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The dynamics of a Lotka–Volterra predator–prey model with state dependent impulsive harvest for predator

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ABSTRACT

According to the economic and biological aspects of renewable resources management, we propose a Lotka–Volterra predator–prey model with state dependent impulsive harvest. By using the Poincaré map, some conditions for the existence and stability of positive periodic solution are obtained. Moreover, we show that there is no periodic solution with order larger than or equal to three under some conditions. Numerical results are carried out to illustrate the feasibility of our main results. The bifurcation diagrams of periodic solutions are obtained by using the numerical simulations, and it is shown that a chaotic solution is generated via a cascade of period-doubling bifurcations, which implies that the presence of pulses makes the dynamic behavior more complex.

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1. Introduction

In population dynamics, the economic and biological aspects of renewable resources has been one of the interesting research topics. The optimal exploitation of renewable resources, which has direct effect on their sustainable development, has been paid much attention (Angelova and Dishliev, 1998b; Clark, 1976; Song and Chen, 2001) and references therein. Generally speaking, sustainable development of a resource implies that the resource can be utilized forever. However, it is always hoped that we can achieve sustain ability at a high level of productivity and good economic profit, and this requires scientific and effective management of the resources. It is a purpose of this paper to analyze the exploitation of the Lotka–Volterra predator–prey model, which is the most basic and important models in population dynamics.

On the other hand, in the natural world, we sometimes need to reasonably control the population of the predator since the predator might lead the prey to decrease and even become extinct. The example of controlling predator can be easily found in ecology, especially in the island ecological systems (for example, the story of the rabbits, birds and cats mentioned in paper; Fan et al., 2005). Usually, the measure of control has two kinds: harvest and release the target. In this paper, we consider the measure of proportionable harvest. In addition, the artificial control is usually put in practice at discrete moments, which in nature are submitted to short temporary perturbations that negligible compared to the process duration. These short-time perturbations are often assumed to be in the form of impulses in the modeling process. Consequently, impulsive differential equations (IDEs) provides a natural description of such processes.

In the last decades, some IDEs have been introduced in population dynamics, such as Gao et al. (2006, 2007), Zhang and Teng (2007) and D'onof (2002) studies the dynamical behavior of epidemic model with pulse vaccination and chemotherapeutic treatment of disease, Tang and Chen (2002) established the conditions for the existence and stability of periodic solution of population dynamic with birth pulses, and Baek (2008), Liu and Rohlf (1998), Liu (1994), Shair Ahmad and Stamova (2007) and Ballinger and Liu (1997) researched the population models with impulsive effects and devoted to the criteria for the existence, stability, orbital stability of periodic solutions. In particular, Bainov and Dishliev (Angelova and Dishliev, 1998b) considered impulsive harvest and resources reasonable exploitation simultaneously for Logistic model. The moments of impulsive harvest and the magnitudes of every impulse are determined so that the biomass of population at terminal time of exploitation is maximum provided the fixed quantity of total impulsive harvest. Angelova and Dishliev (1998a) considered the optimal problem for general n-impulsive population model and determine the optimal number

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of impulsive harvest and quantities of every harvest so that the biomass in the moment T will be maximum. Further, they studied impulsive harvest for general population model for similar optimal aim in Angelova and Dishliev (2000). Dong et al. (2007) study the periodic Gompertz system with harvesting and obtained the optimal impulsive harvesting effort and the corresponding optimal population level. However, the majority of these models only consider impulses introduced at fixed time intervals. We note that impulsive state feedback control strategy is used widely in real life problems. In practical ecological system, the control measures (by catching, poisoning, releasing the natural enemy or harvesting, etc.) are taken only when the amount of species reaches a threshold value, rather than the usual impulsive fixed-time control strategy. Recently, many papers have been devoted to the analysis of mathematical models describing IDEs with state-dependent impulsive effects. For example, Nie et al. (2009a,b), Jiang and Lu (2006, 2007), Jiang et al. (2007), Zeng et al. (2006) and Tang et al. (2005) consider the dynamic behaviors of predator–prey models with state-dependent impulsive effects and obtained the existence and stability of positive periodic solution by using the Poincaré map and the properties of the Lambert W function, Jiang et al. (2005) studied the sufficient conditions of existence and stability of periodic solutions of of a stage-structured pest management system by means of the sequence convergence rule and the analogue of the Poincaré criterion, and Wang et al. (2008) analyzes a autonomous state-dependent IDEs and given some necessary conditions for the existence and stabilities of periodic solutions.

