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# Exponential stability of Cohen–Grossberg neural networks with a general class of activation functions ☆

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**Abstract**

In this Letter, the dynamics of Cohen–Grossberg neural networks model are investigated. The activation functions are only assumed to be Lipschitz continuous, which provide a much wider application domain for neural networks than the previous results. By means of the extended nonlinear measure approach, new and relaxed sufficient conditions for the existence, uniqueness and global exponential stability of equilibrium of the neural networks are obtained. Moreover, an estimate for the exponential convergence rate of the neural networks is precisely characterized. Our results improve those existing ones.


*Keywords:* Cohen–Grossberg neural networks; Exponential stability; Exponential decay estimate; Nonlinear measure

## 1. Introduction

In this Letter, we consider the Cohen–Grossberg neural networks model, which is described by the following differential equations [1]

$$\frac{du_i(t)}{dt} = -a_i\big(u_i(t)\big)\left[b_i\big(u_i(t)\big) - \sum_{j=1}^{n} w_{ij} f_j\big(u_j(t)\big) + I_i\right], \quad i = 1, 2, \dots, n, \tag{1}$$

where $n \geqslant 2$ is the number of neurons in the networks, $u_i(t)$ denotes the neuron state vector, $a_i$ denotes an amplification function, $b_i$ denotes a self-signal function, $W = (w_{ij})_{n\times n}$ is the connection matrix, $f_i$ denotes an activation function, and $I_i$ represents the constant external input.

Model (1) includes a large number of models from neurobiology and population biology [2]. In particular, it includes as a special case the popular Hopfield neural networks [3]. Moreover, Cohen–Grossberg neural networks have potential applications in many areas such as associative memory and optimization. Therefore, the investigation on Cohen–Grossberg neural networks is of fundamental theoretical and practical significance.

As is well known, stability of neural networks is fundamental for the designs and applications of neural networks [4–16]. In this Letter, we are devoted to the exponential stability analysis of (1). We only make the following assumptions:

(H$_1$) Each $a_i$ is continuous and $0 < \grave{\alpha}_i \leqslant a_i(r) \leqslant \acute{\alpha}_i$ for any $r \in \mathbb{R}$.

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(H$_2$) Each $b_i$ is continuous, and there exists a constant $\lambda_i > 0$ such that $(r_1 - r_2)[b_i(r_1) - b_i(r_2)] \geqslant \lambda_i(r_1 - r_2)^2$ for any $r_1, r_2 \in \mathbb{R}$.
(H$_3$) Each $f_i$ is Lipschitz continuous. Denote $m_i$ the minimal Lipschitz constant of $f_i$, i.e., $m_i = \sup_{r_1,r_2 \in \mathbb{R}, r_1 \neq r_2} \frac{|f_i(r_1) - f_i(r_2)|}{|r_1 - r_2|}$.

Recently, Wang et al. [12], Liao et al. [13] and Lu et al. [14], respectively, analyzed the global exponential stability of (1), but they additionally required that each $f_i$ be bounded or monotonically increasing. However, in this letter, not only do we abandon the boundedness condition of $f_i$, but also we remove the differentiability and monotonicity restriction from $f_i$, although this will lead to more difficulty in stability analysis. In addition, we do not impose any restriction such as symmetry on the connection matrix.

To the best of the authors' knowledge, the most popular approach to stability analysis of neural networks is based on Lyapunov direct method (see, for example, [4,7,8,12–16] and references therein). However, no general rule can guide how a Lyapunov function should be constructed for a given system, and thus the construction of a proper Lyapunov function is usually rather difficult. This Letter aims to analyze the stability of Cohen–Grossberg neural networks (1) by means of the extended nonlinear measure approach, and derive a series of new relaxed sufficient conditions for the exponential stability of (1).

## 2. Preliminaries

Let $\mathbb{R}^n$ denote the $n$-dimensional real vector space endowed with vector norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ denote the inner product of vectors in $\mathbb{R}^n$ and $\operatorname{sign}(x) = (\operatorname{sign}(x_1), \operatorname{sign}(x_2), \ldots, \operatorname{sign}(x_n))^T$ denote the sign vector of $x \in \mathbb{R}^n$, where $\operatorname{sign}(r)$ represents the sign function of $r \in \mathbb{R}$. The $l^1$-norm and $l^2$-norm of $x$ are defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$ for any $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Consider the following system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F\big(x(t)\big), \quad t \geqslant t_0, \tag{2}$$

where $F$ is a nonlinear operator from $\Omega$ to $\mathbb{R}^n$ and $x(t) \in \Omega$.

**Definition 1.** Suppose that $x^*$ is an equilibrium point of system (2). System (2) is said to be globally exponentially stable, if there exist constants $c \geqslant 1$ and $\sigma > 0$ such that $\|x(t) - x^*\| \leqslant ce^{-\sigma(t-t_0)}\|x_0 - x^*\|$ $(t \geqslant t_0)$, where $x(t)$ is any solution of (2) initiated from $x_0 = x(t_0) \in \mathbb{R}^n$.

