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Global Exponential Stability Analysis of Cohen-Grossberg Neural Networks*

WAN An-hua(万安华)[1], WANG Mian-sen(王绵森)[2], PENG Ji-gen(彭济根)[2], MAO Wei-hua(毛卫华)[3]

(1. *Institute of Automation, Chinese Academy of Sciences, Beijing* 100080, *China*; 2. *Faculty of Science, Xi'an Jiaotong University, Xi'an* 710049, *China*; 3. *Department of Applied Mathematics, South China Agricultural University, Guangzhou* 510642, *China*)

Abstract: In this paper, the exponential stability of Cohen-Grossberg neural network model is considered. To avoid the difficulty of constructing a proper Lyapunov function, the generalized relative Dahlquist constant approach is employed to analyze the stability of neural networks, and sufficient conditions for the existence of a unique equilibrium and the global exponential stability of Cohen-Grossberg neural networks are presented. Moreover, the exponential convergence rate of the neural networks to stable equilibrium point is estimated. Our results improve the existing ones.

Key words: Exponential stability; Cohen-Grossberg neural networks; Generalized relative Dahlquist constant; Exponential decay estimate

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1. Introduction

The model that we consider in this paper is Cohen-Grossberg neural networks described by the following differential equations

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i(u_i(t))\Big[b_i(u_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + J_i\Big], i = 1, 2, \cdots, n, \tag{1}$$

where $n \geqslant 2$ is the number of neurons in the networks, $u_i(t)$ denotes the neuron state vector, a_i denotes an amplification function, b_i denotes a self-signal function, $W = (w_{ij})$ is the $n \times n$ connection weight matrix, f_i is an activation function, and J_i represents the constant external

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Biography: WAN An-hua, female, Han, Jiangxi, Ph. D., research major: control theory and its applications.

input. The model (1) includes a lot of models from neurobiology, population biology etc[1], and in particular, it takes the popular Hopfield-type neural networks[2] as a special case. Cohen-Grossberg neural networks model has promising potential applications in classification, parallel computing and associative memory. To date, much attention have been paid to the analysis of the stability of Cohen-Grossberg neural networks[1,3].

To the best of the authors' knowledge, the most common approach to stability analysis of neural networks is Lyapunov's direct method[1,3,4]. However, the construction of an appropriate Lyapunov function is usually rather difficult. This paper aims to analyze the exponential stability of Cohen-Grossberg neural networks by means of the generalized relative Dahlquist constant approach developed in [6]. With this powerful approach, we not only derive some new sufficient conditions for the global exponential stability of neural networks (1), but also precisely characterize the exponential decay estimate of the neural networks.

2. Preliminaries

In this section, we recall the generalized relative Dahlquist constant approach for exponential stability analysis of general nonlinear systems developed in [6].

Let $(X, \|\cdot\|)$ be a Banach space, and Ω be an open subset of X. Consider the following system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)), t \geqslant t_0, \tag{2}$$

where $F : \Omega \to X$ is a nonlinear operator, and $x(t) \in \Omega$.

Definition 1 Suppose x^* is an equilibrium point of system (2). System (2) is said to be exponentially stable on a neighborhood Γ of x^*, if there exist constants $c \geqslant 1$ and $\sigma > 0$ such that $\| x(t) - x^* \| \leqslant c\mathrm{e}^{-\sigma(t-t_0)} \| x_0 - x^* \| \ (t \geqslant t_0)$, where $x(t)$ is any solution of (2) initiated from $x_0 = x(t_0) \in \Gamma$.

In particular, if system (2) has a unique equilibrium point, and it is exponentially stable on the whole space X, then system (2) is said to be globally exponentially stable.

Definition 2[6] Suppose that Ω is an open subset of Banach space X, $F : \Omega \to X$ is an operator, and x^0 is any fixed point in Ω. The constant

$$\alpha_\Omega(F, x^0) = \sup_{x \in \Omega, x \neq x^0} \frac{1}{\| x - x^0 \|} \lim_{r \to +\infty} [\, \| (F + rI)x - (F + rI)x^0 \| - r \| x - x^0 \| \,] \tag{3}$$

is called the generalized relative Dahlquist constant of F at x^0.

The following important properties of the generalized relative Dahlquist constant will be useful for developing the main results in Section 3.

Lemma 1[6] Suppose that $x^* \in \Omega$ is an equilibrium point of system (2). If $\alpha_\Omega(F, x^*) < 0$, then there is no other equilibrium point in Ω than x^*.

