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Singular perturbation approach to stability of a SIRS epidemic system

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ABSTRACT

In this paper, a new method named multiple time scales is introduced to estimate the domain of attraction of the endemic equilibrium point of J. Mena-lorca and H. W. Hethcote's SIRS model. Simultaneously, the recurrence formulae established by S. Balint, A. Balint and V. Negru are adopted to construct Lyapunov function for the reduced system. Furthermore, the stability for the non-hyperbolic equilibrium point of the reduced system is discussed. In the end, numerical simulations are carried out.

1. Introduction

Since the 2th century, epidemic disease has endangered the world deeply [1], and huge calamity has been brought to humanity. As a result, millions of people die and social economics are heavily destroyed. Therefore, it is necessary to adopt all kinds of methods to control and eliminate epidemic disease.

Epidemic models can contribute to the design and analysis of epidemiological surveys, suggest crucial data that should be collected, identify trends, make general forecasts, and estimate the uncertainty in forecasts. Therefore, to construct epidemic models and then investigate its qualitative and quantitative properties has become a focus in applied mathematics. In fact, the original work on epidemic models is due to Kermack and McKendrick's paper [2], where they assume the size of the total population is invariable, divide the population into three classes: the infective, the susceptible and the removed, and formulate a SIR epidemic model. Since then there have appeared various epidemic models (see [1,3–12] and references therein) to describe the spread process of different diseases,lots of numerical and analytical results are obtained, which benefit the disease's controlling and defending.

It is well known that the epidemicity of disease is closely related to the stability of the solutions of mathematical models. Generally, Lyapunov's second method is used to analyze the stability of epidemic models. But, sometimes it is difficult to construct a Lyapunov function. Therefore, Lyapunov's second method is not always effective. For example, considering temporary immunity, Mena-lorca and Hethcote [4] founded a SIRS model

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - dS - \beta SI + \delta R, \\ \dfrac{dI}{dt} = \beta SI - dI - \alpha I - \gamma I, \\ \dfrac{dR}{dt} = \gamma I - \delta R - dR, \end{cases} \tag{1.1}$$

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where Λ is the recruitment rate, d the natural mortality rate, β the infection rate, α the death rate due to disease, γ the recovery rate, and δ the rate that the removed return to the susceptible. By Lyapunov's second method, they proved the globally asymptotical stability of the endemic equilibrium only in the special case when $\delta = 0$. Clearly, when $\delta = 0$ the model (1.1) is just the usual SIR model. Up to now, it is still open whether the endemic equilibrium of the SIRS model (1.1) (with $\delta \neq 0$) is globally asymptotically stable. Thus, it is meaningful to discuss the domain of attraction of the endemic equilibrium.

In this paper, the singular perturbation method is adopted to solve this question. The basal equations for the singular perturbation analysis are

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), z(t)), \\ \epsilon \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = g(x(t), z(t)), \end{cases} \tag{1.2}$$

where $(0, 0)$ is an isolated root of $f(x(t), z(t)) = 0, g(x(t), z(t)) = 0$ in $B_x \times B_z$, and ϵ ($\epsilon \ll 1$) is a real parameter. Let f and g be smooth enough so that system (1.2) has a unique solution for given specified initial conditions. It is worth noting that ϵ is a small parameter. Therefore, the dynamics of the variable z evolve on a much faster time scale than the dynamics of the variable x. Usually, z is referred to as fast variable and x is called slow variable. Let $\tau = \frac{t}{\epsilon}$, then system (1.2) can be reformulated as

$$\begin{cases} \dfrac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \epsilon f(x(\tau), z(\tau)), \\ \dfrac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = g(x(\tau), z(\tau)). \end{cases} \tag{1.3}$$

The time scale given by τ is said to be fast whereas t is slow. Sometimes, we name (1.2) slow system and (1.3) fast system.

Obviously, as long as $\tau \neq 0$, system (1.2) is equivalent to system (1.3). The essential idea in the singular perturbation theory is to deduce the behavior of the solutions of system (1.2) or (1.3) by the properties of reduced system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), h(x)), \tag{1.4}$$

and boundary-layer system

$$\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = g(x, z + h(z)), \tag{1.5}$$

where $z = h(x)$ is an isolated root of $g(x, z) = 0$ in $B_x \times B_z$ and x is treated as a fixed parameter.

For any $(x, z) \in B_x \times B_z$, let $V(x)$ be a Lyapunov for system (1.4) such that

$$\begin{aligned} &\frac{\partial V}{\partial x} f(x, h(x)) \le -\alpha_1 \psi_1^2(x), \quad \alpha_1 > 0, \\ &\frac{\partial V}{\partial x}[f(x, z + h(x)) - f(x, h(x))] \le \beta_1 \psi_1(x)\psi_2(z), \end{aligned} \tag{1.6}$$

and $W(x, z)$ be a Lyapunov function for system (1.5) such that

$$\begin{aligned} &W(x, z) > 0, \quad \text{for any } z \neq 0, \qquad W(x, 0) = 0, \\ &W_1(z) \le W(x, z) \le W_2(z), \\ &\frac{\partial W}{\partial z} g(x, z + h(x)) \le -\alpha_2 \psi_2^2(y), \\ &\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial z}\frac{\partial h}{\partial x}\right] f(x, z + h(x)) \le \beta_2 \psi_1(x)\psi_2(z) + \gamma \psi_2^2(z), \end{aligned} \tag{1.7}$$

where ψ_1, ψ_2, W_1 and W_2 are positive definite functions, which vanish only when their arguments are zero, *e.g.* $\psi_1 = 0$ if and only if $x = 0$. Then, the origin is an asymptotically stable equilibrium point of the singular perturbation system (1.2) in $B_x \times B_z$, for all $\epsilon \in (0, \epsilon^*)$, where $\epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$. Moreover, for any $d \in (0, 1)$

$$v(x, z) = (1 - d)V(x) + dW(x, z)$$

is a Lyapunov function for all $\epsilon \in (0, \epsilon_d)$, where $\epsilon_d \le \epsilon^*$. See [13–17] for detailed singular perturbation theory.

