§9.4 Stability; Almost linear system

In the previous sections of this chapter, we have on several occasions referred to the concepts of stability, asymptotic stability, and instability of a solution of the autonomous system

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y). \tag{9.4.1}$$

In this section we will finally give a precise mathematical meaning to these concepts, discuss an important theorem dealing with the stability of an almost linear system, and explore its consequences by considering an illustrative example.

9.4.1 Definition

Let $x = x(t, x_0, y_0), y = y(t, x_0, y_0)$ be the solution of Equs. (9.4.1) with the initial value $x(0) = x_0, y(0) = y_0$, and $x = x^*, y = y^*$ be a critical point of (9.4.1).

**Definition**

The critical point $x = x^*, y = y^*$ of (9.4.1) is said to be stable if, given any $\varepsilon > 0$, it is possible to find a $\delta$ such that the solution $x = x(t, x_0, y_0), y = y(t, x_0, y_0)$ with

$$\sqrt{(x_0 - x^*)^2 + (y_0 - y^*)^2} < \delta \tag{9.4.2}$$

exists and satisfies

$$\sqrt{(x(t, x_0, y_0) - x^*)^2 + (y(t, x_0, y_0) - y^*)^2} < \varepsilon \tag{9.4.3}$$

for all $t > 0$.

A critical point that is not stable is said to be unstable.

A critical point $(x^*, y^*)$ is said to be asymptotically stable if it is stable and

$$\lim_{t \to +\infty} x(t, x_0, y_0) = x^*, \quad \lim_{t \to +\infty} y(t, x_0, y_0) = y^*, \tag{9.4.4}$$

where $(x_0, y_0)$ satisfies (9.3.2).

**Example**

The critical point $(0,0)$ of following systems

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x,$$

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -3y,$$

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = 3y,$$

are stable, asymptotically stable, and unstable, respectively.

The critical point $(0,0)$ of the following system

$$\frac{dx}{dt} = -x - y + x(x^2 + y^2 - 1), \quad \frac{dy}{dt} = x - y + y(x^2 + y^2 - 1)$$

are...
is asymptotically stable since
\[
\frac{dr}{dt} = -r + r(r^2 - 1), \quad \frac{d\theta}{dt} = 1.
\]

The geometrical illustration of the stability:
Stable: All solutions that start “sufficiently close” to \((x^*, y^*)\) stay “close” to \((x^*, y^*)\).
Unstable: There exists at least one solution starting sufficiently close to \((x^*, y^*)\) can not stay close to \((x^*, y^*)\) for all \(t > 0\).
Asymptotically stable: Trajectories that start “sufficiently close” to \((x^*, y^*)\) must not only stay “close” but must eventually approach \((x^*, y^*)\) as \(t \to \infty\).

For the linear system
\[
\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy,
\]  
with \(ad - bc \neq 0\), the type and stability of the critical point \((0,0)\) are determined by the eigenvalues \(\lambda_1\) and \(\lambda_2\) of \(A\). The stability characteristics are summarized in the following theorem.

**Theorem 9.1**
The critical point of the linear system (9.4.5) is
1) asymptotically stable if \(\lambda_1, \lambda_2\) are real and negative or have negative real parts;
2) stable, but not asymptotically stable, if \(\lambda_1, \lambda_2\) are pure imaginary;
3) unstable if \(\lambda_1, \lambda_2\) are real and and either is positive, or if they have positive real parts.

For the linear system (9.4.5) the stable node, the stable spiral are asymptotically stable, the center is stable, and the saddle, unstable node, unstable spiral are unstable.