Lemma 2[6] Suppose that $x^* \in \Omega$ is an equilibrium point of system (2). If $\alpha_\Omega(F, x^*) < 0$, then x^* is the unique equilibrium point of system (2) in Ω, x^* is exponentially stable, and the exponential decay of any solution $x(t)$ initiated from $x_0 = x(t_0) \in \Omega$ satisfies

$$\| x(t) - x^* \| \leqslant \mathrm{e}^{\alpha_\Omega(F, x^*)(t - t_0)} \| x_0 - x^* \|, \ \forall\, t \geqslant t_0. \tag{4}$$

Lemmas 1 and 2 indicate that the generalized relative Dahlquist constant can exactly characterize the uniqueness and exponential stability of equilibrium point of general nonlinear systems.

3. Global Exponential Stability Analysis of Neural Networks

In this section, we will analyze the stability of neural networks model (1) by means of the generalized relative Dahlquist constant approach.

The main difficulties for stability analysis of model (1) result from the nonlinearity of the functions a_i, b_i and f_i. Most of the previous works have been performed under some special assumptions on a_i, b_i and f_i such as differentiability and monotonicity. Wang et al.[3] investigated the exponential stability of model (1) under the following assumptions:

(H_1) Each a_i is bounded, locally Lipschitz continuous, and $0 < \underline{\alpha}_i \leqslant a_i(r) \leqslant \overline{\alpha}_i$;

(H_2) $b_i(\cdot)$ and $b_i^{-1}(\cdot)$, $i = 1,2,\cdots,n$, are locally Lipschitz continuous, and there exists a constant $\lambda_i > 0$ such that for any $u_i \in \mathbf{R}$, $i = 1,2,\cdots,n$,

$$(u_i - u_i^*)[b_i(u_i) - b_i(u_i^*)] \geqslant \lambda_i (u_i - u_i^*)^2; \tag{5}$$

(H_3) $f_i(\cdot)$, $i = 1,2,\cdots,n$ is Lipschitz continuous with the minimal Lipschitz constant

$$m_i = \sup_{r_1, r_2 \in \mathbf{R}, r_1 \neq r_2} \frac{| f_i(r_1) - f_i(r_2) |}{| r_1 - r_2 |},$$

and f_i is bounded.

By assumption (H_1), it follows that $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^{\mathrm{T}}$ is an equilibrium of system (1.1) if and only if $b_i(u_i^*) - \sum_{j=1}^{n} w_{ij} f_j(u_j^*) + J_i = 0$, $i = 1,2,\cdots,n$.

Under assumptions (H_1), (H_2) and (H_3), Wang et al.[3] investigated the exponential stability of (1) by constructing a Lyapunov function. They proved the following existence result of an equilibrium point.

Lemma 3[3] If conditions (H_1), (H_2), (H_3) hold, then for every set of external input J_i, there exists an equilibrium point for system (1).

In this paper, we make the same assumptions (H_1), (H_2), (H_3) as in [3]. Abandoning the usual requirements such as symmetry on the connection weight matrix $W = (w_{ij})_{n\times n}$, we do not impose any supposition on W. Now we investigate the exponential stability of the neural networks model (1) by the generalized relative Dahlquist constant approach.

In this paper, we always use the 1-norm in $\mathbf{R}^n$, i. e., for each $x \in \mathbf{R}^n$, $\| x \|_1 = \sum_{i=1}^{n} | x_i |$. Let $\operatorname{sign}(r)$ denote the sign function of $r \in \mathbf{R}$. In the following, $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^{\mathrm{T}}$ always denotes the equilibrium point of system (1).

Theorem 1 Suppose that conditions (H_1), (H_2), (H_3) hold, and

$$m_i \sum_{j=1}^{n} \overline{\alpha}_j | w_{ji} | < \underline{\alpha}_i \lambda_i, \quad i = 1,2,\cdots,n. \tag{6}$$

Then for each set of external input J_i, u^* is the unique equilibrium point of neural networks (1), which is globally exponentially stable, and the exponential decay estimate of any solution $u(t)$ of (1) initiated from $u_0 = u(t_0)$ is governed by

$$\| u(t) - u^* \|_1 \leqslant \mathrm{e}^{-b_1(t-t_0)} \| u_0 - u^* \|_1, \tag{7}$$

where $b_1 = \min\limits_{1\leqslant i\leqslant n}(\underline{\alpha}_i\lambda_i - m_i\sum\limits_{j=1}^{n}\bar{\alpha}_j \mid w_{ji}\mid)$.