According to above discussion, to obtain an estimation for the zero solution of system (1.2), it is necessary to construct the Lyapunov functions V and W. However, it is usually difficult for nonlinear systems. In the present paper, by choosing a reasonable transformation, we transform system (1.1) into the standard singular perturbation system. Then, using the recurrence formulae established by S. Balint, A. Balint and V. Negru in [21], we construct a Lyapunov function V_2 for the reduced system. Further, we obtain a Lyapunov function W for boundary-layer system and then verify the conditions (1.6) and (1.7) for V_2 and W. Finally, we estimate the domain of attraction of the endemic equilibrium point for system (1.1).

The remainder of the paper is arranged as follows: Section 2 discusses the stability of the non-hyperbolic equilibrium point of the reduced system, then estimates the domain of attraction of the endemic equilibrium point. Section 3 presents numerical simulation results.

2. The domain of attraction of the endemic equilibrium

For convenience, we introduce two notations: $m = d + \alpha + \gamma$, $n = d + \delta$. Then, system (1.2) can be transformed into

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - dS - \beta SI + \delta R, \\ \dfrac{dI}{dt} = \beta SI - mI, \\ \dfrac{dR}{dt} = \gamma I - nR. \end{cases} \tag{2.1}$$

Let $R_0 = \frac{\beta\Lambda}{dm}$, J. Mena-Lorca, et al. have obtained the following qualitative results about system (2.1) (see [4] for details):

Theorem 2.1 ([4]). *If $R_0 \le 1$, there only exists the disease free equilibrium point $E_0(\frac{\Lambda}{d}, 0, 0)$, which is globally asymptotically stable; while if $R_0 > 1$, there exist an unstable disease free equilibrium point E_0 and a unique locally asymptotically stable endemic equilibrium point $E^*(S^*, I^*, R^*)$, where $S^* = \frac{m}{\beta}$, $I^* = \frac{dmn(R_0-1)}{\beta(mn-\delta\gamma)}$, $R^* = \frac{\gamma}{n}I^*$. Particularly, if $R_0 > 1$ and $\delta = 0$, E^* is globally asymptotically stable.*

Suppose $\frac{\gamma}{\Lambda}$ is small enough, and the number of the infected has the same magnitude order as the susceptible. Re-scale time by $\gamma t = \tau$ and define non-dimensional variables: $x_1 = \frac{S}{\Omega}$, $x_2 = \frac{I}{\Omega}$, and $y = \frac{R}{\Omega\epsilon}$ with $\Omega = \frac{\Lambda}{d}$ and $\epsilon = \frac{\gamma}{\Lambda}$. Then, system (2.1) can be rewritten as the following singular perturbation form

$$\begin{cases} \dfrac{dx_1}{d\tau} = a_1(1-x_1) - a_2x_1x_2 + b_1y, \\ \dfrac{dx_2}{d\tau} = a_2x_1x_2 - a_3x_2, \\ \epsilon\dfrac{dy}{d\tau} = x_2 - b_2y, \end{cases} \tag{2.2}$$

where $a_1 = \frac{d}{\gamma}$, $a_2 = \frac{\beta\Omega}{\gamma}$, $a_3 = \frac{m}{\gamma}$, $b_1 = \frac{\delta}{\Lambda}$ and $b_2 = \frac{n}{\Lambda}$. Clearly, we have

$$b_2 > b_1, a_3 > 1 \quad \text{and} \quad a_3 - a_1 - \frac{b_1}{b_2} > 0. \tag{2.3}$$

Let ϵ tend to zero in system (2.2), we get a reduced system

$$\begin{cases} \dfrac{dx_1}{d\tau} = a_1(1-x_1) - a_2x_1x_2 + b_1y, \\ \dfrac{dx_2}{d\tau} = a_2x_1x_2 - a_3x_2, \\ 0 = x_2 - b_2y. \end{cases} \tag{2.4}$$

Note that the third equation of (2.4) has a unique real function root $y = h(x) = \frac{x_2}{b_2}$, where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$. Then, the reduced system (2.4) can also be expressed as

$$\begin{cases} \dfrac{dx_1}{d\tau} = a_1(1-x_1) - a_2x_1x_2 + \dfrac{b_1}{b_2}x_2 := P(x_1, x_2), \\ \dfrac{dx_2}{d\tau} = a_2x_1x_2 - a_3x_2 := Q(x_1, x_2). \end{cases} \tag{2.5}$$

The boundary layer system is defined as

$$\frac{dy}{d\tau'} = x_2 - b_2(y + h(x)) = -b_2y, \tag{2.6}$$

where $\epsilon\tau' = \tau$.

The following Theorems 2.2 and 2.3 characterize the qualitative properties of the solutions of the reduced system (2.5).

Theorem 2.2. *Let $R_0^r = \frac{a_2}{a_3} = \frac{\beta\Lambda}{dm}$. If $R_0^r < 1$, there only exists a globally asymptotically stable equilibrium point $(1, 0)$; if $R_0^r > 1$, there exist an unstable equilibrium point $(1, 0)$ and a unique globally asymptotically stable equilibrium point (x_1^*, x_2^*), where $x_1^* = \frac{a_3}{a_2}$, $x_2^* = \frac{a_1 a_3 b_2 (R_0^r - 1)}{a_2(b_2 a_3 - b_1)}$; if $R_0^r = 1$, there only exists equilibrium point $(1, 0)$.*

Proof. The Jacobian matrix at $(1, 0)$ is

$$M := \begin{pmatrix} -a_1 & -a_2 + \dfrac{b_1}{b_2} \\ 0 & a_2 - a_3 \end{pmatrix}.$$

It is easy to see that the eigenvalues of the matrix are negative if $R_0^r < 1$, and there exists a positive eigenvalue if $R_0^r > 1$. Therefore, the equilibrium $(1, 0)$ is locally asymptotically stable if $R_0^r < 1$ and unstable if $R_0^r > 1$.

The Jacobian matrix at (x_1^*, x_2^*) is

$$J := \begin{pmatrix} -a_1 - a_2 x_2^* & -a_2 x_1^* + \dfrac{b_1}{b_2} \\ a_2 x_2^* & a_2 x_1^* - a_3 \end{pmatrix}.$$

Obviously, all of the eigenvalues of J are negative if $R_0^r > 1$. Hence, if $R_0^r > 1$, the equilibrium (x_1^*, x_2^*) is locally asymptotically stable.

