Delay-independent criteria for exponential stability of generalized Cohen–Grossberg neural networks with discrete delays

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Abstract

The global exponential stability is investigated for a class of generalized Cohen–Grossberg neural networks with discrete delays. By means of the combination of the nonlinear measure approach and constructing a novel Lyapunov functional together with some nonlinear functional analysis and inequality techniques, general sufficient conditions are obtained for the existence, uniqueness and global exponential stability of equilibrium of the delayed neural networks, which are mild and independent of the delays. The new criteria do not require the boundedness, monotonicity and differentiability assumptions of the normal and the delayed activation functions. Our results generalize and improve many existing ones.

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1. Introduction

Cohen–Grossberg neural networks (CGNN, in brief), as an important recurrent neural networks model [3,6], have aroused a tremendous surge of investigation in these years [2,8–10,16–18]. Due to the finite switching speed of neurons and amplifiers, time delays inevitably exist in biological and artificial neural networks. And, a time delay in the response of a neuron can influence the stability of a network and deteriorate the dynamical performance creating oscillatory and unstable characteristics [17]. In this Letter, we consider a new CGNN model with discrete delays (CGNND, in brief), which is described by the following functional differential equations:

\[
\frac{d u_i(t)}{d t} = -a_i(u_i(t)) \left[ b_1(u_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(v_j u_j(t)) - \sum_{j=1}^{n} w_{ij} g_j(v_j^\tau u_j(t - \tau_{ij})) + I_i \right], \quad i = 1, 2, \ldots, n, \tag{1}
\]

where \( n \geq 2 \) is the number of neurons in the networks, \( u_i(t) \) denotes the neuron state vector, \( a_i \) denotes an amplification function, \( b_1 \) denotes a self-signal function, \( W = (w_{ij})_{n \times n} \) and \( W^\tau = (w_{ij}^\tau)_{n \times n} \), respectively, denote the normal and the delayed connection weight matrix, \( f_j \) and \( g_j \), respectively, denote the normal and the delayed activation function, \( v_j \) and \( v_j^\tau \), respectively, denote the normal and the delayed amplifier gain, \( \tau_{ij} \geq 0 \) is the delay caused during the switching and transmission processes, and \( I_i \) represents the constant external input. The initial conditions associated with (1) are of the following form:

\[
u_i(s) = \phi_i(s) \in C\left([t_0 - \tau, t_0], \mathbb{R}\right), \quad s \in [t_0 - \tau, t_0], \quad i = 1, 2, \ldots, n, \tag{2}\]

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where \( \tau = \max_{1 \leq i,j \leq n} \tau_{ij} < +\infty \), and \( C([t_0 - \tau, t_0], \mathbb{R}) \) denotes the space of all real-valued continuous functions defined on \([t_0 - \tau, t_0] \). Let \( \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \).

In particular, when \( g_j = f_j \) and \( v_j = v_j = 1 \) (\( j = 1, 2, \ldots, n \)), CGNND (1) reduces to CGNN with discrete delays of the following form [8,9]

\[
\frac{d u_i(t)}{d t} = -a_i(u_i(t)) \left[ b_i(u_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) - \sum_{j=1}^{n} w_{ij} f_j(u_j(t - \tau_{ij})) + I_i \right], \quad i = 1, 2, \ldots, n,
\]

when in addition all \( w_{ij} = 0, (1) \) further reduces to the pure delayed CGNN [13]

\[
\frac{d u_i(t)}{d t} = -a_i(u_i(t)) \left[ b_i(u_i(t)) - \sum_{j=1}^{n} w_{ij}^T f_j(u_j(t)) + I_i \right], \quad i = 1, 2, \ldots, n.
\]

Moreover, CGNND (1) includes many other famous models as special cases, for example, Hopfield neural networks with discrete delays [4,5,15], cellular neural networks with discrete delays [7,19] and BAM neural networks with discrete delays [1].

It is well known that the stability of neural networks is of crucial importance for the designing and successful applications of networks [1,2,4,5,7–19]. This Letter aims at deriving general sufficient conditions for the existence, uniqueness and global exponential stability (GES, in brief) of equilibrium of CGNND (1).