Motivated by the literature survey, in this paper, we first develop a Lotka–Volterra predator–prey model with state impulsive harvest on predator. The system is modeled by the following equations:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= x(t)[r - ax(t) - by(t)] \\ \frac{dy(t)}{dt} &= y(t)[-d + cx(t)] \end{aligned}\right\} & y \neq h, \\ \left.\begin{aligned} \triangle x(t) &= x(t^+) - x(t) = px(t) + \tau \\ \triangle y(t) &= y(t^+) - y(t) = -qy(t) \end{aligned}\right\} & y = h, \end{cases} \tag{1.1}$$

where $h \in (0, \infty)$, $p \in (-1, \infty)$ and $q \in (0, 1)$. When the amount of the predator reaches the threshold h at time t_h, controlling measures are taken and amount of prey and predator immediately become $(1 + p)x(t_h) + \tau$ and $(1 - q)h$, respectively.

This paper is organized as follows. In the next section, as preliminaries we present an Poincaré map, an important lemma and a basic definition. In Section 3, we state and prove general criterion for the existence and stability of positive periodic solutions of system (1.1). Some specific examples are given to illustrate our results in the Section 4. Finally, a discussion of the ecological implications of our results is provided and some future research directions are outlined in the last section.

2. Preliminaries

The dynamic behaviors for system (1.1) without impulsive effects are studied by considerable investigators. It has a saddle $(0, 0)$, one locally stable focus $(d/c, (rc - ad)/bc)$ and a saddle $(r/a, 0)$ under the condition $rc > ad$.

Throughout this paper, we assume that the condition $rc - ad > 0$ holds. By the biological background of system (1.1), we only consider system (1.1) in the biological meaning region $D = \{(x, y) : x \geq 0, y \geq 0\}$. Obviously, the global existence and uniqueness of solutions of system (1.1) are guaranteed by the smoothness properties of f, which denotes the mapping defined by right-side of system (1.1), for details see Lakshmikantham et al. (1989) and Bainov and Simeonov (1993).

Set $R = (-\infty, \infty)$ and $z(t) = (x(t), y(t))$ be any solution of (1.1). Firstly, we define the positive orbit through the point $z_0 \in R^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$ for $t \geq t_0 \geq 0$ as

$$O^+(z_0, t_0) = \{z \in R^2_+ : z = z(t), t \geq t_0, z(t_0) = z_0\}.$$

To discuss the dynamics of system (1.1), we define two cross-section to the vector field (1.1) by $\sum_1 = \{(x, y) : x > 0, y = (1 - q)h\}$ and $\sum_2 = \{(x, y) : x > 0, y = h\}$. Suppose that the point $S_n(x_n, h)$ is on the section $\sum_2$. Then the trajectory $O^+(S_n, t_n)$ of system (1.1) jumps to the point $S^+((1 + p)x_n + \tau, (1 - q)h)$ on $\sum_1$ due to the impulsive effects, and then reaches the point $S_{n+1}(x_{n+1}, h)$ on the section $\sum_2$ again, where x_{n+1} is decided by x_n and the parameters p and τ. Therefore, we defined the Poincaré map of $\sum_2$ as follows:

$$x_{n+1} = P(p, \tau, x_n). \tag{2.1}$$

Next, we consider the autonomous system with impulsive effects

$$\begin{cases} \dfrac{dx}{dt} = P(x, y), \quad \dfrac{dy}{dt} = Q(x, y), \quad \varphi(x, y) \neq 0, \\ \triangle x = \xi(x, y), \quad \triangle y = \eta(x, y), \quad \varphi(x, y) = 0, \end{cases} \tag{2.2}$$

where $P(x, y)$ and $Q(x, y)$ are continuous differential functions defined on R^2, and $\varphi(x, y)$ is a sufficiently smooth function with $grad\varphi(x, y) \neq 0$. Let $(\mu(t), \nu(t))$ be a positive T-periodic solution of system (2.2). By Corollary 2 of Theorem 1 given in Simeonov and Bainov (1988), there is the following lemma.