Taking the Dulac function

$$D(x_1, x_2) = \frac{1}{x_1 x_2},$$

where $x_1, x_2 > 0$, we have

$$DP = \frac{a_1}{x_1 x_2} - \frac{a_1}{x_2} - a_2 + \frac{b_1}{b_2 x_1}, \qquad DQ = a_2 - \frac{a_3}{x_1}.$$

It is not difficult to obtain

$$\frac{\partial DP}{\partial x_1} + \frac{\partial DQ}{\partial x_2} = -\frac{a_1}{x_1^2 x_2} - \frac{b_1}{b_2 x_1^2} < 0.$$

Therefore, there is not a closed orbit inside the first quadrant. By Bendixson–Dulac Theorem, the equilibrium $(1, 0)$ is globally asymptotically stable if $R_0^r < 1$ and the equilibrium (x_1^*, x_2^*) is globally asymptotically stable if $R_0^r > 1$. □

Clearly, if $R_0^r = 1$, the eigenvalues of matrix M are 0 and $-a_1$. Thus, $(0, 0)$ is not a hyperbolic equilibrium point. Therefore, we investigate the dynamics near $(1, 0)$ by the center manifold theorem [18]. Firstly, shift $(1, 0)$ to the origin via $y_1 = x_1 + 1$ and $y_2 = x_2$, and system (2.4) can be transformed into

$$\begin{cases} \dfrac{dy_1}{dt} = -a_1 y_1 + \left(\dfrac{b_1}{b_2} - a_2\right) y_2 - a_2 y_1 y_2, \\ \dfrac{dy_2}{dt} = a_2 y_1 y_2. \end{cases} \tag{2.7}$$

Secondly, define the transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & -c \end{pmatrix} \quad \text{with } c = \frac{b_2 a_1}{a_2 b_2 - b_1},$$

and which transformed system (2.7) into the following standard form:

$$\begin{cases} \dfrac{dz_1}{dt} = -a_1 z_1 + f_1(z_1, z_2), \\ \dfrac{dz_2}{dt} = f_2(z_1, z_2), \end{cases} \tag{2.8}$$

where $f_1 = a_2(c-1) z_1 (z_1 - z_2)$ and $f_2 = a_2 z_1 (z_1 - z_2)$.

By the existence theorem in the center manifold theory (see [18] for details), there exists a center manifold for system (2.8), which can be expressed locally as follows

$$W^c(0) = \{(z_1, z_2) \in \mathbb{R}^2 | z_1 = h'(z_2), \|z_2\| < \delta, h'(0) = 0, Dh'(0) = 0 | \delta > 0\}$$

with δ sufficiently small, and Dh' is the derivative of h' with respect to z_2.

Now, the first task is to compute the center manifold $W^c(0)$. For the purpose, we assume $h'(z_2)$ has the form

$$z_1 = h'(z_2) = h_1 z_2^2 + h_2 z_2^3 + h_3 z_2^4 + h_4 z_2^5 + h_5 z_2^6 + h_6 z_2^7 \cdots. \tag{2.9}$$

By the invariance of $W^c(0)$ under the dynamics of (2.8), the center manifold satisfies

$$Dh' \cdot f_2(z_2, h') + a_1 h' - f_1(z_2, h') = 0. \tag{2.10}$$

Substituting (2.8) into (2.10), and then equating coefficients on each power of z_2 to zero yields

$$\begin{aligned}
h_1 &= \frac{a_2(c-1)}{a_1}, \\
h_2 &= \frac{a_2^2(c-1)(c-3)}{a_1^2}, \\
h_3 &= \frac{a_2^3(c-1)(-9c+14+c^2)}{a_1^3}, \\
h_4 &= \frac{a_2^4(c-1)(-19c^2+79c-85+c^3)}{a_1^4}, \\
h_5 &= \frac{a_2^5(c-1)(-34c^3+274c^2-742c+621+c^4)}{a_1^5}, \\
h_6 &= \frac{a_2^6(c-1)(743c^3-3717c^2+7544c-55c^4-5236+c^5)}{a_1^6}.
\end{aligned} \tag{2.11}$$

Then, using the Eqs. (2.10) and (2.11), we get a approximation for h':

$$\begin{aligned}
h' = (c-1)\Bigg[&\frac{a_2}{a_1} z_2^2 + \frac{a_2^2(c-3)}{a_1^2} z_2^3 + \frac{a_2^3(-9c+14+c^2)}{a_1^3} z_2^4 \\
&+ \frac{a_2^4(-19c^2+79c-85+c^3)}{a_1^4} z_2^5 + \frac{a_2^5(-34c^3+274c^2-742c+621+c^4)}{a_1^5} z_2^6 \\
&+ \frac{a_2^6(743c^3-3717c^2+7544c-55c^4-5236+c^5)}{a_1^6} z_2^7 \Bigg] + \cdots.
\end{aligned} \tag{2.12}$$

Substituting (2.12) into the second equation of system (2.8), we achieve the vector field reduced to the center manifold

$$\begin{aligned}
\frac{\mathrm{d}z_2}{\mathrm{d}t} = {} & a_2 z_2^2 + \frac{a_2^2(c-1)}{a_1} z_2^3 + \frac{a_2^3(c-1)(c-3)}{a_1^2} z_2^4 + \frac{a_2^4(c-1)(c-2)(c-7)}{a_1^3} z_2^5 \\
&+ \frac{a_2^5(c-1)(-19c^2+79c-85+c^3)}{a_1^4} z_2^6 \cdots.
\end{aligned} \tag{2.13}$$

Therefore, by using (2.13) and the positiveness of a_2, we get

Theorem 2.3. *If $R_0^r = 1$, the equilibrium point $(1, 0)$ of the reduced system (2.5) is unstable.*

Let $y^* = \frac{x_2^*}{b_2}$. Note that (x_1^*, x_2^*, y^*) is a unique positive equilibrium point of system (2.2) and its dynamic properties are equivalent to those of the endemic equilibrium point of system (1.1) for any positive ϵ. Therefore, to analyze the asymptotical stability of the endemic equilibrium point of (1.1), we only need to investigate the stability of the nontrivial equilibrium point of (2.2). For this task, we make some preparations.

