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Delay-independent criteria for exponential stability of generalized Cohen–Grossberg neural networks with discrete delays ☆

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Abstract

The global exponential stability is investigated for a class of generalized Cohen–Grossberg neural networks with discrete delays. By means of the combination of the nonlinear measure approach and constructing a novel Lyapunov functional together with some nonlinear functional analysis and inequality techniques, general sufficient conditions are obtained for the existence, uniqueness and global exponential stability of equilibrium of the delayed neural networks, which are mild and independent of the delays. The new criteria do not require the boundedness, monotonicity and differentiability assumptions of the normal and the delayed activation functions. Our results generalize and improve many existing ones.

Keywords: Cohen–Grossberg neural networks; Global exponential stability; Discrete delays; Nonlinear measure; Lyapunov functional

1. Introduction

Cohen–Grossberg neural networks (CGNN, in brief), as an important recurrent neural networks model [3,6], have aroused a tremendous surge of investigation in these years [2,8–10,16–18]. Due to the finite switching speed of neurons and amplifiers, time delays inevitably exist in biological and artificial neural networks. And, a time delay in the response of a neuron can influence the stability of a network and deteriorate the dynamical performance creating oscillatory and unstable characteristics [17]. In this Letter, we consider a new CGNN model with discrete delays (CGNND, in brief), which is described by the following functional differential equations:

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i\big(u_i(t)\big)\left[b_i\big(u_i(t)\big) - \sum_{j=1}^{n} w_{ij} f_j\big(v_j u_j(t)\big) - \sum_{j=1}^{n} w_{ij}^{\tau} g_j\big(v_j^{\tau} u_j(t-\tau_{ij})\big) + I_i\right], \quad i = 1, 2, \ldots, n, \tag{1}$$

where $n \geqslant 2$ is the number of neurons in the networks, $u_i(t)$ denotes the neuron state vector, a_i denotes an amplification function, b_i denotes a self-signal function, $W = (w_{ij})_{n\times n}$ and $W^{\tau} = (w_{ij}^{\tau})_{n\times n}$, respectively, denote the normal and the delayed connection weight matrix, f_j and g_j, respectively, denote the normal and the delayed activation function, v_j and v_j^{τ}, respectively, denote the normal and the delayed amplifier gain, $\tau_{ij} \geqslant 0$ is the delay caused during the switching and transmission processes, and I_i represents the constant external input. The initial conditions associated with (1) are of the following form:

$$u_i(s) = \phi_i(s) \in C\big([t_0 - \tau, t_0], \mathbb{R}\big), \quad s \in [t_0 - \tau, t_0], \quad i = 1, 2, \ldots, n, \tag{2}$$

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where $\tau = \max_{1 \leqslant i, j \leqslant n} \tau_{ij} < +\infty$, and $C([t_0 - \tau, t_0], \mathbb{R})$ denotes the space of all real-valued continuous functions defined on $[t_0 - \tau, t_0]$. Let $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^{\mathrm{T}}$.

In particular, when $g_j = f_j$ and $v_j^\tau = v_j = 1$ $(j = 1, 2, \ldots, n)$, CGNND (1) reduces to CGNN with discrete delays of the following form [8,9]

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i\big(u_i(t)\big)\left[b_i\big(u_i(t)\big) - \sum_{j=1}^{n} w_{ij} f_j\big(u_j(t)\big) - \sum_{j=1}^{n} w_{ij}^\tau f_j\big(u_j(t - \tau_{ij})\big) + I_i\right], \quad i = 1, 2, \ldots, n, \tag{3}$$

when in addition all $w_{ij} = 0$, (1) further reduces to the pure delayed CGNN [13]

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i\big(u_i(t)\big)\left[b_i\big(u_i(t)\big) - \sum_{j=1}^{n} w_{ij}^\tau f_j\big(u_j(t - \tau_{ij})\big) + I_i\right], \quad i = 1, 2, \ldots, n. \tag{4}$$

Moreover, CGNND (1) includes many other famous models as special cases, for example, Hopfield neural networks with discrete delays [4,5,15], cellular neural networks with discrete delays [7,19] and BAM neural networks with discrete delays [1].

It is well known that the stability of neural networks is of crucial importance for the designing and successful applications of networks [1,2,4,5,7–19]. This Letter aims at deriving general sufficient conditions for the existence, uniqueness and global exponential stability (GES, in brief) of equilibrium of CGNND (1).

