Permanence and stability in multi-species non-autonomous Lotka–Volterra competitive systems with delays and feedback controls

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ABSTRACT
In this paper, we consider a multi-species Lotka–Volterra type competitive system with delays and feedback controls. A general criteria on the permanence is established, which is described by integral form and independent of feedback controls. By constructing suitable Lyapunov functionals, a set of easily verifiable sufficient conditions are derived for global stability of any positive solution to the model.

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1. Introduction

In this paper, we consider the non-autonomous n-species Lotka–Volterra type competitive systems with delays and feedback controls

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= x_i(t) \left( b_i(t) - a_i(t)x_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t)) - d_i(t)u_i(t - \delta_i(t)) \right), \\
\frac{du_i(t)}{dt} &= r_i(t) - e_i(t)u_i(t) + c_i(t)x_i(t - \sigma_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \( x_i(t) \) represents the population density of the \( i \)-th species at time \( t \), \( u_i(t) \) denotes “indirect control” variable (see [1, 14,16]); the functions \( a_i(t), b_i(t), c_i(t), d_i(t), e_i(t), r_i(t), \delta_i(t), \sigma_i(t), \tau_{ij}(t) \) and \( a_{ij}(t) \) (\( i, j = 1, 2, \ldots, n \)) are defined on \( \mathbb{R} = (-\infty, \infty) \).

In particular, when the delays \( \delta_i(t) \equiv 0, \sigma_i(t) \equiv 0 \) and \( \tau_{ij}(t) \equiv 0 \) for all \( t \in \mathbb{R} \) and \( i, j = 1, 2, \ldots, n \), then the system (1.1) will degenerate into the following non-delayed non-autonomous \( n \)-species Lotka–Volterra system

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= x_i(t) \left( b_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t) - d_i(t)u_i(t) \right), \\
\frac{du_i(t)}{dt} &= r_i(t) - e_i(t)u_i(t) + c_i(t)x_i(t), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

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where \( b_{ij}(t) = a_{ij}(t) + a_{ij}(t) \) and \( b_{ij}(t) = a_{ij}(t) \) for \( i, j = 1, 2, \ldots, n \) and \( i \neq j \). We remark that the method we develop for (1.1) can be used to discuss the special system (1.2).

As we well know, systems like (1.1) and (1.2) without feedback controls are very important in the models of multi-species populations dynamics. Many important results on the permanence, extinction, global asymptotical stability for the two species or multi-species non-autonomous Lotka–Volterra systems and their special cases of periodic and almost periodic systems can be found in [3–5,11–13,16,19,23–27] and the references therein.

However, we note that ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. In ecology, a question of practical interest is whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control theory, we call the disturbance functions as control variables. In 1993, Gopalsamy and Weng [1] introduced a feedback control variable into the delayed logistic model and discussed the asymptotic behavior of solutions in logistic models with feedback controls, in which the control variables satisfy certain differential equation. In the last decade, much work has been done on the ecosystem with feedback controls (see [1,2,6–8,10,15,17,21,22,28,29] and the references therein). In particular, Li and Liu [2], Lalli et al. [8], Li and Wang [10], Liu and Xu [21], Chen and Li [28], Tang and Zou [29] have studied delay equations with feedback controls.

The main purpose of this paper is to study the system (1.1) in the general non-autonomous case. We will establish a new general criteria for the permanence of system (1.1), which is described by integral form and independent of feedback controls. In addition, sufficient conditions are obtained for the global stability of any positive solution to system (1.1). The structure of this paper is as follows. In the next section, some useful lemmas are presented. In Section 3, we state and prove the sufficient conditions on the permanence and global stability of any positive solutions for system (1.1). We apply our criteria for some well-known special cases of system (1.1) to illustrate the generality of our results in Section 4.

2. Preliminaries

First we consider the following single-species non-autonomous logistic system

\[
\frac{dy(t)}{dt} = y(t)[a(t) - b(t)y(t)],
\]

where \( a(t) \) and \( b(t) \) are bounded continuous functions defined on \( R \). We note that for any \((t_0, y_0) \in R_+ \times R_+\), system (2.1) has a unique solution \( y(t) \) satisfying \( y(t_0) = y_0 \), where \( R_+ = [0, \infty) \) and \( R_+ = (0, \infty) \). If \( y(t) > 0 \) on the interval of existence, then \( y(t) \) is said to be a positive solution. It is easy to see that \( y(t) \) is positive if and only if the initial value \( y(t_0) > 0 \).

Let \( y^*(t) \) be a positive solution of system (2.1) defined on \( R_+ \). We say that \( y^*(t) \) is globally uniformly attractive on \( R_+ \), if for any constants \( \eta > 1 \) and \( \varepsilon > 0 \), there is a constant \( T(\eta, \varepsilon) > 0 \) such that for any initial time \( t_0 \in R_+ \) and any solution \( y(t) \) of system (2.2) with \( y(t_0) \in [\eta^{-1}, \eta] \), one has \( |y(t) - y^*(t)| < \varepsilon \) for all \( t \geq t_0 + T(\eta, \varepsilon) \). By Lemma 1 given in [9], we have the following result.

**Lemma 2.1.** Suppose that \( \inf_{t \geq 0} b(t) \geq 0 \) and there are positive constants \( \omega_1 \) and \( \omega_2 \) such that \( \lim_{t \to +\infty} \int_t^{t+\omega_1} a(s) \, ds > 0 \) and \( \lim_{t \to +\infty} \int_t^{t+\omega_2} b(s) \, ds > 0 \). Then

(a) there is a constant \( M > 1 \) such that

\[
M^{-1} \leq \liminf_{t \to +\infty} y(t) \leq \limsup_{t \to +\infty} y(t) \leq M
\]

for any positive solution \( y(t) \) of system (2.1).

(b) each fixed positive solution \( y^*(t) \) of system (2.1) is globally uniformly attractive on \( R_+ \).

Now, we consider the following first order linear differential equation with a parameter

\[
\frac{dv(t)}{dt} = g(t, \beta) - c(t)v(t),
\]

where \( g(t, \beta) \) is a continuous bounded function defined on \( (t, \beta) \in R_+ \times [0, \beta_0] \), \( \beta_0 \) a constant, and \( c(t) \) a continuous function defined on \( R \). For system (2.2) we introduce the following assumptions.