Consider the following system

$$\dot{x} = f(x), \tag{2.14}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a analytical function with the following properties:

(1) $f(0) = 0$, i.e. $x = 0$ is an equilibrium point of (2.14).
(2) every eigenvalue of $\frac{\partial f}{\partial x}(0)$ has negative real part, i.e. $x = 0$ is a asymptotically stable equilibrium point.

Let DA be the domain of attraction of the zero solution of (2.14). The following theorem provides a tool to determine DA.

Theorem 2.4 ([19]). *DA coincides with the natural domain of analyticity of the unique V of the problem*

$$\begin{aligned} &\langle \nabla V, f\rangle = -\|x\|^2, \\ &V(0) = 0. \end{aligned} \tag{2.15}$$

The function V is positive on DA and $\lim_{x\to x_0} V(x) = \infty$ *for any* $x_0 \in \partial DA$.

Thus, the problem of determining DA is reduced to finding the natural domain of analyticity of the solution V of (2.15). The function V is called the optimal Lyapunov function for (2.14). Generally, it is not easy to construct the optimal Lyapunov function V and determine its domain of analyticity. But, in the diagonalizable case, we can determine the coefficients of the expansion of $W = V \circ S$ at 0, where S reduces $\frac{\partial f}{\partial x}(0)$ to the diagonal form. By Cauchy–Hadamard type theorem [20], we can obtain the domain of convergence D^0 of the series W, and $DA^0 = S(D^0)$ is a part of the domain of attraction.

For system (2.14), the following theorem holds:

Theorem 2.5 ([21]). *For each isomorphism* $S : \mathbb{C}^n \to \mathbb{C}^n$ *and* $g = S^{-1} \circ f \circ S$, *the problem*

$$\begin{aligned} &\langle \nabla W, g\rangle = -\|Sz\|^2, \\ &W(0) = 0, \end{aligned} \tag{2.16}$$

has a unique analytical solution, namely, $W = V \circ S$, *where V is the optimal Lyapunov function for* (2.14).

Let $S : \mathbb{C}^n \to \mathbb{C}^n$ be one isomorphism which reduces $\frac{\partial f}{\partial x}(0)$ to the diagonal form $S^{-1}\frac{\partial f}{\partial x}(0)S = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Donote $g = S^{-1} \circ f \circ S$ and $W = V \circ S$, where V is the optimal Lyapunov function for (2.14).

For given S, we consider the expansion of W at origin:

$$W(z_1, z_2, \ldots, z_n) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1 j_2 \cdots j_n} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} \tag{2.17}$$

and the expansions of the scalar components g_i of g at origin:

$$g_i(z_1, z_2, \cdot, z_n) = \lambda_i z_i + \sum_{m=2}^{\infty} \sum_{|j|=m} b^i_{j_1 j_2 \cdots j_n} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$$

then the coefficients $B_{j_1 j_2 \cdots j_n}$ of the development (2.17) are given by

$$B_{j_1 j_2 \cdots j_n} = \begin{cases} -\dfrac{1}{2\lambda_{i_0}} \displaystyle\sum_{i=1}^{n} s_{ii_0}^2 & \text{if } |j| = j_{i_0} = 2\,; \\ -\dfrac{2}{\lambda_p + \lambda_q} \displaystyle\sum_{i=1}^{n} s_{ip} s_{iq} & \text{if } |j| = 2 \text{ and } j_p = j_q = 1; \\ -\dfrac{1}{\sum_{i=1}^{n} j_i \lambda_i} \displaystyle\sum_{p=2}^{|j|-1} \sum_{|k|=p, k_i \le j_i} \sum_{i=1}^{n} [(j_i - k_i + 1) b^i_{k_1 k_2 \cdots k_n} B_{j_1-k_1 \cdots j_i-k_i \cdots j_n-k_n}] & \text{if } |j| \ge 3. \end{cases} \tag{2.18}$$

For V_p, $p \ge 2$, we have the following results [22]:

Theorem 2.6. *For any* $p \ge 2$, *there exists* $r_p > 0$ *such that for any* $x \in \overline{B}(r_p)/\{0\}$ *one has:*

$$\begin{aligned} &V_p(x) > 0, \\ &\langle \nabla V_p(x), f(x)\rangle < 0, \end{aligned}$$

where $B(r_p) = \{x \in \mathbb{R}^n \,|\, \|x\| < r_p\}$, *and* $\overline{B}(r_p)$ *is its closure.*

Remark 1. For any positive integer $p \ge 2$, Theorem 2.6 provides a Lyapunov function for system (2.14) in $B(r_p)$.

Define

$$a_{11} = -(a_1 + a_2 x_2^*), \qquad a_{12} = -\left(a_2 x_1^* - \frac{b_1}{b_2}\right), \qquad a_{21} = a_2 x_2^*, \tag{2.19}$$

$$r^0 = \min\left\{ \frac{a_{11} a_{12}}{a_2(a_{21} - a_{12} - a_{11})}, \frac{1}{a_2\left(\frac{a_{21}+a_{11}}{a_{12}a_{21}} - \frac{1}{a_{11}} + \frac{a_{12}}{a_{11}a_{21}}\right)} \right\}. \tag{2.20}$$

Using (2.3), we easily obtain $a_{11}, a_{12} < 0$ and $a_{21} > 0$.

Then, for any $r^* \in (0, r^0)$, we have the following results on the domain of attraction of (x_1^*, x_2^*, y^*):

Theorem 2.7. *Let* $\epsilon^* = \frac{2b_2(1-\frac{r^*}{r^0})}{b_1(a_2+a_{21})\max\{2k_1,k_3\}}$, *where* k_1, k_3 *will be defined in* (2.32). *Then, for any* $\epsilon \in (0, \epsilon^*]$, *the positive equilibrium point* (x_1^*, x_2^*, y^*) *is asymptotically stable in* $\{(x_1, x_2)|(x_1 - x_1^*, x_2 - x_2^*) \in \overline{B}(r^*) \subset \mathbb{R}^2\} \times (-\infty, +\infty)$ *when* $R_0 > 1$.

