} = \{\theta, 1, \ldots, 1-\theta\}, \quad \theta \in [0,1]$$

and $\sum_{j=0}^{k}\omega_j^{(k)} = k,\ k \geqslant 0$. Fig. 1 shows the numerical simulation results for different $r(t)$. In Fig. 1, (a) we choose $r(t) = -1 + 0.5\sin 2t$, (b) we choose $r(t) = 0.5\sin 2t$ and (c) we choose $r(t) = 0.02 + 0.1\sin(t)$ which correspond to $\langle r\rangle^* < 0$, $\langle r\rangle^* = 0$ and $\langle r\rangle^* > 0$, respectively. For the case $r(t) = 0.5 + 0.2\sin 2t$, which correspond to $\langle r\rangle_* > 0$, we also give the an numerical simulation in Fig. 1(d). And the numerical simulations show that species $x(t)$ will be persistent for $\langle r\rangle_* > 0$.

6. Conclusions

In this paper, we consider the asymptotic property of global positive solution, weakly persistence and extinction of a stochastic non-autonomous logistic system with time delays. The model (4) is more general than some existing models, however, the delays make the task more complicated to deal with. The results in the paper show that under some suitable condition, model (4) still retains some well-known properties, for example, nonexplosion and boundedness. Time delays make it difficult to investigates the persistence of the species, we give some weakly persistence instead. And we will consider the persistence of the species $x(t)$ in future work. The numerical simulations show that species will be persistent under some conditions. Finally, numerical simulations are shown to verify the effectiveness of the obtained results.

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