Lemma 2.1 (Analogue of Poincaré's criterion). *If the Floquet multiplier μ satisfies the condition $|\mu| < 1$, where*

$$\mu = \prod_{j=1}^{n} \kappa_j \exp\left[\int_0^T \left(\frac{\partial P(\mu(t), \nu(t))}{\partial x} + \frac{\partial Q(\mu(t), \nu(t))}{\partial y}\right) dt\right]$$

with

$$\kappa_j = \frac{((\partial\eta/\partial y)(\partial\varphi/\partial x) - (\partial\eta/\partial x)(\partial\varphi/\partial y) + (\partial\varphi/\partial x))P_+ + ((\partial\xi/\partial x)(\partial\varphi/\partial y) - (\partial\xi/\partial y)(\partial\varphi/\partial x) + (\partial\varphi/\partial y))Q_+}{(\partial\varphi/\partial x)P + (\partial\varphi/\partial y)Q}$$

and P, Q, $\partial\xi/\partial x$, $\partial\xi/\partial y$, $\partial\eta/\partial x$, $\partial\eta/\partial y$, $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ are calculated at the point $(\mu(\tau_j), \nu(\tau_j))$, $P_+ = P(\mu(\tau_j^+), \nu(\tau_j^+))$, $Q_+ = Q(\mu(\tau_j^+), \nu(\tau_j^+))$, and $\tau_j (j \in N)$ is the time of the jth jump. Then, $(\mu(t), \nu(t))$ is orbitally asymptotically stable.

Let $z(t) = (x(t), y(t))$ be a solution of system (1.1) with initial conditions $z_0 = z(t_0) = (x_0, (1 - q)h) \in R^2_+$. This trajectory $O^+(z_0, t_0)$ starts from the point $E_0(x_0, (1 - q)h)$ first intersects the section $\sum_2$ at the point $F_0(\tilde{x}_0, h)$, next jumps to the point $E_1(x_1, (1 - q)h)$ on the section $\sum_1$ due to the impulsive effects, and then reaches the point $F_1(\tilde{x}_1, h)$ on the section $\sum_2$ again, etc. So, we have two impulsive points' sequences $\{E_k(x_k, (1 - q)h)\}$ and $\{F_k(\tilde{x}_k, h)\} (k = 0, 1, 2, \ldots)$. We notice that the coordinates satisfy the relation $x_k = (1 + p)\tilde{x}_{k-1} + \tau (k = 1, 2, \ldots)$.

Definition 2.1. A trajectory $O^+(z_0, t_0)$ of system (1.1) is said to be order-k periodic if there exists a positive integer $k \geq 1$ such that k is the smallest integer for $x_0 = x_k$.

3. Main Results

From the geometrical construction of the phase space of system (1.1), we note that the trajectory from any initial point (x_0, y_0) with $y_0 < h$ intersects the section $\sum_2$ an infinite number of times if $h \leq (rc - ad)/bc$. However, the trajectory from any initial point (x_0, y_0) with $y_0 < h$ may intersects the section $\sum_2$ a finite number of times if $h > (rc - ad)/bc$. So, in this subsection, we give the sufficient conditions for the existence and stability of positive periodic solutions of system (1.1) in the cases of $h \le (rc - ad)/bc$ and $h > (rc - ad)/bc$.

3.1. The Case of $h \le (rc - ad)/bc$

On the existence of positive periodic solutions of system (1.1), we have the following theorem.

Theorem 3.1. *For any $q \in (0, 1)$, $p > -1$ and $\tau \ge 0$, if*

$$p\frac{r - b(1-q)h}{a} + \tau \le 0, \tag{3.1}$$

then system (1.1) has a positive order-1 periodic solution.

Proof. Let the point $A_1(\varepsilon_0, (1-q)h)$ on the section $\sum_1$, where ε_0 is small enough and $\varepsilon < (1+p)(r - b(1-q)h)/a + \tau$ such that the trajectory $O^+(A_1, t_0)$ of system (1.1) starts from the initial point A_1 intersects the section $\sum_2$ at the point $B_1(\varepsilon_1, h)$ and $\varepsilon_1 > (r - b(1-q)h)/a$. At the state B_1, the trajectory $O^+(A_1, t_0)$ is subjected by impulsive effects to jumps to the point $A_2((1+p)\varepsilon_1 + \tau, (1-q)h)$ on the section $\sum_1$, and then reaches the point $B_2(\varepsilon_2, h)$ on the section $\sum_2$ again. By the geometrical construction of the phase space of system (1.1) and the condition (3.1), we have $\varepsilon_0 < (1+p)\varepsilon_1 + \tau$ and the point A_2 is on the right of the point A_1. Further, the point B_2 is on the left of the point B_1 and $\varepsilon_2 > \varepsilon_1$. So, from (2.1) we have $\varepsilon_2 = P(q, \tau, \varepsilon_1)$ and

$$P(p, \tau, \varepsilon_1) - \varepsilon_1 = \varepsilon_2 - \varepsilon_1 < 0. \tag{3.2}$$