2. Assumptions and preliminaries

In recent investigations, Liao et al. [8] and Wang et al. [13], respectively, analyzed the GES of (3) and (4), but they additionally impose the boundedness assumption on the activation functions; Lu [9] analyzed the GES of (3), but they additionally required that each f_j not only be bounded, but also be monotonically nondecreasing. However, the boundedness or monotonicity assumption on the activation functions will make the results inapplicable to some important engineering problems [4,12]. Unlike these literature, we make neither the boundedness assumption nor the monotonicity or differentiability assumption of the activation functions f_j and g_j. In this Letter, we make the following assumptions:

(H_1) Each a_i is continuous and $0 < \grave{\alpha}_i \leqslant a_i(s) \leqslant \acute{\alpha}_i$, $\forall s \in \mathbb{R}$.
(H_2) Each b_i is continuous, and there exists a constant $\lambda_i > 0$ such that for any $s_1, s_2 \in \mathbb{R}$, $(s_1 - s_2)[b_i(s_1) - b_i(s_2)] \geqslant \lambda_i (s_1 - s_2)^2$, $i = 1, 2, \ldots, n$.
(H_3) Each f_j and g_j are Lipschitz continuous. For convenience, we respectively denote

$$m_j = \sup_{s_1, s_2 \in \mathbb{R}, s_1 \neq s_2} \frac{|f_j(s_1) - f_j(s_2)|}{|s_1 - s_2|} \quad \text{and} \quad m_j^\tau = \sup_{s_1, s_2 \in \mathbb{R}, s_1 \neq s_2} \frac{|g_j(s_1) - g_j(s_2)|}{|s_1 - s_2|},$$

the minimal Lipschitz constant of f_j and g_j, $j = 1, 2, \ldots, n$.

Let $\mathbb{R}^n$ denote the n-dimensional real vector space equipped with Euclidean norm $\|\cdot\|_2$ (i.e., $\|x\|_2^2 = \sum_{i=1}^n x_i^2$ for each vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$). Denote by $\langle \cdot, \cdot \rangle$ the inner product of any two vectors in $\mathbb{R}^n$. In [16], we introduce the following nonlinear measure concept in the sense of Euclidean norm.

Definition 1 *([16]).* Suppose that Ω is an open subset of $\mathbb{R}^n$ and F is an operator from Ω into $\mathbb{R}^n$. The constant

$$m_\Omega(F) = \sup_{x, y \in \Omega, x \neq y} \frac{\langle F(x) - F(y), x - y \rangle}{\|x - y\|_2^2}$$

is called the nonlinear measure of F on Ω.

Lemma 1 *([16]). If $m_\Omega(F) < 0$, then $F : \Omega \to \mathbb{R}^n$ is a one-to-one mapping. In addition, if $\Omega = \mathbb{R}^n$, then F is a homeomorphism of $\mathbb{R}^n$.*

3. Existence and uniqueness of equilibrium of CGNND (1)

The existence and uniqueness of equilibrium is prerequisite for investigating the global stability of neural networks. Now we will apply the nonlinear measure approach together with some nonlinear functional analysis techniques to analyze the existence and uniqueness of equilibrium of CGNND (1).

Theorem 1. *Suppose that* (H_1)–(H_3) *hold and there exist ten sets of real numbers* $c_i > 0$, $e_i > 0$, $r_i > 0$, $\eta_i > 0$, $\tilde{r}_i > 0$, $\tilde{\eta}_i > 0$, k_i, d_i, $\tilde{k}_i$, $\tilde{d}_i$ *such that*

$$m_i v_i \sum_{j=1}^{n} \frac{e_j r_j}{e_i \eta_i} |w_{ji}|^{k_j/d_i} + \sum_{j=1}^{n} m_j v_j \frac{e_i c_j \eta_j}{e_j c_i r_i} |w_{ij}|^{2-k_i/d_j} + m_i^{\tau} v_i^{\tau} \sum_{j=1}^{n} \frac{e_j \tilde{r}_j}{e_i \tilde{\eta}_i} \left|w_{ji}^{\tau}\right|^{\tilde{k}_j/\tilde{d}_i} + \sum_{j=1}^{n} m_j^{\tau} v_j^{\tau} \frac{e_i c_j \tilde{\eta}_j}{e_j c_i \tilde{r}_i} \left|w_{ij}^{\tau}\right|^{2-\tilde{k}_i/\tilde{d}_j} < 2\lambda_i, \quad i = 1, 2, \ldots, n. \tag{5}$$

Then for each set of external input I_i, (1) *has a unique equilibrium point* u^*.

Proof. Define an operator $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_i(u) = -\left[b_i(u_i) - \sum_{j=1}^{n} w_{ij} f_j(v_j u_j) - \sum_{j=1}^{n} w_{ij}^{\tau} g_j\left(v_j^{\tau} u_j\right) + I_i \right],$$

where $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$, $F(u) = (F_1(u), F_2(u), \ldots, F_n(u))^{\mathrm{T}}$. Note that $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^{\mathrm{T}}$ is an equilibrium of (1) if and only if $F(u^*) = 0$.