2. Assumptions and preliminaries

In recent investigations, Liao et al. [8] and Wang et al. [13], respectively, analyzed the GES of (3) and (4), but they additionally impose the boundedness assumption on the activation functions; Lu [9] analyzed the GES of (3), but they additionally required that each \( f_j \) not only be bounded, but also be monotonically nondecreasing. However, the boundedness or monotonicity assumption on the activation functions will make the results inapplicable to some important engineering problems [4,12]. Unlike these literature, we make neither the boundedness assumption nor the monotonicity or differentiability assumption of the activation functions \( f_j \) and \( g_j \). In this Letter, we make the following assumptions:

(H1) Each \( a_i \) is continuous and \( 0 < \hat{a}_i \leq a_i(s) \leq \hat{a}_i, \forall s \in \mathbb{R} \).

(H2) Each \( b_i \) is continuous, and there exists a constant \( \lambda_i > 0 \) such that for any \( s_1, s_2 \in \mathbb{R}, (s_1 - s_2) [b_i(s_1) - b_i(s_2)] \geq \lambda_i (s_1 - s_2)^2, i = 1, 2, \ldots, n \).

(H3) Each \( f_j \) and \( g_j \) are Lipschitz continuous. For convenience, we respectively denote

\[
m_j = \sup_{s_1, s_2 \in \mathbb{R}, s_1 \neq s_2} \frac{|f_j(s_1) - f_j(s_2)|}{|s_1 - s_2|} \quad \text{and} \quad m_j^* = \sup_{s_1, s_2 \in \mathbb{R}, s_1 \neq s_2} \frac{|g_j(s_1) - g_j(s_2)|}{|s_1 - s_2|},
\]

the minimal Lipschitz constant of \( f_j \) and \( g_j, j = 1, 2, \ldots, n \).

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional real vector space equipped with Euclidean norm \( \| \cdot \|_2 \) (i.e., \( \|x\|_2^2 = \sum_{i=1}^{n} x_i^2 \) for each vector \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \)). Denote by \( \langle \cdot, \cdot \rangle \) the inner product of any two vectors in \( \mathbb{R}^n \). In [16], we introduce the following nonlinear measure concept in the sense of Euclidean norm.

**Definition 1** ([16]). Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \) and \( F \) is an operator from \( \Omega \) into \( \mathbb{R}^n \). The constant

\[
m_\Omega(F) = \sup_{x, y \in \Omega, x \neq y} \frac{\langle F(x) - F(y), x - y \rangle}{\|x - y\|_2^2}
\]

is called the nonlinear measure of \( F \) on \( \Omega \).

**Lemma 1** ([16]). If \( m_\Omega(F) < 0 \), then \( F : \Omega \rightarrow \mathbb{R}^n \) is a one-to-one mapping. In addition, if \( \Omega = \mathbb{R}^n \), then \( F \) is a homeomorphism of \( \mathbb{R}^n \).

3. Existence and uniqueness of equilibrium of CGNND (1)

The existence and uniqueness of equilibrium is prerequisite for investigating the global stability of neural networks. Now we will apply the nonlinear measure approach together with some nonlinear functional analysis techniques to analyze the existence and uniqueness of equilibrium of CGNND (1).
Define an operator

In view of (5), we immediately derive

Theorem 1. Suppose that (H1)–(H5) hold and there exist ten sets of real numbers \( c_i > 0, e_i > 0, r_i > 0, \eta_i > 0, \bar{r}_i > 0, \bar{\eta}_i > 0, k_i, d_i, \bar{k}_i, \bar{d}_i \) such that

\[
m_{ij} v_i \sum_{j=1}^{n} \frac{e_{ij}}{e_{ji}} |w_{ji}|^{k_i/d_i} + \sum_{j=1}^{n} m_{ij} v_j \frac{e_{ij}}{e_{ji}} |w_{ij}|^{2-k_i/d_j} + m_{ij} v_i \sum_{j=1}^{n} \frac{e_{ij}}{e_{ji}} |w_{ji}|^{\bar{k}_i/d_i} + \sum_{j=1}^{n} m_{ij} v_j \frac{e_{ij}}{e_{ji}} |w_{ij}|^{2-\bar{k}_i/d_j} < 2\lambda_i,
\]

\( i = 1, 2, \ldots, n. \) (5)

Then for each set of external input \( I_i \), (1) has a unique equilibrium point \( u^* \).