We now want to relate the results for the linear system (9.4.5) to the nonlinear system
\[
\frac{dx}{dt} = ax + by + F_1(x, y), \quad \frac{dy}{dt} = cx + dy + G_1(x, y),
\]  
where \(ad - bc \neq 0\). We assume that the function \(F_1(x, y)\) and \(G_1(x, y)\) be continuous, have continuous first partial derivatives, and
\[
\lim_{r \to 0} \frac{F_1(r \cos \theta, r \sin \theta)}{r} = 0, \quad \lim_{r \to 0} \frac{G_1(r \cos \theta, r \sin \theta)}{r} = 0,
\]
where \(r = \sqrt{x^2 + y^2}\). Such a system is often referred as an almost linear system in the neighborhood of the critical point \((0,0)\).

The conditions on \(F\) and \(G\) are satisfied by many functions of two variables. For example,
\[
\begin{align*}
\frac{dx}{dt} &= -x + lx^2 + mxy + ny^2, \\
\frac{dy}{dt} &= -y + \varepsilon y + ax^2 + bxy,
\end{align*}
\]
\[
\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -\frac{q}{l}x - \frac{q}{l}(\sin x - x),
\end{aligned}
\]

are almost linear systems in the neighborhood of the \((0,0)\).

The type and stability of a critical point of the almost linear system (9.4.6) are closely related in most cases to the type and stability of the critical point of the corresponding linear system (9.4.5).

**Theorem 9.2**

Let \( r_1 \) and \( r_2 \) be the roots of the characteristic equation of the linear system (9.4.5) corresponding to the almost linear system (9.4.6). Then the type and stability of the critical point \((0,0)\) of the linear system (9.4.5) and the almost linear system (9.4.6) are as shown in the table below:

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Linear system</th>
<th>Almost linear system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_2 &lt; r_1 &lt; 0 )</td>
<td>stable node</td>
<td>stable node</td>
</tr>
<tr>
<td>( r_2 &gt; r_1 &gt; 0 )</td>
<td>Unstable node</td>
<td>unstable node</td>
</tr>
<tr>
<td>( r_2 &lt; 0 &lt; r_1 )</td>
<td>Unstable, saddle</td>
<td>unstable, saddle</td>
</tr>
<tr>
<td>( r_2 = \bar{r}_1 = \alpha + i\beta ) ( \alpha &lt; 0 )</td>
<td>stable spiral,</td>
<td>stable spiral</td>
</tr>
<tr>
<td>( r_2 = \bar{r}_1 = \alpha + i\beta ) ( \alpha &gt; 0 )</td>
<td>unstable spiral,</td>
<td>unstable spiral</td>
</tr>
<tr>
<td>( r_2 = \bar{r}_1 = i\beta )</td>
<td>center</td>
<td>unknown</td>
</tr>
</tbody>
</table>

The proof of Theorem 9.2 is beyond this course. The Essentially, Theorem 9.2 says that for \( x \) and \( y \) near zero the nonlinear term \( F_1(x,y) \) and \( G_1(x,y) \) are small and do not affect the stability and type of critical point as determined by the linear terms except in the sensitive case: \( \lambda_1 \) and \( \lambda_2 \) pure imaginary.

Even if the critical point is of the same type as that of the linear system, the trajectories of the almost linear system may be considerably different in appearance from those of the corresponding linear system. This is illustrated in following example.

**Example**

Sketch the trajectories of the system

\[
\begin{aligned}
\frac{dx}{dt} &= -x, \\
\frac{dy}{dt} &= -2y + \alpha y^2.
\end{aligned}
\]  

(9.4.8)

**Solution**

By eliminating the time \( t \) we obtain the equation

\[
\frac{dy}{dx} = 2y + \alpha y^2, \quad \frac{dy}{y(2 + \alpha y)} = \frac{dx}{x}.
\]

\[
[\frac{1}{2y} - \frac{\alpha}{2(2 + \alpha y)}]dy = \frac{dx}{x}, \quad C_1 + \ln x^2 = \ln|\frac{y}{2 + \alpha y}|.
\]
\[ C_2 x^2 = \frac{y}{2 + \alpha y}, \quad y = \frac{2C_2 x^2}{1 - \alpha C_2 x^2}. \] 

(9.4.9)