On the other hand, suppose that the curve $L: -cx + d = 0$ intersects the section $\sum_1$ at the point $E_0(d/c, (1-q)h)$. The trajectory $O^+(E_0, t_0)$ from the initial point E_0 intersects the section $\sum_2$ at the point $F_1(x_1, h)$, next jumps to the point $F_1^+((1+p)x_1 + \tau, (1-q)h)$ on the section $\sum_1$ and then reaches the point $F_2(x_2, h)$ on the section $\sum_2$ again. If there are positive constant p^* and τ^* satisfy the condition (3.1), such that $(1+p^*)x_1 + \tau^* = d/c$. Then, the point F_1^+ coincides with the point E_0 just for $p = p^*$ and $\tau = \tau^*$, the point F_1^+ is on the left of the point E_0 for $(1+p)x_1 + \tau < d/c$ and it is on the right of the point E_0 for $(1+p)x_1 + \tau > d/c$. However, From the geometrical construction of the phase space of system (1.1), we obtain that the point F_2 is on the right of the point F_1 for any $q \in (-1, \infty)$ and $\tau \ge 0$.

From the above discussion, we obtain that

(i) if $x_1 = x_2$, then system (1.1) has a positive order-1 periodic solution,
(ii) if $x_1 < x_2$, then

$$P(p, \tau, x_1) - x_1 = x_2 - x_1 > 0. \tag{3.3}$$

By (3.2) and (3.3), it follows that the Poincaré map (2.1) has a fixed point, that is the system (1.1) has a positive order-1 periodic solution. This completes the proof of this theorem. □

Next, we state and prove our result on the uniqueness and stability of positive order-1 periodic solutions of system (1.1). It is immediate that if each positive order-1 periodic solution of system (1.1) is stable, then system (1.1) admits a unique positive periodic solution.

Theorem 3.2. *Let $(\phi(t), \psi(t))$ be a positive order-1 periodic solution of system (1.1) with periodic T. Suppose the condition*

$$|\mu| = |\kappa \exp \int_0^T [r - c - (2a - d)\phi(t) - b\psi(t)]\,dt| < 1 \tag{3.4}$$

holds, where

$$\kappa = \frac{(1-q)[-c + d((1+p)\phi(0) + \tau)]}{(-c + d\phi(0))}.$$

Then $(\phi(t), \psi(t))$ is a unique positive order-1 periodic solution of system (1.1) and which is orbitally asymptotically stable and has asymptotic phase property.

Proof. Based on the conclusion of Theorem 3.1, we need only to verify the stability of positive order-1 periodic solutions $(\phi(t), \psi(t))$ of system (1.1). Suppose the solution $(\phi(t), \psi(t))$ intersects sections $\sum_1$ and $\sum_2$ at points $E^+((1+p)\phi(0) + \tau, (1-q)h)$ and $E(\phi(0), h)$, respectively. Comparing with system (1.1), we have

$$P(x, y) = x[r - ax - by], \quad Q(x, y) = y[-c + dx],$$

and $\xi(x, y) = px + \tau$, $\eta(x, y) = -qy$, $\varphi(x, y) = y - h$, $(\phi(T), \psi(T)) = (\phi(0), h)$ and $(\phi(T^+), \psi(T^+)) = ((1+p)\phi(0) + \tau, (1-q)h)$. Thus

$$\frac{\partial P}{\partial x} = r - 2ax - by, \quad \frac{\partial Q}{\partial y} = -c + dx, \tag{3.5}$$

and

$$\frac{\partial \xi}{\partial x} = p, \quad \frac{\partial \eta}{\partial y} = -q, \quad \frac{\partial \varphi}{\partial y} = 1, \quad \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial x} = 0. \tag{3.6}$$

Further, from (3.5) and (3.6), it follows that

$$\kappa = \frac{((\partial\eta/\partial y)(\partial\varphi/\partial x) - (\partial\eta/\partial x)(\partial\varphi/\partial y) + (\partial\varphi/\partial x))P_+ ((\partial\xi/\partial x)(\partial\varphi/\partial y) - (\partial\xi/\partial y)(\partial\varphi/\partial x) + (\partial\varphi/\partial y))Q_+}{(\partial\varphi/\partial x)P + (\partial\varphi/\partial y)Q}$$

$$= \frac{Q_+(\phi(T^+), \psi(T^+))(1+p)}{Q(\phi(T), \psi(T))} = \frac{(1-q)[-c + d((1+p)\phi(0) + \tau)]}{(-c + d\phi(0))}$$

and

$$\mu = \kappa \exp \int_0^T [r - c - (2a - d)\phi(t) - b\psi(t)]\,dt.$$

By the condition (3.4), we see that system (1.1) satisfies all conditions of Lemma 2. Therefore, by Lemma 2, the order-1 periodic solution $(\phi(t), \psi(t))$ of system (1.1) is orbitally asymptotically stable and has asymptotic phase property. This completes the proof of this theorem. □