Proof. Define an operator \( F : \mathbb{R}^n \to \mathbb{R}^n \) by

\[
F_i(u) = - \left[ b_i(u_i) - \sum_{j=1}^{n} w_{ij} f_j(v_j u_j) - \sum_{j=1}^{n} w_{ij}^T g_j(v_j^T u_j) + I_i \right],
\]

where \( u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n, F(u) = (F_1(u), F_2(u), \ldots, F_n(u))^T \). Note that \( u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T \) is an equilibrium of (1) if and only if \( F(u^*) = 0 \).

Let \( P = \text{diag}(c_1, c_2, \ldots, c_n) \) and \( Q = \text{diag}(e_1, e_2, \ldots, e_n) \). For all \( y, z \in \mathbb{R}^n \),

\[
(QFQ^{-1}P)y - QFQ^{-1}Pz, y - z \]

\[
= \sum_{i=1}^{n} \left[ e_i \left( b_i(e_i^{-1}c_i y_i) - b_i(e_i^{-1}c_i z_i) - \sum_{j=1}^{n} w_{ij} \left[ f_j(v_j e_j^{-1} c_j y_j) - f_j(v_j e_j^{-1} c_j z_j) \right] \right) \right.
\]

\[
- \sum_{j=1}^{n} w_{ij}^T \left[ g_j(v_j^T e_j^{-1} c_j y_j) - g_j(v_j^T e_j^{-1} c_j z_j) \right] \right]
\]

\[
= \sum_{j=1}^{n} \left[ e_i \left( b_i(e_i^{-1}c_i y_i) - b_i(e_i^{-1}c_i z_i) \right) \right]
\]

\[
- \sum_{j=1}^{n} \left[ w_{ij}^T \left[ g_j(v_j^T e_j^{-1} c_j y_j) - g_j(v_j^T e_j^{-1} c_j z_j) \right] \right] \right]
\]

\[
\leq \sum_{j=1}^{n} \left[ e_i \left( b_i(e_i^{-1}c_i y_i) - b_i(e_i^{-1}c_i z_i) \right) \right]
\]

\[
- \sum_{j=1}^{n} \left[ w_{ij}^T \left[ g_j(v_j^T e_j^{-1} c_j y_j) - g_j(v_j^T e_j^{-1} c_j z_j) \right] \right] \right]
\]

\[
\leq - \sum_{i=1}^{n} \lambda_i c_i (y_i - z_i)^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ e_i m_{ij} v_j e_j^{-1} c_j |w_{ij}| |y_j - z_j||y_i - z_i| \right]
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ e_i m_{ij} v_j e_j^{-1} c_j |w_{ij}| |y_j - z_j||y_i - z_i| \right]
\]

\[
\leq - \sum_{i=1}^{n} \lambda_i c_i (y_i - z_i)^2
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ e_i m_{ij} v_j e_j^{-1} c_j \frac{1}{2} \left[ \frac{r_i}{\eta_j} |w_{ij}|^{k_i/d_j} (y_j - z_j)^2 + \frac{\eta_j}{r_i} |w_{ij}|^{2-k_i/d_j} (y_i - z_i)^2 \right] \right]
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ e_i m_{ij} v_j e_j^{-1} c_j \frac{1}{2} \left[ \frac{\bar{r}_i}{\bar{\eta}_j} |w_{ij}|^{\bar{k}_i/d_j} (y_j - z_j)^2 + \frac{\bar{\eta}_j}{\bar{r}_i} |w_{ij}|^{2-\bar{k}_i/d_j} (y_i - z_i)^2 \right] \right]
\]

\[
= - \sum_{i=1}^{n} \lambda_i c_i \left[ 2\lambda_i - m_{ii} v_i \sum_{j=1}^{n} \frac{e_{ij}}{e_{ji}} |w_{ji}|^{k_i/d_i} - \sum_{j=1}^{n} m_{ij} v_j e_j c_j \frac{e_{ij}}{e_{ji}} |w_{ij}|^{2-k_i/d_j} \right]
\]

\[
- m_{ii} v_i \sum_{j=1}^{n} \frac{e_{ij}}{e_{ji}} |w_{ji}|^{\bar{k}_i/d_i} - \sum_{j=1}^{n} m_{ij} v_j e_j c_j \frac{e_{ij}}{e_{ji}} |w_{ij}|^{2-\bar{k}_i/d_j} \right] \right]
\]