Proof. Let $\bar{x}_1 = x_1 - x_1^*, \bar{x}_2 = x_2 - x_2^*$ and $\bar{y} = y - y^*$, omit the bar, the full system (2.2), the reduced system (2.5) and the boundary-layer system (2.6) are respectively transformed as follows

$$\begin{cases} \dfrac{dx}{d\tau} = f(x, y), \\ \epsilon \dfrac{dy}{d\tau} = x_2 - b_2 y, \end{cases} \tag{2.21}$$

$$\frac{dx}{dt} = f(x, h(x)) \tag{2.22}$$

and

$$\frac{dy}{d\tau'} = -b_2 y, \tag{2.23}$$

where

$$\tau' = \tau/\epsilon, \qquad f(x, y) = \begin{pmatrix} a_{11}x_1 - a_2 x_1^* x_2 - a_2 x_1 x_2 + b_1 y, \\ a_{21}x_1 + a_2 x_1 x_2, \end{pmatrix}, \qquad h(x) = \frac{x_2}{b_2}. \tag{2.24}$$

Clearly, system (2.21) is a new standard singular perturbation system, and (2.22), (2.23) are its reduced system and boundary-layer system, respectively.

Define the following transformation:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad S = \begin{pmatrix} \dfrac{\lambda_1}{a_{21}} & \dfrac{\lambda_2}{a_{21}} \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{2.25}$$

where $\lambda_1 = \frac{a_{11}+\sqrt{a_{11}^2+4a_{12}a_{21}}}{2a_{21}}$, $\lambda_2 = \frac{a_{11}-\sqrt{a_{11}^2+4a_{12}a_{21}}}{2a_{21}}$ are the eigenvalues of the Jacobian matrix of system (2.22) at origin. Then system (2.22) are transformed into the following standard form

$$\frac{dz}{dt} = g'(z), \tag{2.26}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad g'(z) = \begin{pmatrix} \lambda_1 z_1 - \dfrac{a_2}{\lambda_1 - \lambda_2}\left(1 + \dfrac{\lambda_2}{a_{21}}\right)(z_1 + z_2)(\lambda_1 z_1 + \lambda_2 z_2) \\ \lambda_2 z_2 + \dfrac{a_2}{\lambda_1 - \lambda_2}\left(1 + \dfrac{\lambda_1}{a_{21}}\right)(z_1 + z_2)(\lambda_1 z_1 + \lambda_2 z_2) \end{pmatrix}.$$

For the isomorphism $S : \mathbb{C}^n \to \mathbb{C}^n$ and the function $g' : \mathbb{R}^2 \to \mathbb{R}^2$, it follows from Theorem 2.5 that there exists a unique analytical function $W = V \circ S$ such that $\langle \nabla W, g' \rangle = -\|Sx\|^2$ and $W(0) = 0$, where V is the optimal Lyapunov function for the reduced system (2.22).

Let the expansion of W at $(0, 0)$ be

$$W(z_1, z_2) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1 j_2} z_1^{j_1} z_2^{j_2}, \tag{2.27}$$

and denote the components g_i of g as

$$g_i(z_1, z_2) = \lambda_i z_i + \sum_{m=2}^{\infty} \sum_{|j|=m} b^i_{j_1 j_2} z_1^{j_1} z_2^{j_2}, \quad i = 1, 2.$$

The coefficients of expansion for W are given

$$B_{j_1j_2}=\begin{cases}-\dfrac{1}{2\lambda_{i_0}}\displaystyle\sum_{i=1}^{2}s_{ii_0}^2, & \text{if } |j|=j_{i_0}=2,\\ -\dfrac{2}{\lambda_p+\lambda_q}\displaystyle\sum_{i=1}^{2}s_{ip}s_{iq}, & \text{if } |j|=2, j_p=j_q,\\ -\dfrac{1}{\displaystyle\sum_{i=1}^{2}j_i\lambda_i}\displaystyle\sum_{p=2}^{|j|-1}\sum_{i=1}^{2}[(j_i-k_i+1)b_{k_1k_2}^{i}B_{j_1-k_1j_2-k_2}], & \text{if } |j|\geq 3.\end{cases} \tag{2.28}$$

After tedious computation, we obtain

$$B_{20}=-\frac{1}{2\lambda_1}(s_{11}^2+s_{21}^2),$$
$$B_{11}=-\frac{2}{\lambda_1+\lambda_2}(s_{11}s_{12}+s_{21}s_{22}),$$
$$B_{02}=-\frac{1}{2\lambda_2}(s_{12}^2+s_{22}^2). \tag{2.29}$$

Let $W_2(z_1,z_2)$ be the sum of the square terms of the expansion (2.27). Clearly,

$$W_2(z_1,z_2)=B_{20}z_1^2+B_{11}z_1z_2+B_{02}z_2^2.$$

Correspondingly, by the reversibility of S, we obtain a analytical function

$$\begin{aligned}V_2(z_1,z_2)&=(W_2\circ S^{-1})(z_1,z_2)\\&=\frac{1}{(\lambda_1-\lambda_2)^2}\left[a_{21}^2(B_{20}-B_{11}+B_{02})z_1^2+a_{21}(-2B_{20}\lambda_2+(\lambda_1+\lambda_2)B_{11}-2B_{02}\lambda_1)z_1z_2\right.\\&\quad\left.+\ (B_{20}\lambda_2^2-B_{11}\lambda_1\lambda_2+B_{02}\lambda_1^2)z_2^2\right]\\&=k_1z_1^2+k_3z_1z_2+k_2z_2^2,\end{aligned} \tag{2.30}$$

where

$$k_1=\frac{\alpha_{21}^2(B_{20}-B_{11}+B_{02})}{(\lambda_1-\lambda_2)},\qquad k_2=\frac{\lambda_1^2B_{02}+\lambda_2^2B_{20}-\lambda_1\lambda_2B_{11}}{(\lambda_1-\lambda_2)^2},$$
$$k_3=\frac{\lambda_1^2a_{21}^2B_{11}-2\lambda_1^2a_{21}^2B_{02}+\lambda_1^2\lambda_2+\lambda_2a_{21}^2+\lambda_1\lambda_2a_{21}^2B_{11}}{a_{21}\lambda_1(\lambda_2-\lambda_1)^2}. \tag{2.31}$$