(A1) Function \( g(t, \beta) \) satisfies the Lipschitz condition with \( \beta \in [0, \beta_0] \), i.e., there is a constant \( L = L(\beta_0) > 0 \) such that

\[
|g(t, \beta_1) - g(t, \beta_2)| \leq L|\beta_1 - \beta_2| \quad \text{for all } t \in R \text{ and } \beta_1, \beta_2 \in [0, \beta_0].
\]

(A2) Function \( c(t) \) is non-negative bounded on \( R_+ \) and there is a constant \( \omega_3 > 0 \) such that \( \lim_{t \to +\infty} \int_t^{t+\omega_3} c(s) \, ds > 0 \).

In system (2.2), when parameter \( \beta = 0 \) we obtain the following system

\[
\frac{dv(t)}{dt} = g(t, 0) - c(t)v(t).
\]

(2.3)

For any \( \theta > 0 \), let \( v(t) \) be a solution of system (2.3) with initial value \( v(t_0) = 0, \theta \). It is easy to prove that for all \( t \geq t_0 \), \( v(t) \geq 0 \) if the initial value \( v(t_0) \geq 0 \), and \( v(t) > 0 \) if the initial value \( v(t_0) > 0 \). By Lemma 3 given in [9], we have the following results.
Lemma 2.2. Suppose that conditions (A_1) and (A_2) hold. Then

(a) there is a constant $M > 0$ such that $\limsup_{t \to \infty} v(t) \leq M$ for any positive solution $v(t)$ of system (2.3).

(b) if there is a constant $\omega_5 > 0$ such that $\liminf_{t \to \infty} g(s, 0) ds > 0$, then there is a constant $\eta > 1$ such that $\eta^{-1} \leq \liminf_{t \to \infty} v(t) \leq \limsup_{t \to \infty} v(t) \leq \eta$ for any positive solution $v(t)$ of system (2.3).

(c) if the condition of the (b) holds, then each fixed positive solution $v^*(t)$ of system (2.3) is globally uniformly attractive on $R_{+0}$.

Let $v_0 \in R_+, t_0 \in R_{+0}$ and $\beta \in [0, \beta_0]$, and further let $v_\beta(t)$ and $v_0(t)$ be the solutions of systems (2.2) and (2.3) with initial value $v_\beta(t_0) = v_0$ and $v_0(t_0) = v_0$, respectively. So we can get the following result.

Lemma 2.3. Suppose that assumptions (A_1) and (A_2) hold, then $v_\beta(t)$ converges to $v_0(t)$ uniformly for $t \in [t_0, \infty)$ as $\beta \to 0$.

Proof. Let $V(t) = |v_\beta(t) - v_0(t)|$. By assumptions (A_1) and (A_2) we obtain the Dini derivative

\[
D^+ V(t) = \text{sgn}(v_\beta(t) - v_0(t)) \{ -c(t)[v_\beta(t) - v_0(t)] + g(t, \beta) - g(t, 0) \}
\]

\[
\leq -c(t)v_\beta(t) - v_0(t) + |g(t, \beta) - g(t, 0)|
\]

\[
\leq -c(t)V(t) + \beta L
\]

(2.4)

for all $t \geq t_0$ and $\beta \in [0, \beta_0]$. From (A_2) and $v_\beta(t_0) = v_0(t_0) = v_0$, by the famous comparison theorem and the variation of constants formula of solutions for first-order linear differential equation, we can get from (2.4) that $V(t) \to 0$ uniformly for $t \in [t_0, \infty)$ as $\beta \to 0$, that is, $v_\beta(t)$ converges to $v_0(t)$ uniformly for $t \in [t_0, \infty)$ as $\beta \to 0$. This completes the proof. $\square$

Remark 2.1. In system (2.3), if $g(t, 0) \equiv 0$, then system (2.3) has a trivial equilibria $E = 0$, and $E$ is globally asymptotically stable. For any $I > 0$ and $t_0 \in R_{+0}$, let $\beta \in [0, \beta_0]$ and $v_\beta(t)$ be the positive solution of system (2.2) with initial value $v_\beta(t_0) = v_0$, $[0, I]$. By Lemmas 2.2 and 2.3, we further have the following result: the solution $v_\beta(t)$ converges to 0, as $\beta \to 0$ and $t \to \infty$, i.e., for any $\epsilon > 0$, there are positive constants $T = T(\epsilon, I)$ and $\delta = \delta(\epsilon)$ such that $v_\beta(t) < \epsilon$ for all $t \geq t_0 + T$ and $\beta < \delta$.

For the convenience of the statement in the following of this paper, we introduce the definition on permanence.

Definition 2.1. System (1.1) is said to be permanent, if there are positive constants $m$ and $M$ such that

\[
m \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq M \quad (i = 1, 2, \ldots, n)
\]

for any positive solution $(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_n(t))$ of system (1.1).

Remark 2.2. In system (1.1), $u_i(t)$ $(i = 1, 2, \ldots, n)$ are control variables, so we do not consider the permanence of control variables.

3. Main results

For system (1.1), we introduce the following assumptions. Let $t_0 \in R$ be a fixed initial time.

(H_1) Functions $a_i(t), b_i(t), c_i(t), d_i(t), e_i(t), r_i(t), \delta_i(t), si_i(t), t_j(t)$ and $a_{ij}(t), c_{ij}(t), d_{ij}(t), e_{ij}(t), r_{ij}(t), \delta_{ij}(t), si_{ij}(t), t_{ij}(t)$ and $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are non-negative for all $t \geq t_0$.

(H_2) For each $1 \leq i \leq n$, there are positive constants $\omega_i$ and $\alpha_i$, such that

\[
\liminf_{t \to \infty} \int_t^{t+\omega_i} a_i(s) ds > 0, \quad \liminf_{t \to \infty} \int_t^{t+\alpha_i} e_i(s) ds > 0.
\]

Let $\tau = \sup\{t_j(t), \delta_i(t), \sigma_i(t) : t \geq t_0, i, j = 1, 2, \ldots, n\}$. We define $C^n[-\tau, 0]$ the Banach space of bounded continuous functions $\phi : [-\tau, 0] \to R^n$ with the supremum norm defined by

\[
\|\phi\|_c = \sup_{-\tau \leq s \leq 0} |\phi(s)|.
\]

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $|\phi(s)| = \sum_{i=1}^n |\phi_i(s)|$. By the fundamental theory of functional differential equations [13, 16, 18], we know that for any $(\phi, \psi) \in C^n[-\tau, 0] \times C^n[-\tau, 0]$, system (1.1) has a unique solution

\[
X(t, \phi, \psi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), u_1(t, \psi), u_2(t, \psi), \ldots, u_n(t, \psi))
\]

satisfying the initial condition $X_{0\phi}(\cdot, \phi, \psi) = (\phi, \psi)$.