Let $P = \operatorname{diag}(c_1, c_2, \ldots, c_n)$ and $Q = \operatorname{diag}(e_1, e_2, \ldots, e_n)$. For all $y, z \in \mathbb{R}^n$,

$$\begin{aligned}
&\left\langle QFQ^{-1}Py - QFQ^{-1}Pz, y - z \right\rangle \\
&= \sum_{i=1}^{n} \Bigg\{ -e_i \Bigg\{ b_i\left(e_i^{-1} c_i y_i\right) - b_i\left(e_i^{-1} c_i z_i\right) - \sum_{j=1}^{n} w_{ij} \left[f_j\left(v_j e_j^{-1} c_j y_j\right) - f_j\left(v_j e_j^{-1} c_j z_j\right) \right] \\
&\quad - \sum_{j=1}^{n} w_{ij}^{\tau} \left[g_j\left(v_j^{\tau} e_j^{-1} c_j y_j\right) - g_j\left(v_j^{\tau} e_j^{-1} c_j z_j\right) \right] \Bigg\} (y_i - z_i) \Bigg\} \\
&\leqslant \sum_{i=1}^{n} \Bigg\{ -e_i \left[b_i\left(e_i^{-1} c_i y_i\right) - b_i\left(e_i^{-1} c_i z_i\right) \right] (y_i - z_i) + e_i \sum_{j=1}^{n} \left(|w_{ij}| \left| f_j\left(v_j e_j^{-1} c_j y_j\right) - f_j\left(v_j e_j^{-1} c_j z_j\right) \right| |y_i - z_i| \right) \\
&\quad + e_i \sum_{j=1}^{n} \left(\left|w_{ij}^{\tau}\right| \left| g_j\left(v_j^{\tau} e_j^{-1} c_j y_j\right) - g_j\left(v_j^{\tau} e_j^{-1} c_j z_j\right) \right| |y_i - z_i| \right) \Bigg\} \\
&\leqslant -\sum_{i=1}^{n} \lambda_i c_i (y_i - z_i)^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ e_i m_j v_j e_j^{-1} c_j |w_{ij}| |y_j - z_j| |y_i - z_i| \right\} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ e_i m_j^{\tau} v_j^{\tau} e_j^{-1} c_j \left|w_{ij}^{\tau}\right| |y_j - z_j| |y_i - z_i| \right\} \\
&\leqslant -\sum_{i=1}^{n} \lambda_i c_i (y_i - z_i)^2 \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ e_i m_j v_j e_j^{-1} c_j \frac{1}{2} \left[\frac{r_i}{\eta_j} |w_{ij}|^{k_i/d_j} (y_j - z_j)^2 + \frac{\eta_j}{r_i} |w_{ij}|^{2-k_i/d_j} (y_i - z_i)^2 \right] \right\} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ e_i m_j^{\tau} v_j^{\tau} e_j^{-1} c_j \frac{1}{2} \left[\frac{\tilde{r}_i}{\tilde{\eta}_j} \left|w_{ij}^{\tau}\right|^{\tilde{k}_i/\tilde{d}_j} (y_j - z_j)^2 + \frac{\tilde{\eta}_j}{\tilde{r}_i} \left|w_{ij}^{\tau}\right|^{2-\tilde{k}_i/\tilde{d}_j} (y_i - z_i)^2 \right] \right\} \\
&= -\sum_{i=1}^{n} \frac{c_i}{2} \Bigg\{ 2\lambda_i - m_i v_i \sum_{j=1}^{n} \frac{e_j r_j}{e_i \eta_i} |w_{ji}|^{k_j/d_i} - \sum_{j=1}^{n} \frac{m_j v_j e_i c_j \eta_j}{e_j c_i r_i} |w_{ij}|^{2-k_i/d_j} \\
&\quad - m_i^{\tau} v_i^{\tau} \sum_{j=1}^{n} \frac{e_j \tilde{r}_j}{e_i \tilde{\eta}_i} \left|w_{ji}^{\tau}\right|^{\tilde{k}_j/\tilde{d}_i} - \sum_{j=1}^{n} \frac{m_j^{\tau} v_j^{\tau} e_i c_j \tilde{\eta}_j}{e_j c_i \tilde{r}_i} \left|w_{ij}^{\tau}\right|^{2-\tilde{k}_i/\tilde{d}_j} \Bigg\} (y_i - z_i)^2.
\end{aligned}$$

In view of (5), we immediately derive $m_{\mathbb{R}^n}(QFQ^{-1}P) < 0$. By Lemma 1, we conclude that $QFQ^{-1}P$ is a homeomorphism of $\mathbb{R}^n$, which implies that there exists a unique u^* in $\mathbb{R}^n$ such that $QFQ^{-1}P(u^*) = 0$. Since Q and P are invertible, we derive that $F(u) = 0$ has a unique solution u^*, i.e., (1) has a unique equilibrium point u^*. □

Remark 1. Wang et al. [13], Liao et al. [8] and Lu [9] obtained the existence of an equilibrium of (3) and (4) by Brouwer's fixed point theorem, but they additionally required that each f_j be globally bounded. Moreover, they did not provide the uniqueness of the equilibrium.

4. Global exponential stability analysis of CGNND (1)

In this section, we will investigate the global exponential stability of the delayed neural networks (1) by constructing a novel Lyapunov functional together with skillful transformation and flexible inequality techniques.