Substituting (2.29) into (2.31), we get

$$k_1=\frac{a_{21}(a_{21}-a_{12})}{2a_{11}a_{21}a_{12}},\qquad k_2=\frac{a_{12}^2+a_{11}^2-a_{12}a_{21}}{2a_{11}a_{21}a_{12}},\qquad k_3=-\frac{1}{a_{12}}. \tag{2.32}$$

By means of (2.19), it is not difficult to verify that k_1,k_2,$k_3>0$, and V_2 is a positive definite function in $\mathbb{R}^2$. The total derivative of V_2 along the flow of (2.22) is

$$\frac{\partial V_2}{\partial x}f(x,h(x))=c_1x_1^2x_2-x_1^2-x_2^2+c_2x_1x_2^2, \tag{2.33}$$

where $x\in\mathbb{R}^2$, $c_1=\frac{a_2(a_{12}-a_{21}-a_{11})}{a_{11}a_{12}}$, $c_2=\frac{a_2(a_{21}a_{11}+a_{12}^2+a_{11}^2-a_{21}a_{12})}{a_{11}a_{12}a_{21}}$.

By using (2.3) and (2.19), we easily get $c_1=\frac{a_1}{a_{11}a_{12}}(a_1-a_3+\frac{b_1}{b_2})>0$ and $c_2=a_2(\frac{-a_1}{a_{12}a_{21}}+\frac{a_{12}}{a_{11}+a_{21}}-\frac{1}{a_{11}})>0$. Then, we have $r^0>0$. It follows from Eq. (2.33) that, for any $0<r^*<r^0$, the total derivative $\frac{\partial V_2}{\partial x}f(x,h(x))$ is negative definite in $\overline{B}(r^*)$. In fact, for any $x\in\overline{B}(r^*)$, we have

$$\frac{\partial V_2}{\partial x}f(x,h(x))<-\alpha_1(r^*)\psi_1^2(x), \tag{2.34}$$

where $\psi_1(x)=\sqrt{x_1^2+x_2^2}$, $\alpha_1(r^*)=1-\frac{r^*}{r^0}>0$.

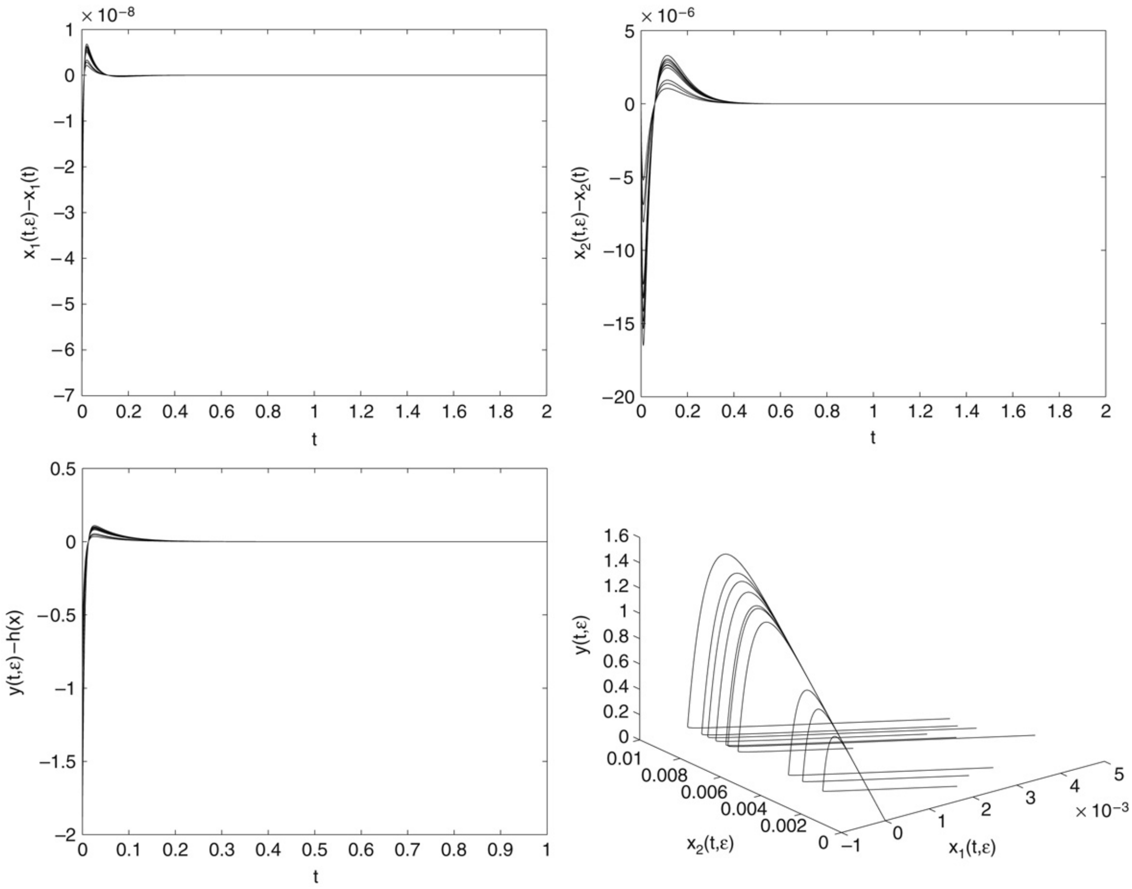

Fig. 1. $\epsilon = 0.00025$, $r^0 \approx 0.00578090651$ and $R_0 \approx 84.25057632$.

Let

$$W(x, y) = y^2, \tag{2.35}$$

where x is considered as a parameter. Clearly, W is a positive definite function in $\mathbb{R}^1$. The total derivative of W along the flow of boundary-layer system (2.22) is

$$\frac{\partial W(x, y)}{\partial y} g(x, y + h(x)) = -\alpha_2 \psi_2^2(y), \quad \text{for all } x \in \mathbb{R} \tag{2.36}$$

where $\alpha_2 = 2b_2 > 0$, $\psi_2(y) = |y|$.