 Defined $C^n[-\tau, 0] = \{\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in C^n[-\tau, 0] : \phi_i(s) \geq 0 \text{ and } \phi_i(0) > 0 \text{ for all } s \in [-\tau, 0] \text{ and } i = 1, 2, \ldots, n\}$. Motivated by the biological background of system (1.1), in this paper we are only concerned with positive solutions of system (1.1). It is not difficult to see that the solution $X(t, \phi, \psi)$ of system (1.1) is positive, if the initial function $(\phi, \psi) \in C^n[-\tau, 0] \times C^n[-\tau, 0]$.

First, we have the theorem on the ultimate boundedness of all positive solutions of system (1.1).
Theorem 3.1. Suppose that assumptions (H1) and (H2) hold, then system (1.1) is ultimately bounded, in the sense that there are positive constants M and T such that if \( t > T \), then \( x_i(t) \leq M \) and \( u_i(t) \leq M \) for all \( i = 1, 2, \ldots, n \) for all positive solutions \( X(t) = (x(t), u(t)) \) of system (1.1).

Proof. Let \( X(t) = (x(t), u(t)) \) be any positive solution of system (1.1). We first prove that the components \( x_i(t) \) for all \( i = 1, 2, \ldots, n \) of system (1.1) is ultimately bounded. From assumption (H1) and the \( i \)-th equation of system (1.1) we have

\[
\frac{dx_i(t)}{dt} \leq x_i(t)[b_i(t) - a_i(t)x_i(t)].
\]

It has been shown (see, for example [30,31]) that under assumptions (H1) and (H2) any positive solution \( y(t) \) of the following nonautonomous system

\[
\frac{dy(t)}{dt} = y(t)[b_i(t) - a_i(t)y(t)]
\]

is ultimately bounded on \( R_{t>0} \). Hence, by the comparison theorem, we can obtain a constant \( M_1 \) such that for any positive solution \( (x(t), u(t)) \) of system (1.1), there is a \( T_1 > 0 \) such that \( x_i(t) < M_1 \) for all \( t \geq T_1 \). Let \( M_1 = \max_{1 \leq i \leq n} \{ M_{11} \} \) and \( T_1 = \max_{1 \leq i \leq n} \{ T_{11} \} \), we have

\[
x_i(t) \leq M_1 \quad \text{for all } t \geq T_1, \ i = 1, 2, \ldots, n.
\]

Further, from the above, assumption (H1) and the \( (n+i) \)-th equation of system (1.1) we have

\[
\frac{du_i(t)}{dt} \leq r_i(t) - e_i(t)u_i(t) + c_i(t)M_1
\]

for all \( t \geq T_1 + \tau \). From assumptions (H1) and (H2), we have that \( r_i(t), e_i(t)c_i(t) \) are non-negative and \( \lim \inf_{t \to \infty} \int_{t-\tau}^{t+\tau} e_i(s) \, ds > 0 \). Hence, using the comparison theorem and the conclusion (a) of Lemma 2.2; further, we can get a constant \( M_2 \) such that \( u_i(t) < M_2 \) for all \( t \geq T_2 \). Now, we let \( M = \max\{ M_1, M_{12}, \ldots, M_{n2} \} \) and \( T = \max\{ T_1, T_{12}, \ldots, T_{n2} \} \), then for all \( t \geq T \)

\[
x_i(t) \leq M, \quad u_i(t) \leq M, \quad i = 1, 2, \ldots, n.
\]

Therefore, the solution \( X(t) = (x(t), u(t)) \) is ultimately bounded. This completes the proof. \( \square \)

In order to obtain the permanence of system (1.1), we consider the following two auxiliary systems

\[
\frac{dx_i(t)}{dt} = x_i(t)[b_i(t) - a_i(t)x_i(t)] \tag{3.1}
\]

and

\[
\frac{du_i(t)}{dt} = r_i(t) - e_i(t)u_i(t), \tag{3.2}
\]

where \( i = 1, 2, \ldots, n \). Let \( x_{i0}(t) \) and \( u_{i0}(t) \) be some positive solutions of systems (3.1) and (3.2), respectively.

Remark 3.1. If \( r_i(t) \equiv 0 \), then system (3.2) has a trivial equilibria \( E = 0 \), and \( E \) is globally asymptotically stable. In this case, we let \( u_{i0}(t) = 0 \) \( (i = 1, 2, \ldots, n) \).

Now, we state and prove our result on the permanence of system (1.1).

Theorem 3.2. Suppose that assumptions (H1) and (H2) hold. If there exist positive constants \( \gamma_i \) and \( \lambda_i \) such that for each \( 1 \leq i \leq n \),

\[
\lim_{t \to \infty} \int_{t}^{t+\gamma_i} \left[ b_i(s) - \sum_{j=1, j \neq i}^{n} a_{ij}(s)x_{j0}(s - r_{ij}(s)) - d_i(s)u_{i0}(s - \delta_i(s)) \right] \, ds > 0 \tag{3.3}
\]

and

\[
\lim_{t \to \infty} \int_{t}^{t+\lambda_i} r_i(s) \, ds > 0, \tag{3.4}
\]

then system (1.1) is permanent.
Proof. Let $X(t) = (x(t), u(t))$ be any positive solution of system (1.1). From Theorem 3.1, there is a constant $M > 0$ satisfying for any positive solution $X(t)$ of system (1.1), there is a $T_1 \geq t_0$

$$0 < x_i(t) < M, \quad 0 < u_i(t) < M, \quad \text{for all } t \geq T_1, \ i = 1, 2, \ldots, n.$$ 

Therefore, from assumption $(H_1)$ and the $i$-th equation of system (1.1) we have

$$\frac{dx_i(t)}{dt} \geq x_i(t) \left[ b_i(t) - a_i(t)M - \sum_{j=1}^{n} a_{ij}(t)M - d_i(t)M \right]$$

$$\geq -\alpha_i x_i(t)$$

for all $t \geq T_1 + \tau$, where $\alpha_i = \sup_{t \geq t_0} \{|b_i(t) - a_i(t)M - \sum_{j=1}^{n} a_{ij}(t)M - d_i(t)M|\}$. For any $t \geq T_1 + \tau$, integrating (3.5) from $t - \sigma_i(t)$ to $t$ we obtain

$$x_i(t - \sigma_i(t)) \leq x_i(t) \exp(\alpha_i \sigma_i(t)) \leq x_i(t) \exp(\alpha \tau) \leq x_i(t) \exp(\alpha \tau),$$

where $\alpha = \max_{1 \leq i \leq n} \{\alpha_i\}$ and $i = 1, 2, \ldots, n$.