\( (y_i - z_i)^2. \)

In view of (5), we immediately derive \( m_{\mathbb{R}^n}(QFQ^{-1}P) < 0 \). By Lemma 1, we conclude that \( QFQ^{-1}P \) is a homeomorphism of \( \mathbb{R}^n \), which implies that there exists a unique \( u^* \) in \( \mathbb{R}^n \) such that \( QFQ^{-1}P(u^*) = 0 \). Since \( Q \) and \( P \) are invertible, we derive that \( F(u) = 0 \) has a unique solution \( u^* \), i.e., (1) has a unique equilibrium point \( u^* \).
Remark 1. Wang et al. [13], Liao et al. [8] and Lu [9] obtained the existence of an equilibrium of (3) and (4) by Brouwer’s fixed point theorem, but they additionally required that each \( f_j \) be globally bounded. Moreover, they did not provide the uniqueness of the equilibrium.

4. Global exponential stability analysis of CGNND (1)

In this section, we will investigate the global exponential stability of the delayed neural networks (1) by constructing a novel Lyapunov functional together with skillful transformation and flexible inequality techniques.

Theorem 2. Suppose that (H1)–(H3) and (5) hold. Then for each set of external input \( I_i \), CGNND (1) has a unique equilibrium \( u^* \) and is globally exponentially stable, and there exists a constant \( \sigma > 0 \) such that the exponential decay estimate of any solution satisfies

\[
\|u(t) - u^*\|_2 \leq \sqrt{c} e^{-\sigma(t-t_0)} \sup_{s \in [t_0-t_0]} \|\phi(s) - u^*\|_2, \quad t \geq t_0,
\]

where

\[
c = \max_{1 \leq i \leq n} \left\{ \frac{e_i^2}{\alpha_i^2} + \frac{e_i}{2\sigma} \sum_{j=1}^{n} m^2_i v^2_j \sqrt{\frac{e_i}{e_j}} \frac{1}{c^2_i} \right\} \left( \min_{1 \leq i \leq n} \left\{ \frac{e_i^2}{\alpha_i^2} \right\} \right).
\]

Proof. By Theorem 1, it follows directly from conditions (H1)–(H3) and (5) that CGNND (1) has a unique equilibrium \( u^* = (u^*_1, u^*_2, \ldots, u^*_n)^T \).

Substitution of \( u_i(t) = \frac{\sqrt{c_i}}{e_i} x_i(t) + u^*_i \) into (1) leads to

\[
\frac{dx_i(t)}{dt} = -\frac{e_i}{\sqrt{c_i}} a_i \left( \frac{\sqrt{c_i}}{e_i} x_i(t) + u^*_i \right) \left\{ b_i \left( \frac{\sqrt{c_i}}{e_i} x_i(t) + u^*_i \right) - b_i \left( u^*_i \right) - \sum_{j=1}^{n} w_{ij} [f_j (\frac{\sqrt{c_i}}{e_j} x_j(t) + u^*_j) - f_j (u^*_j)] \right\} - \sum_{j=1}^{n} w_{ij} \left[ g_j (\frac{\sqrt{c_i}}{e_j} x_j(t - \tau_{ij}) + u^*_j) - g_j (u^*_j) \right], \quad i = 1, 2, \ldots, n.
\]