3.2. The Case of $h > (rc - ad)/bc$

In view of the geometrical construction of the phase space of the system (1.1), there is a trajectory Γ which crosses $\sum_1$ at the points $D_1(\tilde{x}_1, (1-q)h)$ and $D_2(\tilde{x}_2, (1-q)h)(\tilde{x}_1 < \tilde{x}_2)$, and tangents to the line $L_2: y = h$ at the point $D_3(d/c, h)$. So, the trajectory of system (1.1) which starts from the point $(x, (1-q)h)$ with $x \in (\tilde{x}_1, \tilde{x}_2)$ will not intersect $\sum_2$.

Let $(1+p)d/c + \tau < \tilde{x}_1$, then the trajectory of system (1.1) which starts from the point $((1+p)d/c + \tau, (1-q)h)$ will intersect $\sum_2$ at the point (x_1^*, h). On the other hand, there is a x_2^* such that $(1+p)x_2^* + \tau = \tilde{x}_1$. Let $\alpha = \min\{x_1^*, x_2^*\}$, then the trajectory of system (1.1) which starts from the point $E_0(x_0, h)(x_0 \in (c/d, \alpha))$ will intersects with $\sum_2$ infinite times due to the impulsive effects with conditions $(1+p)d/c + \tau < \tilde{x}_1$ and $h > (rc - ad)/bc$.

Suppose the trajectory $O^+(E_0, t_0)$ of system (1.1) which starts from the initial point $E_0(x_0, h)$ jumps to the point $E_0^+(x_0^+, (1-q)h)$ at $\sum_1$ due to the impulsive effects $\triangle x(t) = px(t) + \tau$ and $\triangle y(t) = -qy(t)$. At the state E_0^+, the trajectory $O^+(E_0, t_0)$ intersects $\sum_1$ and $\sum_2$ at the points $A_1(\bar{x}_1, (1-q)h)(x_0^+ < \bar{x}_1)$ and $E_1(x_1, h)$, respectively. Then $x_1 \in (d/c, \alpha)$. Further, by the Poincaré map (2.1) of the section $\sum_2$, it follows that $x_1 = P(p, \tau, x_0)$. Repeating the above process, we have $x_{n+1} = P(p, \tau, x_n)(n = 0, 1, \ldots)$. On the other hand, for any two points $E_i(x_i, h)$ and $E_j(x_j, h)$ on $\sum_2$, where $x_i, x_j \in (d/c, \alpha)$ and $x_i < x_j$. In view of the impulsive effects, the points $E_i^+((1+p)x_i + \tau, (1-q)h)$ is on the left of the point $E_j^+((1+p)x_j + \tau, (1-$

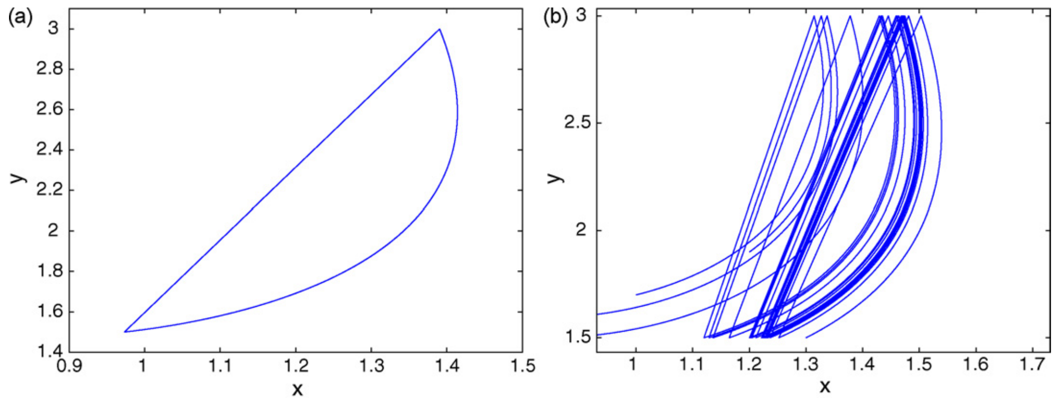

Fig. 1. Dynamical behavior of system (1.1) with $r = 0.8$, $a = 0.2$, $b = 0.2$, $d = 0.5$, $c = 0.6$, $p = -0.3$, $q = 0.5$, $\tau = 0.2$ and $h = 3$: (a) the existence of positive order-1 periodic solutions, (b) the stability of positive order-1 periodic solution.