On the other hand, from assumption $(H_1)$ and the $i$-th equation of system (1.1) we have

$$\frac{dx_i(t)}{dt} \leq x_i(t) [b_i(t) - a_i(t) \chi_i(t)].$$

The comparison equation is system (3.1). From assumptions $(H_1)$ and $(H_2)$ we have $a_i(t) \geq 0$ and $\lim \inf_{t \to +\infty} \int_{t}^{t+\alpha} a_i(t) \, dt > 0$. Further, by (3.3) it follows that $\lim \inf_{t \to +\infty} \int_{t}^{t+\alpha} b_i(t) \, dt > 0$. So, the system (3.1) satisfies all conditions of Lemma 2.1. Hence, by Lemma 2.1, each positive solution of system (3.1) is globally asymptotically stable. Therefore, by the comparison theorem, and since $x_0(t)$ is be globally uniformly attractive positive solution of system (3.1), we obtain for any $\epsilon > 0$ that there is a constant $t_1 = t_1(\epsilon) > 0$ such that

$$x_i(t) \leq x_0(t) + \epsilon \quad \text{for all } t \geq t_1.$$

Let $t_1 = \max_{1 \leq i \leq n} \{t_1\}$, then for all $t \geq t_1$

$$x_i(t) \leq x_0(t) + \epsilon, \quad i = 1, 2, \ldots, n.$$ (3.7)

For any $t_2, t_3$ and $t_3 \geq t_2 \geq 0$, integrating directly system (1.1) we have

$$x_i(t_2) = x_i(t_3) \exp \int_{t_2}^{t_3} \left[ b_i(t) - a_i(t) x_i(t) - \sum_{j=1}^{n} a_{ij}(t) x_j(t - \tau_j(t)) - d_i(t) u_i(t - \delta_i(t)) \right] dt.$$ (3.8)

Claim 3.1. There is a constant $\beta > 0$ such that $\lim \sup_{t \to +\infty} x_i(t) > \beta \ (i = 1, 2, \ldots, n)$ for any positive solution $X(t) = (x(t), u(t))$ of system (1.1).

In fact, by (3.3), we can choose positive constants $T_2 \geq T_1, \epsilon$ and $\delta$ such that

$$\int_{t}^{t+\gamma} \left[ b_i(s) - \alpha_i(s) + a_i(s) \right] \, ds - \sum_{j=1, j \neq i}^{n} a_{ij}(s) \left[ x_{n}(s - \tau_j(s)) + \epsilon \right] - d_i(s) \left[ u_{n}(s - \delta_i(s)) + \epsilon \right] \geq 0$$

for all $t \geq T_2$ and $i = 1, 2, \ldots, n$.

By assumptions $(H_1), (H_2)$ and (3.4), we have $r_i(t)$ and $e_i(t)$ are non-negative and $\lim \inf_{t \to +\infty} \int_{t}^{t+\gamma} r_i(t) \, dt > 0,$ $\lim \inf_{t \to +\infty} \int_{t}^{t+\gamma} e_i(t) \, dt > 0 \ (i = 1, 2, \ldots, n)$. Therefore, system (3.2) satisfies all conditions of the (b) and (c) of Lemma 2.2. So, by Lemma 2.2, each positive solution of system (3.2) is globally asymptotically stable.

Consider the following system with one parameter

$$\frac{du_i(t)}{dt} = r_i(t) - e_i(t)u_i(t) + c_i(t)\beta \exp(\alpha \tau),$$ (3.10)

where $\beta \in [0, \beta_0]$ is the parameter and $i = 1, 2, \ldots, n$. Let $u_\beta(t)$ be a solution of system (3.10). Similar, by assumptions $(H_1), (H_2)$ and (3.4), we see that system (3.10) satisfies all conditions of the (c) of Lemmas 2.2 and 2.3. By the conclusion (c) of Lemma 2.2, $u_\beta(t)$ is globally asymptotically stable. Further, by Lemma 2.3 we obtain that $u_\beta(t)$ uniformly for $t \in \mathbb{R}_+$ converges to $u_{\beta_0}(t)$, as $\beta \to 0$. Hence, there is a constant $\beta > 0$ and $\beta < \epsilon$ such that for all $t \geq T_2$

$$u_i(t) \leq u_{\beta_0}(t) + \frac{\epsilon}{2}, \quad i = 1, 2, \ldots, n.$$ (3.11)

If Claim 3.1 is not true, then there is an integer $k \in \{1, 2, \ldots, n\}$ and a positive solution $(x(t), u(t))$ of system (1.1) such that $\lim \sup_{t \to +\infty} x_i(t) < \beta$. Hence, there is a constant $T_3 > T_2$ such that $x_i(t) < \beta$ for all $t \geq T_3$. From assumption $(H_1), (3.6)$ and the $(n + k)$-th equation of system (1.1) we obtain
\[
\frac{du_k(t)}{dt} \leq r_k(t) - e_k(t)u_k(t) + c_k(t)\beta \exp(\alpha t) \quad \text{for all } t \geq T_3.
\]

Using the comparison theorem and globally asymptotically stability of solution \(u_{k\beta}(t)\), we obtain that there is a \(T_4 \geq T_3\) such that
\[
u_k(t) \leq u_{k\beta}(t) + \varepsilon \quad \text{for all } t \geq T_4.
\] (3.12)

Therefore, from (3.11) and (3.12) it follows that
\[
u_k(t) \leq u_{i0}(t) + \varepsilon \quad \text{for all } t \geq T_4.
\] (3.13)

On the other hand, by (3.7) there is a \(T_5 \geq T_4\) such that
\[
u_i(t) \leq x_{i0}(t) + \varepsilon \quad \text{for all } t \geq T_5,
\] (3.14)

where \(i = 1, 2, \ldots, n\) and \(i \neq k\).