It follows from (9.4.9) that the each trajectories of (9.4.8) is a parabola for \( \alpha = 0 \), which is the trajectory of linear system. If \( \alpha \neq 0 \) system (9.4.8) is a almost linear system, and there is big difference for the trajectories between the linear and almost linear systems. For example if we choose \( \alpha = 0, \alpha = 1 \), and consider the solution with the initial condition \( y(1) = 1 \). The solutions for the linear and almost linear system are

\[ y = x^2, \quad y = \frac{\frac{2}{3}x^2}{1 - \frac{1}{3}x^2}, \]

respectively. The solution of the linear system exists for all \( x \), whereas, the existence interval for the almost linear system is \( x^2 < 3 \). Even there is big difference for the solutions of the two systems, they are quite close in the neighborhood of the origin.

**Example Damped pendulum**

We consider the motion of a damped pendulum for which the damping is proportional to the speed. It is not difficult to show, by reasoning similar to that used in the derivation of the spring-mass equation, that the governing equation is

\[ ml^2 \frac{d^2 \theta}{dt^2} + cl \frac{d\theta}{dt} + mgl \sin \theta = 0, \]

where the damping constant \( c > 0 \). Letting \( x = \theta \) and \( y = \frac{d\theta}{dt} \) gives the system

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{g}{l} \sin x - \frac{c}{ml} y. \] 

(9.4.10)

The point \( x = 0, y = 0 \) is a critical point of the system (9.4.10). Because of the damping mechanism, we expect any small motion about \( \theta = 0 \) to decay in amplitude. Thus intuitively the equilibrium point \((0,0)\) should be asymptotically stable. To show this we rewrite the system (9.4.10) as

\[ \begin{cases} 
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = -\frac{g}{l} x - \frac{c}{ml} y. 
\end{cases} \] 

(9.4.11)

It is easy to show that \( \lim_{r \to 0} \frac{\sin r\phi - r\phi}{r} = 0 \), so the system (9.4.11) is an almost linear system; hence Theorem 9.2 is applicable. The roots of the characteristic equation associated with the corresponding linear system

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{g}{l} x - \frac{c}{ml} y \]

are

\[ r_1 = \frac{1}{2} \left[ \frac{-c}{ml} + \sqrt{\left( \frac{c}{ml} \right)^2 - \frac{4g}{l}} \right], \quad r_2 = \frac{1}{2} \left[ \frac{-c}{ml} - \sqrt{\left( \frac{c}{ml} \right)^2 - \frac{4g}{l}} \right]. \]
The roots $r_1$ and $r_2$ are real, negative, or complex with negative real parts. The critical point $(0,0)$ is an asymptotically stable node or spiral. Therefore, $(0,0)$ is a stable critical point of the almost linear system. Some trajectories of the system are sketched by Maple (see Figure 9.4.1)

```
With(DEtools):
DE941 := [diff(x(t),t)=y(t),
           diff(y(t),t)=-sin(x(t))-0.1*y(t)];
DEplot( DE941, [x(t),y(t)], t=-10..30,
         [[x(0)=3,y(0)=0], [x(0)=-3,y(0)=0],
         [x(0)=0,y(0)=3], [x(0)=0,y(0)=-3],
         [x(0)=0,y(0)=4], [x(0)=0,y(0)=-4],
         [x(0)=4,y(0)=0], [x(0)=-4,y(0)=0]],
         x=-3*Pi..3*Pi, y=-5..5, stepsize=0.05,
         dirgrid=[21,21], color=red,
         linecolor=blue, axes=BOXED,
         title="Damped pendulum: Stable spiral ",
         arrows=SLIM);
```

**Example Two competing species**

Suppose that we have two similar species competing for a limited food supply; for example, two species of fish in a pond that do not prey on each other but do compete for the available food.
Let $x$ and $y$ be the populations of the two species at time $t$. Then $x$ and $y$ satisfy the following system of equations

$$\begin{align*}
\frac{dx}{dt} &= x(2 - 2x - y), \\
\frac{dy}{dt} &= y(2 - x - 2y).,
\end{align*}$$