Next, we verify that $W(x, y)$ and $V_2(x)$ satisfy the remainder conditions of (1.6) and (1.7) for any $(x, y) \in \overline{B}(r^*) \times \overline{B}(r)$, where r is an arbitrary positive number. In fact, for any $(x, y) \in \overline{B}(r^*) \times \overline{B}(r)$, we have

$$\begin{aligned} \nabla V_2 \cdot [f(x, y + h(x)) - f(x, h(x))] &= (2k_1x_1 + k_3x_2, k_3x_1 + 2k_2x_2) \cdot \begin{pmatrix} b_1 y \\ 0 \end{pmatrix} \\ &= b_1 y\,(2k_1x_1 + k_3x_2) \\ &\le \beta_1 \psi_1(x)\psi_2(y) \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\partial W(x, y)}{\partial x} - \frac{\partial W(x, y)}{\partial y}\frac{\partial h(x)}{\partial x}\right) f(x, y + h(x)) &= -y(a_{21}x_1 + a_2x_1x_2) \\ &\le \beta_2 \psi_1(x)\psi_2(y), \end{aligned}$$

where $\beta_1 = \max\{2k_1, k_3\} b_1$, $\beta_2 = (a_2 + a_{21})$.

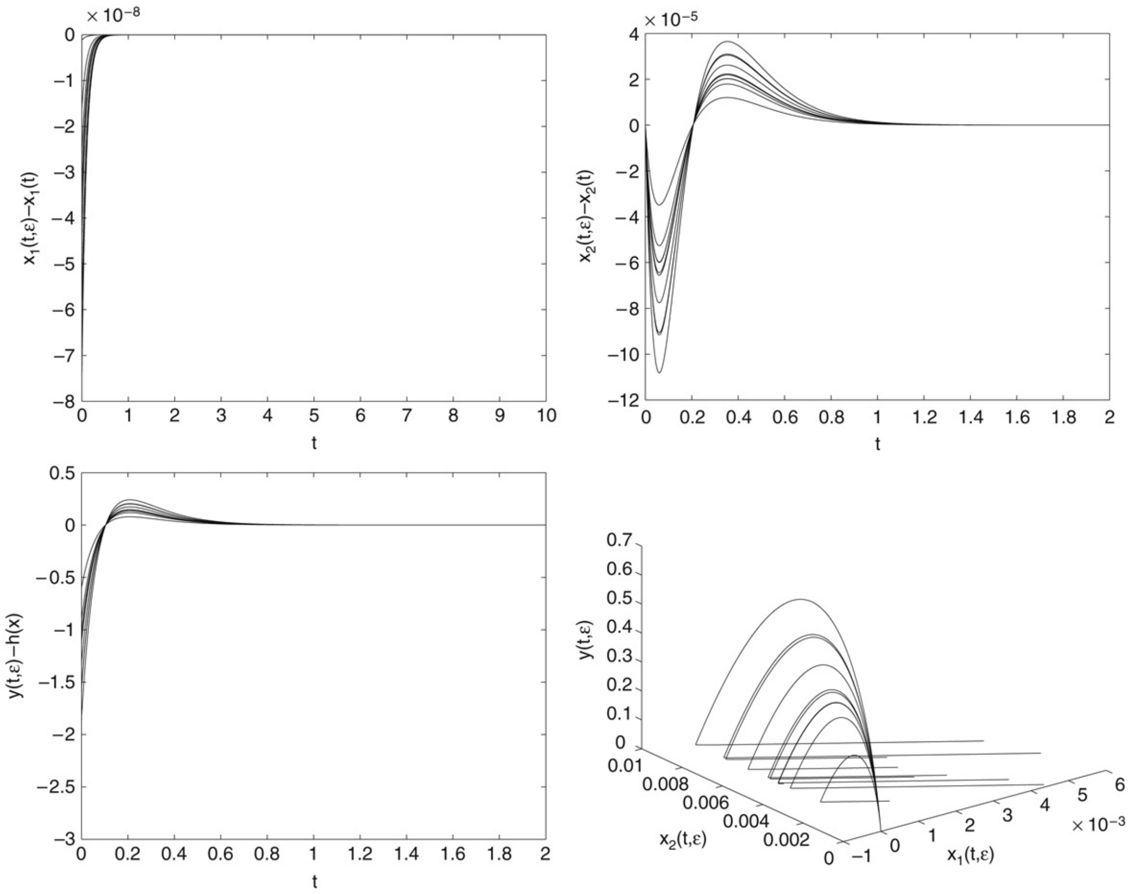

Fig. 2. $\epsilon = 0.0005$, $r^0 \approx 0.00593709846$ and $R_0 \approx 80$.

Let $\epsilon^* = \frac{\alpha_1(r^*)\alpha_2}{\beta_1\beta_2} = \frac{2b_2(1-\frac{r^*}{r^0})}{(a_2+a_{21})b_1\max\{2k_1,k_3\}}$. By the results presented in the introduction, for any $\epsilon \in (0, \epsilon^*]$, the equilibrium point (x_1^*, x_2^*, y^*) is asymptotically stable in $\{(x_1, x_2)|(x_1 - x_1^*, x_2 - x_2^*) \in \overline{B}(r^*)\} \times \{y|y - y^* \in \overline{B}(r)\}$. $\square$

In fact, by the arbitrariness of r^* and r, we have

Remark 2. Under the assumptions of Theorem 2.7, the positive equilibrium point (x_1^*, x_2^*, y) is asymptotically stable in $\{(x_1, x_2)|(x_1 - x_1^*, x_2 - x_2^*) \in B(r^0)\} \times (-\infty, +\infty)$.

For the endemic equilibrium point (S^*, I^*, R^*) of system (1.1), we have

Theorem 2.8. *If ϵ is small enough and $R_0 > 1$, the endemic equilibrium point (S^*, I^*, R^*) is asymptotically stable in* $\{(S, I) \in R_+^2|(S - S^*, I - I^*) \in B(\frac{\Lambda r^0}{d})\} \times (0, \infty)$.