By (3.8), (3.13) and (3.14) and assumption (H1) we obtain
\[
\begin{align*}
x_k(t) &= x_k(T_5 + \tau) \exp \int_{T_5 + \tau}^t \left[ b_i(s) - a_i(s)x_i(s) - \sum_{j=1}^n a_{ij}(s)x_j(s - \tau_j(s)) - d_i(s)u_i(s - \delta_i(s)) \right] \, ds \\
&\geq x_k(T_5 + \tau) \exp \int_{T_5 + \tau}^t \left[ b_i(t) - [a_i(t) + a_{ij}(t)] \right] \varepsilon \\
&- \sum_{j=1,j \neq i}^n a_{ij}(s) \left[ x_{i0}(s - \tau_j(s)) + \varepsilon \right] - d_i(s)u_{i0}(s - \delta_i(s)) + \varepsilon \right] \, ds
\end{align*}
\]

for all \(t \geq T_5 + \tau\). Thus from (3.9) we finally obtain \(\lim_{t \to \infty} x_k(t) = \infty\), which leads to a contradiction. Therefore, Claim 3.1 is true.

**Claim 3.2.** There is a constant \(\gamma > 0\) such that \(\lim_{t \to \infty} x_k(t) > \gamma (i = 1, 2, \ldots, n)\) for any positive solution \(x(t)\) of system (1.1).

In fact, if Claim 3.2 is not true, then there is an integer \(k \in \{1, 2, \ldots, n\}\) and a sequence of the initial value \(\{x_m = (\phi^{(m)}, \psi^{(m)})\} \subset C^\infty_0 \times C^\infty_0\) such that, for the solution \((x(t, X_m), u(t, X_m))\) of system (1.1),
\[
\liminf_{t \to \infty} x_k(t, X_m) < \frac{\beta}{m^2}, \quad m = 1, 2, \ldots,
\]

where constant \(\beta\) is given in Claim 3.1. By Claim 3.1, for every \(m\) there are two time sequences \(\{s_q^{(m)}\}\) and \(\{t_q^{(m)}\}\), satisfying
\[
0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots \text{ and } \lim_{q \to \infty} s_q^{(m)} = \infty,
\]

such that
\[
x_k(s_q^{(m)}, X_m) = \frac{\beta}{m}, \quad x_k(t_q^{(m)}, X_m) = \frac{\beta}{m^2},
\] (3.15)

and
\[
\frac{\beta}{m^2} < x_k(t, X_m) < \frac{\beta}{m} \quad \text{for all } t \in (s_q^{(m)}, t_q^{(m)}).
\] (3.16)

From Theorem 3.1, we obtain that system system (1.1) is ultimate boundedness. Therefore, we can choose a positive constant \(T^{(m)}\) for every \(m\) such that \(x_i(t, X_m) < M\) and \(u_i(t, X_m) < M\) for all \(t > T^{(m)}\) and \(i = 1, 2, \ldots, n\). Further, there is an integer \(K_1^{(m)} > 0\) such that \(s_q^{(m)} > T^{(m)} + \tau\) for all \(q > K_1^{(m)}\). Let \(q > K_1^{(m)}\) for any \(t \in [s_q^{(m)}, t_q^{(m)}]\), by assumption (H1) we have
\[
\frac{dx_k(t, X_m)}{dt} \geq x_k(t, X_m) \left[ b_k(t) - a_k(t)M + \sum_{j=1}^n a_{ij}(t)M - d_k(t)M \right] \\
\geq -\gamma_0 x_k(t, X_m),
\]

where \(\gamma_0 = \sup_{t \geq 0} \{ |b_k(t) - a_k(t)M + \sum_{j=1}^n a_{ij}(t)M - d_k(t)M| \} \).

Integrating the above inequality from \(s_q^{(m)}\) to \(t_q^{(m)}\), we further have
\[
x_k(t_q^{(m)}, X_m) \geq x_k(s_q^{(m)}, X_m) \exp(-\gamma_0(t_q^{(m)} - s_q^{(m)})).
\]

Consequently, by (3.15)
\[
\frac{\beta}{m^2} \geq \frac{\beta}{m} \exp(-\gamma_0(t_q^{(m)} - s_q^{(m)})).
\]
Then,
\[ t_q^{(m)} - s_q^{(m)} \geq \frac{\ln m}{\gamma_0} \quad \text{for all } q > K_1^{(m)}. \]

By (3.9), there are positive constants \( P \) and \( \rho \) such that
\[ \int_t^{t + \epsilon} \left\{ b_i(t) - [a_i(t) + \alpha_i(t)] \right\} \epsilon - \sum_{j=1, j \neq i}^n a_{ij}(s) \left[ x_{ij}(s - \tau_i(s)) + \epsilon \right] - d_i(s) [u_{ij}(s - \delta_i(s)) + \epsilon] \right\} \mathrm{d}s > \rho. \]
for all \( t \geq 0, \kappa \geq P \) and \( i = 1, 2, \ldots, n \).