(9.4.12)

$E(\frac{2}{3}, \frac{2}{3})$ is an equilibrium point of (9.4.12). To examine the stability of this critical point, let $x = u + \frac{2}{3}$, $y = v + \frac{2}{3}$. Substituting for $x$ and $y$ in (9.4.12) and simplifying, we obtain

$$\begin{align*}
\frac{du}{dt} &= -\frac{2}{3}(2u + v) - u(2u + v), \\
\frac{dv}{dt} &= -\frac{2}{3}(u + 2v) - v(u + 2v).
\end{align*}$$

(9.4.13)

It is not difficult to show that the system (9.4.13) is an almost linear system. The corresponding linear system is

$$\begin{align*}
\frac{du}{dt} &= -\frac{4}{3}u + \frac{3}{3}v, \\
\frac{dv}{dt} &= -\frac{2}{3}u - \frac{4}{3}v.
\end{align*}$$

(9.4.14)

For System (9.4.14) $q = \frac{4}{3} > 0$, $p = -\frac{8}{3} < 0$, and $p^2 - 4q = \frac{16}{3} > 0$. Hence $E$ is a asymptotically stable node of (9.4.14). And from Theorem 9.2 it follows that the critical point $E$ is an asymptotically stable node of the almost linear system (9.4.12). The direction field and some of the orbits of (9.4.12) are shown in Figure 9-4-2. We can show that $E$ is globally asymptotically stable in the next section.

Theorem 9.2 says that for $x$ and $y$ near zero the nonlinear terms $F_1(x, y)$ and $G_1(x, y)$ are small and do not affect the stability and type of critical point as determined by the linear terms except in two sensitive cases: $r_1$ and $r_2$ pure imaginary, and $r_1$ and $r_2$ real and equal. We see that small perturbations in the coefficients of linear system, and hence in the roots $r_1$ and $r_2$ can alter the type and stability of the critical point only in these two sensitive cases. When $r_1$ and $r_2$ are pure imaginary, a small perturbation can change the stable center into either an asymptotically stable or unstable spiral point or even leave it as a center.

When $r_1 = r_2$ a small perturbation does not affect the stability of the critical point, but may change the node into a spiral point. It is reasonable to expect that the small nonlinear terms might have a similarly substantial effect, at least in these two sensitive cases.
Example
\[
\begin{align*}
\frac{dx}{dt} &= -y + y(x^2 + y^2), \\
\frac{dy}{dt} &= x - x(x^2 + y^2).
\end{align*}
\]
The system can be changed to
\[
\begin{align*}
\frac{dr}{dt} &= 0, \\
\frac{d\theta}{dt} &= 1 - r^2.
\end{align*}
\]
The critical point (0,0) is a center.

Example
\[
\begin{align*}
\frac{dx}{dt} &= -y - x\sqrt{x^2 + y^2}, \\
\frac{dy}{dt} &= x - y\sqrt{x^2 + y^2}.
\end{align*}
\]
The system can be changed to
\[
\begin{align*}
\frac{dr}{dt} &= -r^2, \\
\frac{d\theta}{dt} &= 1.
\end{align*}
\]
The critical point (0,0) is a stable node.

Example
\[
\begin{align*}
\frac{dx}{dt} &= -y + x(x^2 + y^2)^8\sin\frac{\pi}{\sqrt{x^2 + y^2}}, \\
\frac{dy}{dt} &= x + y(x^2 + y^2)^8\sin\frac{\pi}{\sqrt{x^2 + y^2}}.
\end{align*}
\]
The system can be changed to
\[
\begin{align*}
\frac{dr}{dt} &= r^{17}\sin\frac{\pi}{r}, \\
\frac{d\theta}{dt} &= 1.
\end{align*}
\]
The critical point (0,0) is a center-focus.