Proof Define an operator $F: \mathbf{R}^n \to \mathbf{R}^n$ as $F(u) = (F_1(u), F_2(u), \cdots, F_n(u))^{\mathrm{T}}$, $\forall\, u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} \in \mathbf{R}^n$, where

$$F_i(u) = -a_i(u_i)[b_i(u_i) - \sum_{j=1}^{n} w_{ij} f_j(u_j) + J_i].$$

Then neural networks (1) is reformulated as the following nonlinear system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(u(t)), \quad t \geqslant t_0. \tag{8}$$

Since u^* is an equilibrium of (1), $F(u^*) = (F_1(u^*), F_2(u^*), \cdots, F_n(u^*))^{\mathrm{T}} = 0$.

Now consider the generalized relative Dahlquist constant of F at u^*

$$\begin{aligned}\alpha_{\mathbf{R}^n}(F, u^*) &= \sup_{u\in\mathbf{R}^n, u\neq u^*} \frac{1}{\| u - u^* \|_1} \lim_{r\to+\infty}[\,\|(F+rI)u - (F+rI)u^*\|_1 - r\| u - u^*\|_1] \\ &= \sup_{u\in\mathbf{R}^n, u\neq u^*} \frac{1}{\| u - u^* \|_1} \lim_{r\to+\infty}\sum_{i=1}^{n}[\,| F_i(u) + r(u_i - u_i^*)| - r\mid u_i - u_i^*\mid].\end{aligned}$$

For sufficiently large $r > 0$, we can claim that for any $i = 1, 2, \cdots, n$, when $u_i \neq u_i^*$,

$$\begin{aligned}| F_i(u) + r(u_i - u_i^*)| - r\mid u_i - u_i^*\mid &= F_i(u)\operatorname{sign}(u_i - u_i^*) \\ &= -a_i(u_i)\Big[b_i(u_i) - \sum_{j=1}^{n} w_{ij} f_j(u_j) + J_i\Big]\operatorname{sign}(u_i - u_i^*).\end{aligned}$$

In view of $b_i(u_i^*) - \sum\limits_{j=1}^{n} w_{ij} f_j(u_j^*) + J_i = 0$ and (5), we derive

$$\begin{aligned}&| F_i(u) + r(u_i - u_i^*)| - r\mid u_i - u_i^*\mid \\ =& -a_i(u_i)[b_i(u_i) - b_i(u_i^*)]\operatorname{sign}(u_i - u_i^*) \\ &+ \operatorname{sign}(u_i - u_i^*)a_i(u_i)\sum_{j=1}^{n} w_{ij}[f_j(u_j) - f_j(u_j^*)] \\ \leqslant& -\underline{\alpha}_i\lambda_i\mid u_i - u_i^*\mid + \Big| a_i(u_i)\sum_{j=1}^{n} w_{ij}[f_j(u_j) - f_j(u_j^*)]\Big|.\end{aligned}$$

When $u_i = u_i^*$, we readily deduce that

$$\begin{aligned}&| F_i(u) + r(u_i - u_i^*)| - r\mid u_i - u_i^*\mid \\ =& \Big| a_i(u_i)\sum_{j=1}^{n} w_{ij}[f_j(u_j) - f_j(u_j^*)]\Big| \\ =& -\underline{\alpha}_i\lambda_i\mid u_i - u_i^*\mid + \Big| a_i(u_i)\sum_{j=1}^{n} w_{ij}[f_j(u_j) - f_j(u_j^*)]\Big|.\end{aligned}$$

Consequently, for any $u_i \in \mathbf{R}, i = 1, 2, \cdots, n$, we infer that

$$\begin{aligned}&| F_i(u) + r(u_i - u_i^*)| - r\mid u_i - u_i^*\mid \\ \leqslant& -\underline{\alpha}_i\lambda_i\mid u_i - u_i^*\mid + a_i(u_i)\Big|\sum_{j=1}^{n} w_{ij}[f_j(u_j) - f_j(u_j^*)]\Big| \\ \leqslant& -\underline{\alpha}_i\lambda_i\mid u_i - u_i^*\mid + \bar{\alpha}_i\sum_{j=1}^{n}(\mid w_{ij}\mid \cdot m_j \mid u_j - u_j^*\mid).\end{aligned}$$

Then for all $u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} \in \mathbf{R}^n$,

$$\sum_{i=1}^{n}(|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*|)$$
$$\leqslant\sum_{i=1}^{n}\{-\underline{\alpha}_i\lambda_i|u_i-u_i^*|+\bar{\alpha}_i\sum_{j=1}^{n}(m_j|w_{ij}||u_j-u_j^*|)\}$$
$$=\sum_{i=1}^{n}(-\underline{\alpha}_i\lambda_i+m_i\sum_{j=1}^{n}\bar{\alpha}_j|w_{ji}|)|u_i-u_i^*|.$$