Let \( \tilde{u}_{k \beta}(t) \) be the solution of system (3.10) with the initial condition \( \tilde{u}_{k \beta}(t) = u_k(s_q^{(m)}, X_m) \). By (3.6) and (3.16) and assumption (H1), we have
\[ \frac{\mathrm{d}u_k(t, X_m)}{\mathrm{d}t} \leq \tilde{u}_{k \beta}(t) \quad \text{for all } t \in [s_q^{(m)}, t_q^{(m)}]. \]

Using the comparison theorem it follows that
\[ u_k(t, X_m) \leq \tilde{u}_{k \beta}(t) \quad \text{for all } t \in [s_q^{(m)}, t_q^{(m)}]. \quad (3.17) \]

By Lemma 2.2, the solution \( u_{k \beta}(t) \) of system (3.10) is globally uniformly attractive on \( R_{+0} \), and so there is a constant \( T_2 \geq P \), which is independent of any \( m \) and \( q \geq K^{(m)} \), such that
\[ \tilde{u}_{k \beta}(t) \leq u_{k \beta}(t) + \frac{\epsilon}{2} \quad \text{for all } t \geq s_q^{(m)} + T_2. \quad (3.18) \]

On the other hand, by (3.6) there is a \( T_3 \geq T_2 \) which is independent of any \( m \) and \( q \geq K^{(m)} \), such that
\[ x_i(t, X_m) \leq x_{i0}(t) + \epsilon \quad \text{for all } t \geq T_3, \quad (3.19) \]
where \( i = 1, 2, \ldots, n \) and \( i \neq k \). Choose an integer \( N_0 > 0 \) such that for \( m \geq N_0 \) and \( q \geq K^{(m)} \),
\[ t_q^{(m)} - s_q^{(m)} > T_3 + P. \]

Further, from (3.11), (3.17) and (3.18) we obtain
\[ u_k(t, X_m) \leq u_{i0}(t) + \epsilon \quad \text{for all } t \in [s_q^{(m)} + T_3, t_q^{(m)}]. \quad (3.20) \]

So, when \( m \geq N_0 \) and \( q \geq K^{(m)} \), by (3.8), (3.19) and (3.20) and assumption (H1) it follows
\[ \frac{\beta}{m^2} = x_k(s_q^{(m)} + T_3, X_m) \exp \int_{s_q^{(m)} + T_3}^{t_q^{(m)}} \left[ b_k(t) - a_k(t)x_k(t, X_m) \right. \]
\[ - \sum_{j=1}^n a_{ij}(t)x_j(t - \tau_j(t), X_m) - d_k(t)u_k(t - \delta_k(t), X_m) \right] \mathrm{d}t \]
\[ \geq x_k(s_q^{(m)} + T_3, X_m) \exp \int_{s_q^{(m)} + T_3}^{t_q^{(m)}} \left\{ b_k(t) - [a_k(t) + \alpha_k(t)] \epsilon \right. \]
\[ - \sum_{j=1, j \neq k}^n a_{ij}(s) \left[ x_{ij}(s - \tau_i(s)) + \epsilon \right] - d_k(s) [u_{ij}(s - \delta_i(s)) + \epsilon] \right\} \mathrm{d}s \]
\[ > \frac{\beta}{m^2} \]
which leads to a contradiction. Therefore, Claim 3.2 is true.

From Claims 3.1 and 3.2 we complete the proof of this theorem. \( \square \)

**Remark 3.2.** From the proof of Theorem 3.2, we note that \( u_{i0}(t) \) be some fixed positive solution of system (3.2), which is independent of the feedback controls. So, the feedback controls have no influence on the permanence of system (1.1).

In system (1.1), when functions \( r_i(t) \equiv 0 \) \( i = 1, 2, \ldots, n \), we have the following system
\begin{align*}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= x_i(t) \left[ b_i(t) - a_i(t)x_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t - \tau_j(t)) - d_i(t)u_i(t - \delta_i(t)) \right] \quad (3.21) \\
\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} &= -e_i(t)u_i(t) + c_i(t)x_i(t - \sigma_i(t)), \quad i = 1, 2, \ldots, n.
\end{align*}
Theorems 3.1

Suppose that assumptions (1.1) hold. Then there exist positive constants $m_i, M_i$ and $T_i$ such that for any solution $x(t)$ of system (1.1) we have $m_i \leq x_i(t) \leq M_i$ for all $t \geq T_i$ and $i = 1, 2, \ldots, n$. Further, from this and the (n + i)-th equation of system (1.1) we have

$$r_i(t) - e_i(t)u_i(t) + c_i(t)m_1 \leq \frac{du_i(t)}{dt} \leq r_i(t) - e_i(t)u_i(t) + c_i(t)M_1$$

for all $t \geq T_1 + \tau$. It is easy to test that the conditions of the (b) of Lemma 2.2 are satisfied. So, using the comparison theorem and the conclusion (b) of Lemma 2.2, we further can obtain that there are positive constants $m_2, M_2$ and $T_2 \geq T_1 + \tau$ such that for any positive solution $x(t)$ of system (1.1) we have $m_2 \leq u_i(t) \leq M_2$ for all $t \geq T_2$ and $i = 1, 2, \ldots, n$. Now, we let $m = \min\{m_1, m_2\}$ and $M = \max\{M_1, M_2\}$, then for all $t \geq T_2$

$$m \leq x_i(t) \leq M, \quad m \leq u_i(t) \leq M \quad (i = 1, 2, \ldots, n).$$

Finally, by Theorem 1 in [18], it follows that system has at least a positive $\omega$-periodic solution and this completes the proof. \qed

We proceed to the discussion on the global stability of any positive solution to system (1.1). We first introduce an assumption.

(H3) Functions $\tau_{ij}(t), \delta_i(t)$ and $\sigma_j(t)$ are differentiable on $t \in R_+$ and the derivatives $\dot{\tau}_{ij}(t) < 1, \dot{\delta}_i(t) < 1$ and $\dot{\sigma}_j(t) < 1$ for all $t \in R_+$.

On the global stability of any positive solution to system (1.1), we get the following result.

Theorem 3.5. Let $(x^*(t), u^*(t))$ be some fixed positive solution of system (1.1). Suppose assumptions (H1)–(H3) and (3.3) hold. If there are positive constants $\mu_i$ and $\kappa_j$ such that

$$\liminf_{t \to \infty} A_i(t) > 0, \quad \liminf_{t \to \infty} B_i(t) > 0$$

where

$$A_i(t) = \mu_i a_i(t) - \kappa_j \frac{c_i(\beta_i^{-1}(t))}{1 - \delta_i(\beta_i^{-1}(t))} - \sum_{j=1}^{n} \mu_j a_{ij}(t) \frac{a_{ij}(\theta_j^{-1}(t))}{1 - \tau_j(\theta_j^{-1}(t))}$$

$$B_i(t) = \kappa_i e_i(t) - \mu_i \frac{d_i(\gamma_i^{-1}(t))}{1 - \delta_i(\gamma_i^{-1}(t))}$$

(3.23)

in which $i, j = 1, 2, \ldots, n, \theta_j^{-1}(t), \beta_i^{-1}(t)$ and $\gamma_i^{-1}(t)$ denote the inverse function of $t - \tau_j(t), t - \delta_i(t)$ and $t - \sigma_i(t)$, respectively, then $(x^*(t), u^*(t))$ is globally asymptotically stable.