Theorem 2. *Suppose that* (H_1)–(H_3) *and* (5) *hold. Then for each set of external input* I_i, *CGNND* (1) *has a unique equilibrium* u^*, u^* *is globally exponentially stable, and there exists a constant* $\sigma > 0$ *such that the exponential decay estimate of any solution satisfies*

$$\left\|u(t) - u^*\right\|_2 \leqslant \sqrt{c}e^{-\sigma(t-t_0)} \sup_{s\in[t_0-\tau,t_0]} \left\|\phi(s) - u^*\right\|_2, \quad t \geqslant t_0, \tag{6}$$

where

$$c = \max_{1\leqslant i\leqslant n}\left\{\frac{e_i^2}{\grave{\alpha}_i c_i} + \frac{e_i}{2\sigma}\sum_{j=1}^n m_j^\tau v_j^\tau \frac{e_j \tilde{r}_i}{c_j \tilde{\eta}_j}\left|w_{ij}^\tau\right|^{\tilde{k}_i/\tilde{d}_j}\left(1 - e^{-2\sigma\tau_{ij}}\right)\right\} \Big/ \min_{1\leqslant i\leqslant n}\left\{\frac{e_i^2}{\acute{\alpha}_i c_i}\right\}.$$

Proof. By Theorem 1, it follows directly from conditions (H_1)–(H_3) and (5) that CGNND (1) has a unique equilibrium $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$.

Let

$$x_i(t) = \frac{e_i}{\sqrt{c_i}}\left(u_i(t) - u_i^*\right), \quad i = 1, 2, \ldots, n, \quad \text{and} \quad x(t) = \left(x_1(t), x_2(t), \ldots, x_n(t)\right)^T.$$

Substitution of $u_i(t) = \frac{\sqrt{c_i}}{e_i}x_i(t) + u_i^*$ into (1) leads to

$$\begin{aligned}\frac{dx_i(t)}{dt} = -\frac{e_i}{\sqrt{c_i}}a_i\left(\tfrac{\sqrt{c_i}}{e_i}x_i(t) + u_i^*\right)\Bigg\{&b_i\left(\tfrac{\sqrt{c_i}}{e_i}x_i(t) + u_i^*\right) - b_i\left(u_i^*\right) - \sum_{j=1}^n w_{ij}\left[f_j\left(v_j\left(\tfrac{\sqrt{c_j}}{e_j}x_j(t) + u_j^*\right)\right) - f_j\left(v_j u_j^*\right)\right] \\ &- \sum_{j=1}^n w_{ij}^\tau\left[g_j\left(v_j^\tau\left(\tfrac{\sqrt{c_j}}{e_j}x_j(t - \tau_{ij}) + u_j^*\right)\right) - g_j\left(v_j^\tau u_j^*\right)\right]\Bigg\}, \quad i = 1, 2, \ldots, n.\end{aligned} \tag{7}$$

Let

$$\begin{aligned}&p_i\left(x_i(t)\right) = a_i\left(\tfrac{\sqrt{c_i}}{e_i}x_i(t) + u_i^*\right), \qquad q_i\left(x_i(t)\right) = b_i\left(\tfrac{\sqrt{c_i}}{e_i}x_i(t) + u_i^*\right) - b_i\left(u_i^*\right), \\ &s_j\left(x_j(t)\right) = f_j\left(v_j\left(\tfrac{\sqrt{c_j}}{e_j}x_j(t) + u_j^*\right)\right) - f_j\left(v_j u_j^*\right), \qquad s_j^\tau\left(x_j(t - \tau_{ij})\right) = g_j\left(v_j^\tau\left(\tfrac{\sqrt{c_j}}{e_j}x_j(t - \tau_{ij}) + u_j^*\right)\right) - g_j\left(v_j^\tau u_j^*\right).\end{aligned}$$

Then system (7) reduces to

$$\frac{dx_i(t)}{dt} = -\frac{e_i}{\sqrt{c_i}}p_i\left(x_i(t)\right)\left\{q_i\left(x_i(t)\right) - \sum_{j=1}^n w_{ij}s_j\left(x_j(t)\right) - \sum_{j=1}^n w_{ij}^\tau s_j^\tau\left(x_j(t - \tau_{ij})\right)\right\}, \quad i = 1, 2, \ldots, n. \tag{8}$$

It is clear that 0 is an equilibrium of system (8).