In accordance with condition (6), we deduce that

$$\frac{\sum_{i=1}^{n}\{|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*|\}}{\|u-u^*\|_1}\leqslant-\min_{1\leqslant i\leqslant n}(\underline{\alpha}_i\lambda_i-m_i\sum_{j=1}^{n}\bar{\alpha}_j|w_{ji}|).$$

Let $r\to+\infty$, and then take the supremum of the left side of the inequality for all $u\in\mathbf{R}^n(u\neq u^*)$, therefore we deduce that $\alpha_{\mathbf{R}^n}(F,u^*)\leqslant-b_1<0$. Hence, by virtue of Lemmas 1 and 2, we infer that model (1) has a unique equilibrium point u^*, which is globally exponentially stable, and the exponential decay estimate satisfies (7).

Remark 1 Under the same assumptions as Theorem 1, in [3], by the traditional Lyapunov direct method, also proved the global exponential stability of (1), and the exponential decay estimate obeys to

$$\sum_{i=1}^{n}|u_i(t)-u_i^*|\leqslant e^{-\bar{b}_1(t-t_0)}\sum_{i=1}^{n}|u_i(t_0)-u_i^*|, \tag{9}$$

where $\bar{b}_1=\min_{1\leqslant i\leqslant n}\{\underline{\alpha}_i\lambda_i-(\max_{1\leqslant j\leqslant n}\bar{\alpha}_j)m_i\sum_{j=1}^{n}|w_{ji}|\}$. Clearly, $\bar{b}_1\leqslant b_1$, which indicates that under the same assumptions, our result provides more precise estimation for the exponential convergence rate of the neural networks to stable equilibrium point than the result in [3].

In the following, denote $[y]^+=\max\{y,0\}$ for any $y\in\mathbf{R}$. It is clear that $\forall y\in\mathbf{R}, y\leqslant[y]^+$, and for any $p>0, [y]^+<p$ if and only if $y<p$. We will show that when in particular each f_i is monotonically nondecreasing, condition (6) can be further relaxed.

Theorem 2 Suppose that conditions (H_1), (H_2), (H_3) hold. If each f_i is monotonically nondecreasing, and

$$\underline{\alpha}_i w_{ii}^+ +\sum_{j=1,j\neq i}^{n}\bar{\alpha}_j|w_{ji}|<\frac{\underline{\alpha}_i\lambda_i}{m_i},\quad i=1,2,\cdots,n, \tag{10}$$

then for each set of external input J_i, u^* is the unique equilibrium point of neural networks (1), which is globally exponentially stable, and the exponential decay estimate is governed by

$$\|u(t)-u^*\|_1\leqslant e^{-b_2(t-t_0)}\|u_0-u^*\|_1, \tag{11}$$

where $b_2=\min_{1\leqslant i\leqslant n}\{\underline{\alpha}_i\lambda_i-m_i[\underline{\alpha}_i w_{ii}^+ +\sum_{j=1,j\neq i}^{n}\bar{\alpha}_j|w_{ji}|]\}$, and $u(t)$ is any solution of system (1) initiated from $u_0=u(t_0)$.

Proof Define the operator F and system (8) the same as in the proof of Theorem 1. Analogous to the proof of Theorem 1, we will prove that $\alpha_{\mathbf{R}^n}(F,u^*)\leqslant-b_2<0$. We have

$$\alpha_{\mathbf{R}^n}(F,u^*)=\sup_{u\in\mathbf{R}^n,u\neq u^*}\frac{1}{\|u-u^*\|_1}\lim_{r\to+\infty}\sum_{i=1}^{n}[|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*|].$$

For sufficiently large $r>0$, when $u_i\neq u_i^*$, we infer that

$$
\begin{aligned}
& |F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*| \\
=& -a_i(u_i)[b_i(u_i)-b_i(u_i^*)]\mathrm{sign}(u_i-u_i^*) \\
& +\mathrm{sign}(u_i-u_i^*)a_i(u_i)\sum_{j=1}^n w_{ij}[f_j(u_j)-f_j(u_j^*)] \\
\leqslant & -a_i(u_i)\lambda_i|u_i-u_i^*|+a_i(u_i)[w_{ii}|f_i(u_i)-f_i(u_i^*)| \\
& +\sum_{j=1,j\neq i}^n |w_{ij}||f_j(u_j)-f_j(u_j^*)|].
\end{aligned}
$$