Proof. Let $(x(t), u(t))$ be any positive solution of system (1.1). It follows from Theorem 3.1 that there exist positive constants $T$ and $M$ such that for all $t \geq T$

$$0 < x_i^*(t), x_i(t) \leq M, \quad 0 < u_i^*(t), u_i(t) \leq M, \quad i = 1, 2, \ldots, n.$$

We define a Lyapunov functional $V_i(t)$ as

$$V_i(t) = \mu_i |\ln x_i(t) - \ln x_i^*(t)| + \kappa_i |u_i(t) - u_i^*(t)|.$$  

(3.24)
Calculating the upper right derivative of $V_{1i}(t)$ along solutions of system (1.1), it follows that

$$
D^+ V_{1i}(t) = \mu_i \text{sgn}(x_i(t) - x_i^*(t)) \left\{ -a_i(t) \left[ x_i(t) - x_i^*(t) \right] - \sum_{j=1}^{n} a_{ij}(t) \left[ x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t)) \right] \\
- d_i(t) \left[ u_i(t - \delta_i(t)) - u_i^*(t - \delta_i(t)) \right] \right\} + \kappa_i \text{sgn}(u_i(t))
$$

$$
- u_i^*(t) \left\{ -e_i(t) \left[ u_i(t) - u_i^*(t) \right] - c_i(t) \left[ x_i(t - \sigma_i(t)) - x_i^*(t - \sigma_i(t)) \right] \right\}
$$

$$
\leq -\mu_i a_i(t) |x_i(t) - x_i^*(t)| - \kappa_i e_i(t) |u_i(t) - u_i^*(t)| + \sum_{j=1}^{n} \mu_j a_{ij}(t) |x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t))| \\
+ \mu_i d_i(t) |u_i(t - \delta_i(t)) - u_i^*(t - \delta_i(t))| + \kappa_i c_i(t) |x_i(t - \sigma_i(t)) - x_i^*(t - \sigma_i(t))|. \quad (3.25)
$$

Define

$$
V_i(t) = V_{1i}(t) + V_{2i}(t) + V_{3i}(t),
$$

where

$$
V_{2i} = \mu_i \int_{t-\tau_{ij}(t)}^{t} \sum_{j=1}^{n} \frac{a_{ij}(t)}{1 - \tilde{t}_{ij}(t)} |x_j(s) - x_j^*(s)| \, ds + \kappa_i \int_{t-\tau_{ij}(t)}^{t} \frac{c_i(t)}{1 - \tilde{c}_{ij}(t)} |x_i(s) - x_i^*(s)| \, ds \quad (3.27)
$$

and

$$
V_{3i} = \mu_i \int_{t-\delta_i(t)}^{t} \frac{d_i(t)}{1 - \delta_i(t)} |u_i(s) - u_i^*(s)| \, ds. \quad (3.28)
$$

From (3.24)–(3.28), we have

$$
D^+ V_i(t) = D^+ V_{1i}(t) + \dot{V}_{2i}(t) + \dot{V}_{3i}(t)
$$

$$
\leq -\mu_i a_i(t) |x_i(t) - x_i^*(t)| - \kappa_i e_i(t) |u_i(t) - u_i^*(t)| + \sum_{j=1}^{n} \mu_j a_{ij}(t) \frac{a_{ij}(t)}{1 - \tilde{t}_{ij}(t)} |x_j(t) - x_j^*(t)| \left\{ \\
\right.
$$

$$
+ \kappa_i c_i(t) |x_i(t - \sigma_i(t)) - x_i^*(t - \sigma_i(t))| + \mu_i d_i(t) |u_i(t - \delta_i(t)) - u_i^*(t - \delta_i(t))| \right. \\
\left. = - \left[ \mu_i a_i(t) - \kappa_i c_i(t) \frac{(\beta_i)^{-1}(t)}{1 - \tilde{c}_{ij}(t)} \right] |x_i(t) - x_i^*(t)| - \left[ \kappa_i e_i(t) - \mu_i \frac{d_i(t)}{1 - \delta_i(t)} \right] |u_i(t) - u_i^*(t)| \\
+ \sum_{j=1}^{n} \mu_j a_{ij}(t) \frac{a_{ij}(t)}{1 - \tilde{t}_{ij}(t)} |x_j(t) - x_j^*(t)| \right. \quad (3.29)
$$

We now define a Lyapunov function $V(t)$ as

$$
V(t) = \sum_{i=1}^{n} V_i(t). \quad (3.30)
$$

So, from (3.26)–(3.30), we get that

$$
D^+ V(t) = \sum_{i=1}^{n} D^+ V_i(t)
$$

$$
\leq - \sum_{i=1}^{n} \left\{ \mu_i a_i(t) - \kappa_i \frac{c_i(t)}{1 - \tilde{c}_{ij}(t)} - \sum_{j=1}^{n} \mu_j a_{ij}(t) \frac{a_{ij}(t)}{1 - \tilde{t}_{ij}(t)} \right\} |x_i(t) - x_i^*(t)| \\
+ \left[ \kappa_i e_i(t) - \mu_i \frac{d_i(t)}{1 - \delta_i(t)} \right] |u_i(t) - u_i^*(t)| \right\}. \quad (3.31)
$$
From (3.23), we get that

$$D^+ V(t) \leq - \sum_{i=1}^{n} \left[ A_i(t) \left| x_i(t) - x_i^*(t) \right| + B_i(t) \left| u_i(t) - u_i^*(t) \right| \right],$$

where $A_i(t)$ and $B_i(t)$ ($i = 1, 2, \ldots, n$) are defined in (3.23).