$q)h$). Therefore, from the geometrical construction of the phase space of the system (1.1) we have

$$\frac{d}{c} < x_{j+1} < x_{i+1} < \alpha. \quad (3.7)$$

Now, for any $x_0 \in (d/c, \alpha)$, from the Poincaré map (2.1) of the section $\sum_2$ we have $x_1 = P(p, \tau, x_0)$, $x_2 = P(p, \tau, x_1)$, and $x_{n+1} = P(p, \tau, x_n)(n = 3, 4, \ldots)$. In particular, if $y_0 = y_1$, then system (1.1) has a positive order-1 periodic solution, and if $x_0 \neq x_1$ and $x_0 = x_2$, then system (1.1) has a positive order-2 periodic solution.

Next, we discuss the general circumstance, that is $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n (n > 2)$.

(a) If $x_0 < x_1$, from (3.7) we obtain that $x_1 > x_2$. In this case, the relation of y_0, y_1 and y_2 is one of the following:
 (i) $x_2 < x_0 < x_1$
 If $x_2 < x_0 < x_1$, it follows that $x_3 > x_1 > x_2$ by (3.7). Repeating the above process, we have
 $\frac{d}{c} < \cdots < x_{2n} < \cdots < x_2 < x_0 < x_1 < \cdots < x_{2n+1} < \cdots < \alpha.$
 (ii) $x_0 < x_2 < x_1$
 If $x_0 < x_2 < x_1$, similar to (i) we have
 $x_0 < x_2 < \cdots < x_{2n} < \cdots < x_{2n+1} < \cdots < x_3 < x_1 < \alpha.$

(b) If $x_0 > x_1$, from (3.7) we obtain that $x_1 < x_2$. In this case, the relation of x_0, x_1 and x_2 is one of the following:
 (i) $x_1 < x_0 < x_2$
 If $x_1 < x_0 < x_2$, it follows that $x_2 > x_1 > x_3$ by (3.7). Repeating the above process, we have
 $\frac{d}{c} < \cdots < x_{2n+1} < \cdots < x_1 < x_0 < x_2 < \cdots < x_{2n} < \cdots < \alpha.$
 (ii) $x_1 < x_2 < x_0$
 If $x_1 < x_2 < x_0$, similar to (i) we have
 $\frac{d}{c} < x_1 < \cdots < x_{2n+1} < \cdots < x_{2n} < \cdots < x_2 < x_0 < \alpha.$

Further, in case (i) of (a), it follows that $\lim_{n\to\infty} x_{2n} = \theta_2$ and $\lim_{n\to\infty} x_{2n+1} = \theta_1$, where $d/c < \theta_2 < \theta_1 < \alpha$. Therefore, we have $\theta_1 = P(p, \tau, \theta_2)$ and $\theta_2 = P(p, \tau, \theta_1)$. So, system (1.1) has a orbitally asymptotically stable positive order-2 periodic solution. Similarly, in case (ii) of (a) and (ii) of (b), system (1.1) has a orbitally asymptotically stable positive order-1 periodic solution. In case (i) of (b), system (1.1) has a orbitally asymptotically stable positive order-2 periodic solution. Further, we obtain that there exists no order-k $(k \geq 3)$ in system (1.1) with $(1-q)d/c + \tau < \tilde{x}_1$.

From the above, we know that $(1-q)d/c + \tau < \tilde{x}_1$, in where $\tilde{x}_1$ depends on the threshold value h as a function $\varphi(h)$, is a sufficient condition for system (1.1) has a orbitally asymptotically stable positive order-1 or order-2 periodic solution. To summarize the above results, we give the following result.