When $u_i=u_i^*$, we deduce that

$$
\begin{aligned}
|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*| &= a_i(u_i)\Big|\sum_{j=1,j\neq i}^n w_{ij}[f_j(u_j)-f_j(u_j^*)]\Big| \\
&\leqslant -a_i(u_i)\lambda_i|u_i-u_i^*|+a_i(u_i)[w_{ii}|f_i(u_i)-f_i(u_i^*)| \\
&\quad +\sum_{j=1,j\neq i}^n |w_{ij}||f_j(u_j)-f_j(u_j^*)|].
\end{aligned}
$$

Therefore, for any $u\in\mathbf{R}^n$, we derive

$$
\begin{aligned}
& \sum_{i=1}^n[|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*|] \\
\leqslant & -\sum_{i=1}^n a_i(u_i)\lambda_i|u_i-u_i^*|+\sum_{i=1}^n a_i(u_i)[w_{ii}|f_i(u_i)-f_i(u_i^*)] \\
& +\sum_{j=1,j\neq i}^n |w_{ij}||f_j(u_j)-f_j(u_j^*)|] \\
=& -\sum_{i=1}^n a_i(u_i)\lambda_i|u_i-u_i^*|+\sum_{j=1}^n |f_j(u_j)-f_j(u_j^*)| \\
& \cdot\Big[a_j(u_j)w_{jj}+\sum_{i=1,i\neq j}^n a_i(u_i)|w_{ij}|\Big] \\
\leqslant & -\sum_{i=1}^n a_i(u_i)\lambda_i|u_i-u_i^*| \\
& +\sum_{j=1}^n\Big\{m_j|u_j-u_j^*|\Big[a_j(u_j)w_{jj}+\sum_{i=1,i\neq j}^n a_i(u_i)|w_{ij}|\Big]\Big\} \\
=& \sum_{j=1}^n\{(-\lambda_j+m_jw_{jj}^+)a_j(u_j)|u_j-u_j^*|+m_j\sum_{i=1,i\neq j}^n a_i(u_i)|w_{ij}||u_j-u_j^*|\} \\
\leqslant & -\sum_{j=1}^n\Big\{\underline{\alpha}_j(\lambda_j-m_jw_{jj}^+)-m_j\sum_{i=1,i\neq j}^n\bar{\alpha}_i|w_{ij}|\Big\}|u_j-u_j^*|.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
& \sum_{i=1}^n[|F_i(u)+r(u_i-u_i^*)|-r|u_i-u_i^*|] \\
\leqslant & -\sum_{j=1}^n\{\underline{\alpha}_j\lambda_j-m_j\Big[\underline{\alpha}_jw_{jj}^++\sum_{i=1,i\neq j}^n\bar{\alpha}_i|w_{ij}|\Big]\}|u_j-u_j^*| \\
\leqslant & -b_2\|u-u^*\|_1. \qquad (12)
\end{aligned}
$$

Hence, we immediately derive that $\alpha_{\mathbf{R}^n}(F,u^*)\leqslant -b_2<0$. By virtue of Lemmas 1 and 2, we infer that model (1) has a unique equilibrium point u^*, which is globally exponentially stable, and the exponential decay estimate satisfies (11).

Remark 2 Theorem 2 indicates that, in some cases, the negative diagonal connection weights are of some help to stabilize neural networks.

Theorems 1 and 2 show that, with the generalized relative Dahlquist constant approach, we can obtain better exponential stability criteria compared to those in [3] deduced by Lyapunov's direct method.

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Cohen-Grossberg 神经网络的全局指数稳定性分析

万安华[1]，王绵森[2]，彭济根[2]，毛卫华[3]

(1. 中国科学院自动化研究所，北京 100080；2. 西安交通大学理学院，陕西 西安 710049；3. 华南农业大学应用数学系，广东 广州 510642)

摘要：本文研究了 Cohen-Grossberg 神经网络模型的指数稳定性. 为避免构造 Lyapunov 函数的困难，我们采用广义相对 Dalquist 数方法来分析神经网络的稳定性. 借助这一方法，我们不但得到了 Cohen-Grossberg 神经网络模型平衡解的存在性、唯一性和全局指数稳定性的新的充分条件，而且给出了神经网络的指数衰减估计. 所获结论改进了已有文献的相关结果.

关键词：指数稳定性；Cohen-Grossberg 神经网络；广义相对 Dalquist 数；指数衰减估计