In view of (5), we deduce that for any fixed $\tau_{ij} \geqslant 0$, we can find a constant $\sigma > 0$ such that for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}&2\lambda_i - m_i v_i\sum_{j=1}^n \frac{e_j r_j}{e_i \eta_i}|w_{ji}|^{k_j/d_i} - \sum_{j=1}^n \frac{m_j v_j e_i c_j \eta_j}{e_j c_i r_i}|w_{ij}|^{2-k_i/d_j} - m_i^\tau v_i^\tau\sum_{j=1}^n \frac{e_j \tilde{r}_j}{e_i \tilde{\eta}_i}\left|w_{ji}^\tau\right|^{\tilde{k}_j/\tilde{d}_i} \\ &- \frac{2\sigma}{\grave{\alpha}_i} - \sum_{j=1}^n \frac{m_j^\tau v_j^\tau e_i c_j \tilde{\eta}_j}{e_j c_i \tilde{r}_i}\left|w_{ij}^\tau\right|^{2-\tilde{k}_i/\tilde{d}_j} e^{2\sigma\tau_{ij}} > 0.\end{aligned} \tag{9}$$

We construct the following Lyapunov functional as a candidate

$$V\left(x(t)\right) = \sum_{i=1}^n\left\{2e^{2\sigma t}\int_0^{x_i(t)}\frac{s}{p_i(s)}\,ds + \sum_{j=1}^n m_j^\tau v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j}\left|w_{ij}^\tau\right|^{\tilde{k}_i/\tilde{d}_j}\int_{t-\tau_{ij}}^t e^{2\sigma s}x_j^2(s)\,ds\right\}. \tag{10}$$

Estimating the differential of V along the trajectory, we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\\
&= -\sum_{i=1}^{n} 2e^{2\sigma t} x_i(t)\frac{e_i}{\sqrt{c_i}}\left\{q_i\big(x_i(t)\big) - \sum_{j=1}^{n} w_{ij}s_j\big(x_j(t)\big) - \sum_{j=1}^{n} w_{ij}^{\tau}s_j^{\tau}\big(x_j(t-\tau_{ij})\big)\right\}\\
&\quad + \sum_{i=1}^{n} 4\sigma e^{2\sigma t}\int_0^{x_i(t)} \frac{s}{p_i(s)}\,\mathrm{d}s + \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j} e^{2\sigma t}x_j^2(t)\\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j} e^{2\sigma (t-\tau_{ij})}x_j^2(t-\tau_{ij})\\
&\leqslant e^{2\sigma t}\left\{\sum_{i=1}^{n} 2\left[-\lambda_i x_i^2(t) + \frac{e_i}{\sqrt{c_i}}\sum_{j=1}^{n}\big|x_i(t)\big|\big|w_{ij}\big|\big|s_j\big(x_j(t)\big)\big| + \frac{e_i}{\sqrt{c_i}}\sum_{j=1}^{n}\big|x_i(t)\big|\big|w_{ij}^{\tau}\big|\big|s_j^{\tau}\big(x_j(t-\tau_{ij})\big)\big|\right]\right.\\
&\quad \left. + \sum_{i=1}^{n}\frac{2\sigma}{\grave{\alpha}_i}x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}x_j^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}e^{-2\sigma\tau_{ij}}x_j^2(t-\tau_{ij})\right\}\\
&\leqslant e^{2\sigma t}\left\{-2\sum_{i=1}^{n}\lambda_i x_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_i}{\sqrt{c_i}}m_jv_j\frac{\sqrt{c_j}}{e_j}|w_{ij}|\big|x_i(t)\big|\big|x_j(t)\big| + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_i}{\sqrt{c_i}}m_j^{\tau}v_j^{\tau}\frac{\sqrt{c_j}}{e_j}\big|w_{ij}^{\tau}\big|\big|x_i(t)\big|\big|x_j(t-\tau_{ij})\big|\right.\\
&\quad \left. + \sum_{i=1}^{n}\frac{2\sigma}{\grave{\alpha}_i}x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i}{e_j}\frac{\tilde{r}_i}{\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}x_j^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i}{e_j}\frac{\tilde{r}_i}{\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}e^{-2\sigma\tau_{ij}}x_j^2(t-\tau_{ij})\right\}\\
&\leqslant e^{2\sigma t}\left\{-2\sum_{i=1}^{n}\lambda_i x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_i}{e_j}m_jv_j\left[\frac{c_j}{c_i}\frac{\eta_j}{r_i}|w_{ij}|^{2-k_i/d_j}x_i^2(t) + \frac{r_i}{\eta_j}|w_{ij}|^{k_i/d_j}x_j^2(t)\right]\right.\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_i}{e_j}m_j^{\tau}v_j^{\tau}\left[\frac{c_j}{c_i}\frac{\tilde{\eta}_j}{\tilde{r}_i}e^{2\sigma\tau_{ij}}\big|w_{ij}^{\tau}\big|^{2-\tilde{k}_i/\tilde{d}_j}x_i^2(t) + \frac{\tilde{r}_i}{\tilde{\eta}_j}e^{-2\sigma\tau_{ij}}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}x_j^2(t-\tau_{ij})\right]\\
&\quad \left. + \sum_{i=1}^{n}\frac{2\sigma}{\grave{\alpha}_i}x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i}{e_j}\frac{\tilde{r}_i}{\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}x_j^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i}{e_j}\frac{\tilde{r}_i}{\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}e^{-2\sigma\tau_{ij}}x_j^2(t-\tau_{ij})\right\}\\
&= -e^{2\sigma t}\sum_{i=1}^{n}\left\{2\lambda_i - \sum_{j=1}^{n}\frac{m_jv_je_ic_j\eta_j}{e_jc_ir_i}|w_{ij}|^{2-k_i/d_j} - m_iv_i\sum_{j=1}^{n}\frac{e_jr_j}{e_i\eta_i}|w_{ji}|^{k_j/d_i}\right.\\
&\quad \left. - \frac{2\sigma}{\grave{\alpha}_i} - \sum_{j=1}^{n}\frac{m_j^{\tau}v_j^{\tau}e_ic_j\tilde{\eta}_j}{e_jc_i\tilde{r}_i}\big|w_{ij}^{\tau}\big|^{2-\tilde{k}_i/\tilde{d}_j}e^{2\sigma\tau_{ij}} - m_i^{\tau}v_i^{\tau}\sum_{j=1}^{n}\frac{e_j\tilde{r}_j}{e_i\tilde{\eta}_i}\big|w_{ji}^{\tau}\big|^{\tilde{k}_j/\tilde{d}_i}\right\}x_i^2(t)\\
&\leqslant -e^{2\sigma t}\min_{1\leqslant i\leqslant n}\left\{2\lambda_i - \sum_{j=1}^{n}\frac{m_jv_je_ic_j\eta_j}{e_jc_ir_i}|w_{ij}|^{2-k_i/d_j} - m_iv_i\sum_{j=1}^{n}\frac{e_jr_j}{e_i\eta_i}|w_{ji}|^{k_j/d_i}\right.\\
&\quad \left. - \frac{2\sigma}{\grave{\alpha}_i} - \sum_{j=1}^{n}\frac{m_j^{\tau}v_j^{\tau}e_ic_j\tilde{\eta}_j}{e_jc_i\tilde{r}_i}\big|w_{ij}^{\tau}\big|^{2-\tilde{k}_i/\tilde{d}_j}e^{2\sigma\tau_{ij}} - m_i^{\tau}v_i^{\tau}\sum_{j=1}^{n}\frac{e_j\tilde{r}_j}{e_i\tilde{\eta}_i}\big|w_{ji}^{\tau}\big|^{\tilde{k}_j/\tilde{d}_i}\right\}\big\|x(t)\big\|_2^2 < 0.
\end{aligned}$$