Let

\[
p_i(x_i(t)) = a_i \left( \frac{\sqrt{c_i}}{e_i} x_i(t) + u^*_i \right), \quad q_i(x_i(t)) = b_i \left( \frac{\sqrt{c_i}}{e_i} x_i(t) + u^*_i \right) - b_i \left( u^*_i \right),
\]

\[
s_j(x_j(t)) = f_j (\frac{\sqrt{c_i}}{e_j} x_j(t) + u^*_j) - f_j (u^*_j), \quad s_j^*(x_j(t - \tau_{ij})) = g_j (\frac{\sqrt{c_i}}{e_j} x_j(t - \tau_{ij}) + u^*_j) - g_j (u^*_j).
\]

Then system (7) reduces to

\[
\frac{dx_i(t)}{dt} = -\frac{e_i}{\sqrt{c_i}} p_i(x_i(t)) \left\{ q_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} s_j(x_j(t)) - \sum_{j=1}^{n} w_{ij} s_j^*(x_j(t - \tau_{ij})) \right\}, \quad i = 1, 2, \ldots, n.
\]

It is clear that 0 is an equilibrium of system (8).

In view of (5), we deduce that for any fixed \( \tau_{ij} \geq 0 \), we can find a constant \( \sigma > 0 \) such that for all \( i = 1, 2, \ldots, n, \)

\[
2\lambda_i - m_i v_i \sum_{j=1}^{n} \frac{e_j}{c_j} \frac{1}{\tau_{ij}} |w_{ij}|^{2-\tilde{k}_i/\tilde{d}_i} - m_i v_i \sum_{j=1}^{n} \frac{e_j^{\tilde{k}_i}}{c_j^{\tilde{d}_i}} |w_{ij}|^{2-\tilde{k}_i/\tilde{d}_i} - m_i v_i \sum_{j=1}^{n} \frac{e_j^{\tilde{k}_i}}{c_j^{\tilde{d}_i}} |w_{ij}|^{2-\tilde{k}_i/\tilde{d}_i} e^{2\sigma \tau_{ij}} > 0.
\]

We construct the following Lyapunov functional as a candidate

\[
V(x(t)) = \sum_{i=1}^{n} \left\{ 2e^{2\sigma t} \int_{0}^{s} \frac{x_i(s)}{p_i(s)} ds + \sum_{j=1}^{n} m^2_i v^2_j \frac{e_i}{l_j} |w_{ij}|^{2-\tilde{k}_i/\tilde{d}_i} \int_{t-\tau_{ij}}^{t} e^{2\sigma s} x^2_j(s) ds \right\}.
\]
Estimating the differential of $V$ along the trajectory, we deduce that

$$\frac{dV(x(t))}{dt} = -\sum_{i=1}^{n} 2e^{2\sigma x_i(t)} x_i(t) \frac{e_i}{\sqrt{c_i}} \left\{ q_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} s_j(x_j(t)) - \sum_{j=1}^{n} w_{ij}^\tau s_j^\tau(x_j(t - \tau_{ij})) \right\}$$

$$+ \sum_{i=1}^{n} 4\sigma e^{2\sigma x_i(t)} \int_{0}^{\frac{s}{p_i(s)}} ds + \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i r_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_i} e^{2\sigma x_j(t)}$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i r_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_i} e^{2\sigma(t - \tau_{ij})} x_j^2(t - \tau_{ij})$$

$$\leq e^{2\sigma t} \left\{ \sum_{i=1}^{n} 2 \left[ -\lambda_i x_i^2(t) + \frac{e_i}{\sqrt{c_i}} \sum_{j=1}^{n} x_j(t) ||w_{ij}|| s_j(x_j(t)) + \frac{e_i}{\sqrt{c_i}} \sum_{j=1}^{n} x_i(t) ||w_{ij}|| s_j^\tau(x_j(t - \tau_{ij})) \right] \right\}$$

$$+ \sum_{i=1}^{n} \frac{2\sigma}{\lambda_i} x_i^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_j} x_j^2(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_j} e^{2\sigma t_{ij}} x_j^2(t - \tau_{ij})$$