For each $1 \leq i \leq n$, by (3.22), there exist positive constants $\alpha_i$, $\eta_i$, and $T^* \geq T$ such that

$$A_i(t) \geq \alpha_i > 0, \quad B_i(t) \geq \eta_i > 0, \quad \text{for all } t \geq T^*.$$

Integrating both sides of (3.31) on interval $[T^*, t]$, yields that

$$V(T^*) \geq V(t) + \sum_{i=1}^{n} \int_{T^*}^{t} \left[ A_i(s) \left| x_i(s) - x_i^*(s) \right| + B_i(t) \left| u_i(t) - u_i^*(t) \right| \right] ds.$$

From (3.32) and (3.33), we have that

$$V(T^*) \geq V(t) + \sum_{i=1}^{n} \int_{T^*}^{t} \left[ \alpha_i \left| x_i(s) - x_i^*(s) \right| + \eta_i \left| u_i(t) - u_i^*(t) \right| \right] ds.$$

Therefore, $V(t)$ is bounded on $[T^*, \infty]$ and

$$\int_{T^*}^{\infty} \left| x_i(s) - x_i^*(s) \right| ds < \infty, \quad \int_{T^*}^{\infty} \left| u_i(t) - u_i^*(t) \right| ds < \infty, \quad i = 1, 2, \ldots, n.$$

By Theorem 3.1, $|x_i(s) - x_i^*(s)|$ and $|u_i(t) - u_i^*(t)|$ ($i = 1, 2$) are bounded on $[T^*, \infty]$.

On the other hand, it is easy to see that $x_i(t)$, $\hat{x}_i(t)$, $\hat{u}_i(t)$ ($i = 1, 2, \ldots, n$) are bounded for $t \geq T^*$. Therefore, $|\hat{x}_i(t) - x_i^*(t)|$ and $|\hat{u}_i(t) - u_i^*(t)|$ ($i = 1, 2, \ldots, n$) are uniformly continuous on $[T^*, \infty]$. By Barbalat’s Lemma [13, Lemmas 1.2.2 and 1.2.3], we conclude that

$$\lim_{t \to \infty} \left| \hat{x}_i(t) - x_i^*(t) \right| = 0, \quad \lim_{t \to \infty} \left| \hat{u}_i(t) - u_i^*(t) \right| = 0, \quad i = 1, 2, \ldots, n.$$

This completes the proof. \( \square \)

4. Applications

In this section, we will apply the results given in the last section to particular competition systems with delays and feedback controls or without delays and feedback controls, which have been studied extensively in the literature. The following two applications show that the derived sufficient conditions are easily verifiable and more general than those given in the literature, thus we improve and generalize some well-known results.

Example 4.1. We consider the Lotka–Volterra competition system

$$\frac{dx_i(t)}{dt} = x_i(t) \left[ b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) \right], \quad i = 1, 2, \ldots, n,$$

where the functions $b_i(t)$ and $a_{ij}(t)$ ($i, j = 1, 2, \ldots, n$) are defined on $R_{+0}$ and are bounded continuous on $R_{+0}$, and $a_{ij}(t) \geq 0$ for all $t \in R_{+0}$ and $i, j = 1, 2, \ldots, n$.

On the permanence of system (4.1), applying Theorems 3.1 and 3.2 we have

Theorem 4.1. Suppose that there exist positive constants $\omega_i$ and $\lambda_i$ such that for each $i = 1, 2, \ldots, n$,

$$\liminf_{t \to \infty} \int_{t}^{t+\omega_i} a_i(s) ds > 0$$

and

$$\liminf_{t \to \infty} \int_{t}^{t+\lambda_i} \left[ b_i(s) - \sum_{j=1, j \neq i}^{n} a_{ij}(s)x_j(s) \right] ds > 0$$

where $x_j(t)$ is some positive solution of system $dx_j(t)/dt = x_j(t)[b_j(t) - a_{ij}(t)x_i(t)]$, then system (4.1) is permanent.

Remark 4.1. When system (1.1) degenerates into system (4.1), the Theorem 4.1 is exactly the result of [23, Theorem 2], so our result is a generalization of the result of [23].
Example 4.2. Consider the permanence and global stability of any positive solutions to system (1.2).

In system (1.2), we assume that

(B1) Functions $b_i(t)$, $b_{ij}(t)$, $d_i(t)$, $r_i(t)$, $e_i(t)$ and $c_i(t)$ are defined on $R_+ = [0, \infty)$ and bounded continuous functions on $R_+$. $b_i(t)$, $d_i(t)$, $r_i(t)$ and $e_i(t)$ are non-negative for all $t \in R_+$ and $i, j = 1, 2, \ldots, n$.

(B2) For each $1 \leq i \leq n$, there are positive constants $\alpha_i$ and $\theta_i$ such that

$$\liminf_{t \to \infty} \int_t^{t+\alpha_i} r_i(s) \, ds > 0, \quad \liminf_{t \to \infty} \int_t^{t+\theta_i} e_i(t) > 0.$$ 

Theorem 4.2. Suppose assumptions (B1) and (B2) hold. If there are positive constants $\mu_i$ and $\nu_i$ such that for each $i = 1, 2, \ldots, n$,

$$\liminf_{t \to \infty} \int_t^{t+\nu_i} \left[ b_i(s) - \sum_{j=1, j\neq i}^n b_{ij}(s) x_{i0}(s) - d_i(s) u_{i0}(s) \right] \, ds > 0$$

and

$$\liminf_{t \to \infty} \int_t^{t+\mu_i} b_{ii}(s) \, ds > 0,$$

where $x_{i0}(t)$, $u_{i0}(t)$ denote some positive solution of systems $dx_i(t)/dt = x_i(t) [b_i(t) - b_{ii}(t) x_i(t)]$ and $du_i(t)/dt = r_i(t) - e_i(t) u_i(t)$, respectively, then

(a) system (1.2) is permanent.
(b) if there exist positive constants $\alpha_i$ and $\beta_i$ ($i = 1, 2, \ldots, n$) such that

$$\liminf_{t \to \infty} A_i(t) > 0 \quad \text{and} \quad \liminf_{t \to \infty} B_i(t) > 0,$$

where

$$A_i(t) = \alpha_i b_{ii}(t) - \beta_i c_i(t) - \sum_{j=1, j\neq i}^n \alpha_j b_{ij}(t),$$

$$B_i(t) = \beta_i e_i(t) - \alpha_i d_i(t),$$

then each positive solution of system (1.2) is globally asymptotically stable.

Remark 4.2. Similar results have been obtained by Chen in [17, Theorem 2.1]. However, it is obvious to see that the method in [17] is totally different from ours adopted in this paper, and our results can be more easily checked.

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References