Thus, $V(x(t)) \leqslant V(x(t_0))$ for any $t \geqslant t_0$.

By the form of (10) and $x_i(t) = \frac{e_i}{\sqrt{c_i}}(u_i(t) - u_i^*)$, we have

$$\begin{aligned}
&V\big(x(t_0)\big)\\
&= \sum_{i=1}^{n}\left\{2e^{2\sigma t_0}\int_0^{x_i(t_0)}\frac{s}{p_i(s)}\,\mathrm{d}s + \sum_{j=1}^{n} m_j^{\tau}v_j^{\tau}\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}\big|w_{ij}^{\tau}\big|^{\tilde{k}_i/\tilde{d}_j}\int_{t_0-\tau_{ij}}^{t_0} e^{2\sigma s}x_j^2(s)\,\mathrm{d}s\right\}
\end{aligned}$$

$$\leqslant \sum_{i=1}^{n}\left\{\frac{1}{\acute{\alpha}_i}e^{2\sigma t_0}\frac{e_i^2}{c_i}\big(u_i(t_0)-u_i^*\big)^2+\sum_{j=1}^{n}m_j^\tau v_j^\tau\frac{e_i\tilde{r}_i}{e_j\tilde{\eta}_j}|w_{ij}^\tau|^{\tilde{k}_i/\tilde{d}_j}\sup_{s\in[t_0-\tau,t_0]}\frac{e_j^2}{c_j}\big(u_j(s)-u_j^*\big)^2\int_{t_0-\tau_{ij}}^{t_0}e^{2\sigma s}\,\mathrm{d}s\right\}$$

$$\leqslant \max_{1\leqslant i\leqslant n}\left\{\frac{1}{\acute{\alpha}_i}\frac{e_i^2}{c_i}e^{2\sigma t_0}+\sum_{j=1}^{n}m_j^\tau v_j^\tau\frac{e_ie_j\tilde{r}_i}{c_j\tilde{\eta}_j}\big|w_{ij}^\tau\big|^{\tilde{k}_i/\tilde{d}_j}\frac{1}{2\sigma}\big(e^{2\sigma t_0}-e^{2\sigma(t_0-\tau_{ij})}\big)\right\}\sup_{s\in[t_0-\tau,t_0]}\big\|\phi(s)-u^*\big\|_2^2$$

$$=e^{2\sigma t_0}\max_{1\leqslant i\leqslant n}\left\{\frac{e_i^2}{\acute{\alpha}_i c_i}+\frac{e_i}{2\sigma}\sum_{j=1}^{n}m_j^\tau v_j^\tau\frac{e_j\tilde{r}_i}{c_j\tilde{\eta}_j}\big|w_{ij}^\tau\big|^{\tilde{k}_i/\tilde{d}_j}\big(1-e^{-2\sigma\tau_{ij}}\big)\right\}\sup_{s\in[t_0-\tau,t_0]}\big\|\phi(s)-u^*\big\|_2^2,$$

and

$$\min_{1\leqslant i\leqslant n}\left\{\frac{1}{\acute{\alpha}_i}\left(\frac{e_i}{\sqrt{c_i}}\right)^2\right\}e^{2\sigma t}\sum_{i=1}^{n}\big(u_i(t)-u_i^*\big)^2\leqslant V\big(x(t)\big). \tag{11}$$