$$\leq e^{2\sigma t} \left\{ -2 \sum_{i=1}^{n} \lambda_i x_i^2(t) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| x_j^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} e_i m_i v_j \frac{c_j \tilde{\eta}_j}{c_i r_i} |w_{ij}| \frac{k_i/d_j}{e_i} x_j^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} e_i m_i v_j \frac{c_j \tilde{\eta}_j}{c_i r_i} |w_{ij}| \frac{k_i/d_j}{e_i} e^{2\sigma t_{ij}} x_j^2(t - \tau_{ij}) \right\}$$

$$+ \sum_{i=1}^{n} \frac{2\sigma}{\lambda_i} x_i^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_j} x_j^2(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_j} e^{2\sigma t_{ij}} x_j^2(t - \tau_{ij})$$

$$\leq e^{2\sigma t} \left\{ 2\lambda_i - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_i}{e_j r_j} - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_j}{e_j r_j} e^{2\sigma t_{ij}} \right\} x_i^2(t)$$

$$- \frac{2\sigma}{\lambda_i} - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_i}{e_j r_j} e^{2\sigma t_{ij}} - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_j}{e_j r_j} x_i^2(t)$$

$$\leq e^{2\sigma t} \min_{1 \leq i \leq n} \left\{ 2\lambda_i - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_i}{e_j r_j} - \sum_{j=1}^{n} m_j v_j e_j c_j \tilde{\eta}_j |w_{ij}| \frac{k_i/d_j}{e_j r_j} \right\} \|x(t)\|_2^2 < 0.$$

Thus, $V(x(t)) \leq V(x(t_0))$ for any $t \geq t_0$. By the form of (10) and $x_i(t) = \frac{e_i}{\sqrt{c_i}} (u_i(t) - u_i^\tau(t))$, we have

$$V(x(t_0)) = \sum_{j=1}^{n} \int_{0}^{\frac{s}{p_i(s)}} ds + \sum_{j=1}^{n} m_j v_j^\tau \frac{e_i \tilde{r}_i}{e_j \tilde{\eta}_j} |w_{ij}| \frac{k_i/d_j}{e_j} \int_{t_0 - \tau_{ij}}^{t_0} e^{2\sigma t_{ij}} x_j^2(t) ds$$
follows directly from Theorem 1.

Then for each set of external input function

Remark 2. Under the Lipschitz assumption on the activation functions, Theorem 2 presents general mild sufficient conditions for the GES of CGNND (1), which are independent of the delays. Note that the convergence rate \( \sigma \), which is determined by (9), depends on the delays. Also, the global asymptotic stability (GAS, in brief) of CGNND (1) is naturally implied by Theorem 2.

Remark 3. It is worth emphasizing the significance of the incorporation of the ten sets of adjustable parameters into condition (5), which endows our criteria with extensive flexibility and applicability. Through specific choice of the parameters, a series of new criteria can be obtained for the GAS and GES of those models included by (1), and they generalize and improve many existing results such as those in \([1,4,5,7,8,10,13,15,18,19]\).

Denote (H3)’: Each \( f_j \) is Lipschitz continuous. Let \( m_j \) represent the minimal Lipschitz constant of \( f_j \).

Corollary 1. Suppose that (H1), (H2), (H3)’ hold, and there exist ten sets of real numbers \( \text{e}_i > 0, \text{l}_i > 0, r_i > 0, \eta_i > 0, \tilde{r}_i > 0, \tilde{\eta}_i > 0, k_i, d_i, \tilde{k}_i, \tilde{d}_i \) such that

\[
\sum_{j=1}^{n} \left( e_j \text{e}_j \frac{r_j}{\eta_i} |w_{ij}|^{k_j/d_j} + \tilde{r}_j \tilde{\eta}_i |w_{ij}|^{\tilde{k}_j/\tilde{d}_j} \right) + \sum_{j=1}^{n} \left( \eta_j |w_{ij}|^{2-k_j/d_j} + \tilde{\eta}_j |w_{ij}|^{2-\tilde{k}_j/\tilde{d}_j} \right) < 2 \frac{\lambda_i}{m_i}, \quad i = 1, 2, \ldots, n. \tag{12}
\]