Theorem 3.3. *Suppose that $h > (rc - ad)/bc$, then there is a positive constant $\alpha = \varphi(h) > 0$ such that for any $q \in (0, 1)$ and $(1 + p)d/c + \tau < \alpha$, system (1.1) only has a orbitally asymptotically stable positive order-1 or order-2 periodic solution, which is asymptotic orbital stability. Furthermore, system (1.1) has no order-k $k \geq 3$ periodic solution.*

4. Example, Numerical Simulation and Discussing

In this paper, we investigate a class of Lotka–Volterra predator–prey models with state dependent impulsive harvest. By using Poincaré map we give the criteria for the existence and stability of the positive periodic solution of system (1.1).

In the following, we present some numerical simulation of examples which validate these theoretical results obtained in this paper. In order to testify the validity of our results on the uniqueness and stability of positive order-1 periodic solutions, in system (1.1), let $r = 0.8$, $a = 0.2$, $b = 0.2$, $d = 0.5$, $c = 0.6$, $p = -0.3$, $q = 0.5$, $\tau = 0.2$ and $h > 0$. Obviously, we obtain that system (1.1) without impulsive effects has a stable focus (0.8333, 3.1667). Set the section $\sum_1 = \{(x, y) : y = 0.5h, x \geq 0\}$ and the section $\sum_2 = \{(x, y) : y = h, x \geq 0\}$.

It is easy to see that system (1.1) with $h = 3$ satisfies the condition (3.1) of Theorem 3.1. So, by Theorem 3.1, system (1.1) has a positive order-1 periodic solution, which shown in Fig. 1(a). Further, if the conditions of Theorem 3.2 hold, then the positive order-1 periodic solution is orbitally asymptotically stable and has asymptotic phase property, which is shown in Fig. 1(b).

Furthermore, if we choose $h = 3.5$ in system (1.1). By Theorem 3.3, system (4.1) has a positive order-1 periodic solution and which is orbitally asymptotically stable and has asymptotic phase property, which are shown in Fig. 2(a) and (b), respectively.

Fig. 3 shows the existence and stability of positive order-2 periodic solutions in the cases of $h > (rc - ad)/bc$ with $r = 1.8$, $a = 0.15$, $b = 0.2$, $d = 0.55$, $c = 0.6$, $p = 0.86$, $q = 0.72$, $\tau = 0$ and $h = 15$. By Theorem 3.3, system (1.1) has a positive order-2 periodic solution and which is orbitally asymptotically stable and has asymptotic phase property.

Viewing p as a control parameter, the bifurcation diagram of periodic solution of system (1.1) is shown in Fig. 4 with $r = 0.8$, $a = 0.2$, $b = 0.2$, $c = 0.6$, $d = 0.5$, $q = 0.5$, $\tau = 0.2$ and $h = 4$. It is seen from the bifurcation diagram that the order-1 periodic solution is stable for $p \in (-1, p_1)$ and unstable for $p \geq p_1$, where $p_1 \approx 1.8675$. A positive order-2 period solution bifurcates from the order-1 periodic solution via a fold bifurcation at $p \approx 1.8675$. This positive

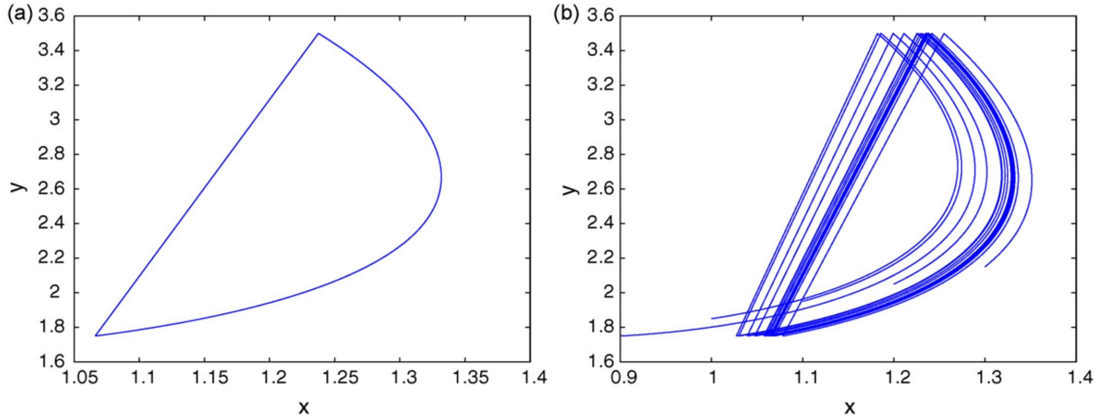

Fig. 2. Dynamical behavior of system (1.1) with $r = 0.8$, $a = 0.2$, $b = 0.2$, $d = 0.5$, $c = 0.6$, $p = -0.3$, $q = 0.5$, $\tau = 0.2$ and $h = 3.5$: (a) the existence of positive order-1 periodic solutions, (b) the stability of positive order-1 periodic solution.