Therefore, we derive

$$\big\|u(t)-u^*\big\|_2^2\leqslant ce^{-2\sigma(t-t_0)}\sup_{s\in[t_0-\tau,t_0]}\big\|\phi(s)-u^*\big\|_2^2,\quad t\geqslant t_0,$$

which directly yields the exponential decay estimation (6). □

Remark 2. Under the Lipschitz assumption on the activation functions, Theorem 2 presents general mild sufficient conditions for the GES of CGNND (1), which are independent of the delays. Note that the convergence rate σ, which is determined by (9), depends on the delays. Also, the global asymptotic stability (GAS, in brief) of CGNND (1) is naturally implied by Theorem 2.

Remark 3. It is worth emphasizing the significance of the incorporation of the ten sets of adjustable parameters into condition (5), which endows our criteria with extensive flexibility and applicability. Through specific choice of the parameters, a series of new criteria can be obtained for the GAS and GES of those models included by (1), and they generalize and improve many existing results such as those in [1,4,5,7,8,10,13,15,18,19].

Denote $(\mathrm{H}_3)'$: Each f_j is Lipschitz continuous. Let m_j represent the minimal Lipschitz constant of f_j.

Corollary 1. *Suppose that* (H_1), (H_2), $(\mathrm{H}_3)'$ *hold, and there exist ten sets of real numbers* $e_i>0$, $l_i>0$, $r_i>0$, $\eta_i>0$, $\tilde{r}_i>0$, $\tilde{\eta}_i>0$, k_i, d_i, $\tilde{k}_i$, $\tilde{d}_i$ *such that*

$$\sum_{j=1}^{n}\frac{e_j}{e_i}\left(\frac{r_j}{\eta_i}|w_{ji}|^{k_j/d_i}+\frac{\tilde{r}_j}{\tilde{\eta}_i}\big|w_{ji}^\tau\big|^{\tilde{k}_j/\tilde{d}_i}\right)+\sum_{j=1}^{n}\frac{l_j}{l_i}\left(\frac{\eta_j}{r_i}|w_{ij}|^{2-k_i/d_j}+\frac{\tilde{\eta}_j}{\tilde{r}_i}\big|w_{ij}^\tau\big|^{2-\tilde{k}_i/\tilde{d}_j}\right)<2\frac{\lambda_i}{m_i},\quad i=1,2,\ldots,n. \tag{12}$$

Then for each set of external input I_i, (3) *has a unique equilibrium* u^*, u^* *is globally exponentially stable, and there exists a constant* $\sigma>0$ *such that the exponential decay estimate of any solution satisfies*

$$\big\|u(t)-u^*\big\|_2\leqslant\sqrt{c}e^{-\sigma(t-t_0)}\sup_{s\in[t_0-\tau,t_0]}\big\|\phi(s)-u^*\big\|_2,\quad t\geqslant t_0, \tag{13}$$

where

$$c=\max_{1\leqslant i\leqslant n}\left\{\frac{m_ie_i}{\acute{\alpha}_il_i}+\frac{e_i\tilde{r}_i}{2\sigma}\sum_{j=1}^{n}\frac{m_j^2}{l_j\tilde{\eta}_j}\big|w_{ij}^\tau\big|^{\tilde{k}_i/\tilde{d}_j}\big(1-e^{-2\sigma\tau_{ij}}\big)\right\}\Big/\min_{1\leqslant i\leqslant n}\left\{\frac{m_ie_i}{\acute{\alpha}_il_i}\right\}.$$

Proof. Condition (12) implies condition (5) holds for $m_i^\tau=m_i$, $v_i^\tau=v_i=1$ and $c_i=\frac{e_il_i}{m_i}$, $i=1,2,\ldots,n$, and thus this corollary follows directly from Theorem 1. □

Remark 4. Liao et al. [8, Theorem 2 and Corollary 1] are special cases of Corollary 1 when $l_i=m_i$, $k_i=\tilde{k}_i=d_i=\tilde{d}_i=r_i=\tilde{r}_i=1$, $\eta_i=m_i^{2\rho_1-1}$, $\tilde{\eta}_i=m_i^{2\rho_2-1}$ $(i=1,2,\ldots,n)$ and $\rho_1,\rho_2\in[0,1]$, and meanwhile, they additionally required that each activation function f_j be bounded.