Then for each set of external input \( I_i \), (3) has a unique equilibrium \( u^* \), and it is globally exponentially stable, and there exists a constant \( \sigma > 0 \) such that the exponential decay estimate of any solution satisfies

\[
\left\| u(t) - u^* \right\|_2 \leq c e^{-\sigma (t-t_0)} \sup_{s \in [t_0-t_0, t_0]} \left\| \phi(s) - u^* \right\|_2, \quad t \geq t_0, \tag{13}
\]

where

\[
c = \max_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i l_i}, \frac{e_i \tilde{r}_i}{2 \sigma} \sum_{j=1}^{n} \frac{m_j^2}{l_j \eta_j} |w_{ij}|^{\tilde{k}_j/\tilde{d}_j} \left( 1 - e^{-2 \sigma t_i} \right) \right\} / \min_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i l_i} \right\}.
\]

Proof. Condition (12) implies condition (5) holds for \( m_j^2 = m_i, v_j^2 = v_i = 1 \) and \( c_i = \frac{e_i \tilde{r}_i}{m_i}, \quad i = 1, 2, \ldots, n \), and thus this corollary follows directly from Theorem 1.

Remark 4. Liao et al. [8, Theorem 2 and Corollary 1] are special cases of Corollary 1 when \( l_i = m_i, k_i = \tilde{k}_i = d_i = \tilde{d}_i = r_i = \tilde{r}_i = 1, \eta_i = \rho_1, \tilde{\eta}_i = \rho_2 \) \((i = 1, 2, \ldots, n)\) and \( \rho_1, \rho_2 \in [0, 1] \), and meanwhile, they additionally required that each activation function \( f_j \) be bounded.

Corollary 2. Suppose that (H1), (H2), (H3)’ hold, and there exist two sets of positive numbers \( e_i, l_i \) and two real numbers \( p, q \) such that

\[
\sum_{j=1}^{n} \left[ e_j \left( |w_{ij}|^p + |w_{ij}^\tau|^q \right) + \frac{l_j}{l_i} \left( |w_{ij}|^{2-p} + |w_{ij}^\tau|^{2-q} \right) \right] < 2 \frac{\lambda_i}{m_i}, \quad i = 1, 2, \ldots, n. \tag{14}
\]
Then for each set of external input $I_i$, (3) has a unique equilibrium $u^*$, $u^*$ is globally exponentially stable, and there exists a constant $\sigma > 0$ such that the exponential decay estimate of any solution satisfies (13), where

$$c = \max_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i I_i} + \frac{e_i}{2\sigma} \sum_{j=1}^{n} \frac{m_j^2}{l_{ij}} |w_{ij}^r|^q \left(1 - e^{-2\sigma \tau_{ij}}\right) \right\} / \min_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i I_i} \right\}.$$  

**Corollary 3.** Suppose that $(H_1), (H_2), (H_3)'$ hold, and there exist two sets of positive numbers $e_i, I_i$ and $q \in \mathbb{R}$ such that

$$m_i \sum_{j=1}^{n} \left( \frac{e_j}{e_i} |w_{ji}^r|^q + \frac{l_j}{l_i} |w_{ij}^r|^{2-q} \right) < 2\lambda_i, \quad i = 1, 2, \ldots, n. \quad (15)$$

Then for each set of external input $I_i$, (4) has a unique equilibrium $u^*$, $u^*$ is globally exponentially stable, and there exists a constant $\sigma > 0$ such that the exponential decay estimate of any solution satisfies (13), where

$$c = \max_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i I_i} + \frac{e_i}{2\sigma} \sum_{j=1}^{n} \frac{m_j^2}{l_{ij}} |w_{ij}^r|^q \left(1 - e^{-2\sigma \tau_{ij}}\right) \right\} / \min_{1 \leq i \leq n} \left\{ \frac{m_i e_i}{\alpha_i I_i} \right\}.$$  

5. Conclusions

This Letter is concerned with the global exponential stability of a new generalized Cohen–Grossberg neural networks model with discrete delays. Only assuming the activation functions to satisfy Lipschitz condition, we apply the nonlinear measure approach and construct a novel Lyapunov functional, thus derive general sufficient conditions for the GES of the delayed neural networks, which greatly extend and improve the existing results. It is believed that our new criteria will be significant for the designs and applications of delayed neural networks.

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