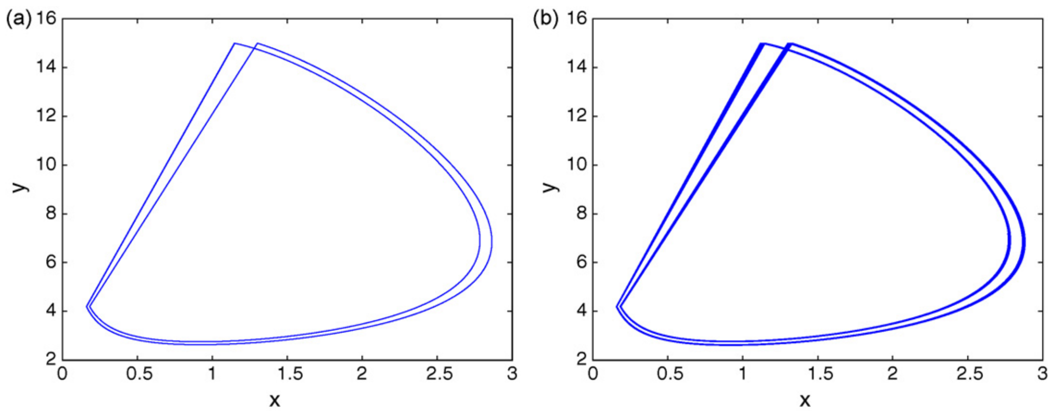

Fig. 3. Dynamical behavior of system (1.1) with $r = 1.8$, $a = 0.15$, $b = 0.2$, $d = 0.55$, $c = 0.6$, $p = -0.86$, $q = 0.72$, $\tau = 0$ and $h = 15$: (a) the existence of positive order-2 periodic solutions, (b) the stability of positive order-2 periodic solution.

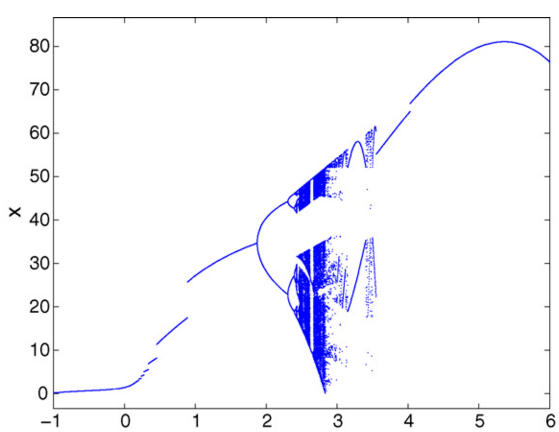

Fig. 4. The bifurcation diagram of stable periodic solution of system (2.2) with $r = 0.8$, $a = 0.2$, $b = 0.2$, $c = 0.6$, $d = 0.5$, $q = 0.5$, $\tau = 0.2$ and $h = 4$.

order-2 period solution is stable for $p \in (p_1, p_2)$ and unstable for $p \geq p_2$, where $p_2 \approx 2.3055$. The period-doubling bifurcation leads to chaos, which implies that the presence of pulses makes the dynamic behavior more complex.

5. Conclusion

In this paper, a Lotka–Volterra model described through a state-dependent impulsive harvesting for predator is proposed and investigated. This model describes that in the economic and biological aspects of renewable resources management for controlling population dynamics, people always take such a strategy that when the predator arrive at a given economic threshold h they will begin to harvest the predator and release prey. Taking such a strategy people can not only control the predator but also protect the natural environment.

By means of the Poincaré map, we obtain the general criterion for the existence and stability of positive periodic solutions of system (1.1). If control parameters satisfy some conditions, then the system is permanent, that is, prey and predator coexist forever. It is also seen that this loss of stability is due to the onset of a nontrivial periodic solution which appears via a supercritical bifurcation. Numerical simulations show that the dynamic behavior of the impulsive system we consider more complex.

Finally, we mention some future directions of work extending previous works:

(i) some situations not detailed in our results which may lead to a chaotic behavior of the system need further study (specifically, make use of impulsive control to stabilize the unstable periodic solution);
(ii) optimal harvesting strategies for impulsive control problem are also worthwhile of study;
(iii) more complicated prey–predator models may also be considered.

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