Corollary 2. *Suppose that* (H_1), (H_2), $(\mathrm{H}_3)'$ *hold, and there exist two sets of positive numbers* e_i, l_i *and two real numbers* p,q *such that*

$$\sum_{j=1}^{n}\left\{\frac{e_j}{e_i}\big(|w_{ji}|^p+\big|w_{ji}^\tau\big|^q\big)+\frac{l_j}{l_i}\big(|w_{ij}|^{2-p}+\big|w_{ij}^\tau\big|^{2-q}\big)\right\}<2\frac{\lambda_i}{m_i},\quad i=1,2,\ldots,n. \tag{14}$$

Then for each set of external input I_i, (3) has a unique equilibrium u^*, u^* is globally exponentially stable, and there exists a constant $\sigma > 0$ such that the exponential decay estimate of any solution satisfies (13), where

$$c = \max_{1 \leqslant i \leqslant n} \left\{ \frac{m_i e_i}{\grave{\alpha}_i l_i} + \frac{e_i}{2\sigma} \sum_{j=1}^{n} \frac{m_j^2}{l_j} \left| w_{ij}^{\tau} \right|^q \left(1 - e^{-2\sigma \tau_{ij}}\right) \right\} \Big/ \min_{1 \leqslant i \leqslant n} \left\{ \frac{m_i e_i}{\acute{\alpha}_i l_i} \right\}.$$

Corollary 3. *Suppose that* (H_1), (H_2), $(\mathrm{H}_3)'$ *hold, and there exist two sets of positive numbers* e_i, l_i *and* $q \in \mathbb{R}$ *such that*

$$m_i \sum_{j=1}^{n} \left(\frac{e_j}{e_i} \left| w_{ji}^{\tau} \right|^q + \frac{l_j}{l_i} \left| w_{ij}^{\tau} \right|^{2-q} \right) < 2\lambda_i, \quad i = 1, 2, \ldots, n. \tag{15}$$

Then for each set of external input I_i, (4) has a unique equilibrium u^*, u^* is globally exponentially stable, and there exists a constant $\sigma > 0$ such that the exponential decay estimate of any solution satisfies (13), where

$$c = \max_{1 \leqslant i \leqslant n} \left\{ \frac{m_i e_i}{\grave{\alpha}_i l_i} + \frac{e_i}{2\sigma} \sum_{j=1}^{n} \frac{m_j^2}{l_j} \left| w_{ij}^{\tau} \right|^q \left(1 - e^{-2\sigma \tau_{ij}}\right) \right\} \Big/ \min_{1 \leqslant i \leqslant n} \left\{ \frac{m_i e_i}{\acute{\alpha}_i l_i} \right\}.$$

5. Conclusions

This Letter is concerned with the global exponential stability of a new generalized Cohen–Grossberg neural networks model with discrete delays. Only assuming the activation functions to satisfy Lipschitz condition, we apply the nonlinear measure approach and construct a novel Lyapunov functional, thus derive general sufficient conditions for the GES of the delayed neural networks, which greatly extend and improve the existing results. It is believed that our new criteria will be significant for the designs and applications of delayed neural networks.

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References

[1] A.P. Chen, J.D. Cao, L.H. Huang, Neurocomputing 57 (2004) 435.
[2] T.P. Chen, L.B. Rong, Phys. Lett. A 317 (2003) 436.
[3] M.A. Cohen, S. Grossberg, IEEE Trans. Systems Man Cybernet. SMC-13 (1983) 815.
[4] P. Van Den Driessche, X. Zou, SIAM J. Appl. Math. 58 (1998) 1878.
[5] K. Gopalsamy, X. He, Physica D 76 (1994) 344.
[6] S. Grossberg, Neural Networks 1 (1988) 17.
[7] M.P. Joy, J. Math. Anal. Appl. 232 (1999) 61.
[8] X.F. Liao, C.G. Li, K.W. Wong, Neural Networks 17 (2004) 1401.
[9] H. Lu, IEEE Trans. Circuits Systems II 52 (9) (2005) 476.
[10] W.L. Lu, T.P. Chen, Neural Comput. 15 (2003) 1173.
[11] J.G. Peng, H. Qiao, Z.-B. Xu, Neural Networks 15 (2002) 95.
[12] H. Qiao, J.G. Peng, Z.-B. Xu, IEEE Trans. Neural Networks 12 (2) (2001) 360.
[13] L. Wang, X.F. Zou, Neural Networks 15 (2002) 415.
[14] L. Wang, X.F. Zou, Physica D 170 (2) (2002) 162.
[15] L.S. Wang, D.Y. Xu, J. Vibration Control 8 (2002) 13.
[16] A.H. Wan, M.S. Wang, J.G. Peng, H. Qiao, Phys. Lett. A 350 (2006) 96.
[17] J. Wu, Introduction to Neural Dynamics and Signal Transmission Delay, Walter de Gruyter, Berlin, 2001.
[18] H. Ye, A.N. Michel, K. Wang, Phys. Rev. E 51 (1995) 2611.
[19] J.Y. Zhang, Comput. Math. Appl. 45 (2003) 1707.