3. Numerical simulation

In this section, we present some numerical simulation results to show how the reduced system (2.5) approximates to the full system (2.2) and how the small parameter ϵ affects the stability of zero solution of system (2.2) when $R_0 > 1$. By the equivalence of system (2.2) and (2.21), we only focus on the numerical analysis of system (2.21) and its reduced system (2.22).

Let $(x_1(t, \epsilon), x_2(t, \epsilon), y(t, \epsilon))$ be the solution of system (2.21), $(x_1(t), x_2(t))$ the solution of system (2.22), $\beta = 0.002$, $\delta = 0.005$, $d = 0.005$ and $\Lambda = 2$. Let ϵ be evaluated at 0.00025, 0.0005 and 0.001, and denote them as ϵ_1, ϵ_2 and ϵ_3, respectively. Correspondingly, by Theorem 2.7, we obtain three different r^0, and every one of them results in a domain of attraction of the zero solution of system (2.21). In each domain, we randomly select 10 points as the initial values of system (2.21). For ϵ_1, ϵ_2 and ϵ_3, the results are respectively sketched in Figs. 1–3. In each Fig., the errors $x_1(t, \epsilon) - x_1(t)$, $x_2(t, \epsilon) - x_2(t)$, $y(t, \epsilon) - h(x_1(t), x_2(t))$ and the solutions $(x_1(t, \epsilon), x_2(t, \epsilon), y(t, \epsilon))$ are showed in turn, where $h(x)$ is defined by (2.24).

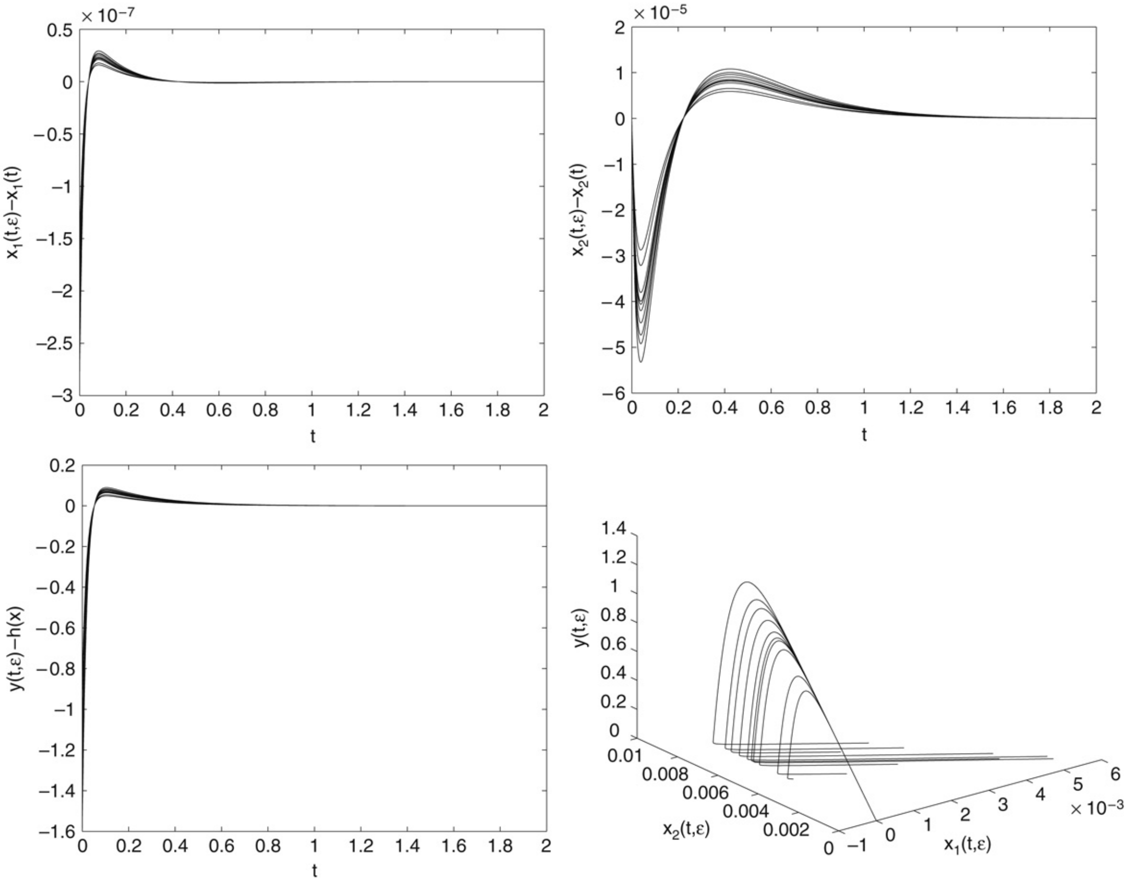

Fig. 3. $\epsilon = 0.001$, $r^0 \approx 0.006549455546$ and $R_0 \approx 72.727273$.

If ϵ is small enough, by using Figs. 1–3, we make the following conclusions: for a fixed ϵ, the solutions of the reduced system (2.22) closely approximate to the solutions of the full system (2.21) and the errors (i.e. $x_1(t,\epsilon)-x_1(t)$, $x_2(t,\epsilon)-x_2(t)$, $y(t,\epsilon)-h(x_1(t),x_2(t))$) quickly converge to zero after first oscillation, and all solutions of system (2.21) approach to zero solution.

4. Conclusion

In this paper, based on the singular perturbed method and the recurrence formulae established by S. Balint, A. Balint and V. Negru, a domain of attraction of the endemic equilibrium point of J. Mena-lorca and H. W. Hethcote's SIRS epidemic model is estimated, when the parameter ϵ is small enough. Furthermore, by the center manifold theorem, the stability of the non-hyperbolic equilibrium point of the reduced system (2.5) is discussed. Finally, numerical simulation results are presented, which help to understand how the induced system (2.21) approximates to the full system (2.22) and how the small parameter ϵ affects the stability of the zero solution of the full system (2.21).

Owing to the limitation of space and time, the global stability of the zero solution of system (2.21) will not be dealt with here, however, this will be considered in the future.

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