**Definition 2** [11]. Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$, $F$ is an operator from $\Omega$ to $\mathbb{R}^n$, and $x^0$ is any fixed point in $\Omega$. The constant

$$M_\Omega(F) = \sup_{x,y \in \Omega, x \neq y} \frac{\langle F(x) - F(y), \operatorname{sign}(x-y)\rangle}{\|x-y\|_1} \tag{3}$$

is called the nonlinear measure of $F$ on $\Omega$. The constant

$$M_\Omega\big(F, x^0\big) = \sup_{x \in \Omega, x \neq x^0} \frac{\langle F(x) - F(x^0), \operatorname{sign}(x-x^0)\rangle}{\|x-x^0\|_1} \tag{4}$$

is called the relative nonlinear measure of $F$ at $x^0$.

Inspired by Definition 2, we introduce the following extended nonlinear measure concepts of $F$ in the sense of the $l^2$-norm, namely:

**Definition 3.** Denote $\Omega$, $F$ and $x^0$ the same as in Definition 2. The constant

$$m_\Omega(F) = \sup_{x,y \in \Omega, x \neq y} \frac{\langle F(x) - F(y), x-y\rangle}{\|x-y\|_2^2} \tag{5}$$

is called the nonlinear measure of $F$ on $\Omega$. The constant

$$m_\Omega\big(F, x^0\big) = \sup_{x \in \Omega, x \neq x^0} \frac{\langle F(x) - F(x^0), x-x^0\rangle}{\|x-x^0\|_2^2} \tag{6}$$

is called the relative nonlinear measure of $F$ at $x^0$.

It is clear that $m_\Omega(F, x^0) \leqslant m_\Omega(F)$. Now we will show that, the above nonlinear measures induced by the $l^2$-norm have the following important properties analogous to Lemma 1 and Theorem 2 in [11].

**Lemma 1.** *If $m_\Omega(F) < 0$, then $F$ is a one-to-one mapping on $\Omega$. If in addition $\Omega = \mathbb{R}^n$, then $F$ is a homeomorphism of $\mathbb{R}^n$.*

**Proof.** Suppose $x_1, x_2 \in \Omega$ satisfy $F(x_1) = F(x_2)$ while $x_1 \neq x_2$. Then

$$m_\Omega(F) \geqslant \frac{\langle F(x_1) - F(x_2), x_1 - x_2\rangle}{\|x_1 - x_2\|_2^2} = 0,$$

which contradicts $m_\Omega(F) < 0$. Thus, $F$ is one-to-one on $\Omega$.

When $\Omega = \mathbb{R}^n$, for all $x, y \in \mathbb{R}^n$, we have

$$\left|\left\langle F(x) - F(y), x - y\right\rangle\right| \leqslant \left\|F(x) - F(y)\right\|_2 \|x - y\|_2. \tag{7}$$

Since $m_{\mathbb{R}^n}(F) < 0$, we immediately derive

$$\left\|F(x) - F(y)\right\|_2 \geqslant -m_{\mathbb{R}^n}(F)\|x - y\|_2. \tag{8}$$

Therefore we deduce $\lim_{\|x\|_2 \to +\infty} \|F(x)\|_2 = +\infty$, i.e., $F$ is norm-coercive on $\mathbb{R}^n$. By the norm-coerciveness theorem [17], we infer that $F$ is homeomorphic. □

**Lemma 2.** *Suppose that $x^*$ is an equilibrium point of system (2). If $m_{\mathbb{R}^n}(F, x^*) < 0$, then $x^*$ is globally exponentially stable, and the exponential decay estimate of any solution $x(t)$ initiated from $x_0 = x(t_0)$ satisfies*

$$\left\|x(t) - x^*\right\|_2 \leqslant e^{m_{\mathbb{R}^n}(F,x^*)(t-t_0)} \left\|x_0 - x^*\right\|_2, \quad t \geqslant t_0. \tag{9}$$

**Proof.** In view of $F(x^*) = 0$ and (2), we can readily deduce that for almost all $t > t_0$, $\frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - x^*\|_2^2 \leqslant 2\langle x(t) - x^*, F(x)\rangle$. Therefore,

$$\frac{\mathrm{d}\|x(t) - x^*\|_2^2}{\mathrm{d}t} \leqslant 2m_{\mathbb{R}^n}\left(F, x^*\right)\left\|x(t) - x^*\right\|_2^2. \tag{10}$$

Integrating the above differential inequality, we then obtain

$$\left\|x(t) - x^*\right\|_2^2 \leqslant e^{2m_{\mathbb{R}^n}(F,x^*)(t-t_0)} \left\|x_0 - x^*\right\|_2^2, \quad t \geqslant t_0, \tag{11}$$

which immediately indicates the exponential decay estimation (9). □

## 3. Main results

In this section, we will investigate the existence, uniqueness and global exponential stability of equilibrium of neural networks (1) by the nonlinear measure approach developed in Section 2.

**Theorem 1.** *Suppose that* (H$_1$), (H$_2$), (H$_3$) *hold. If there exist two sets of real numbers $d_i > 0$ and $l_i > 0$ such that*

$$\sum_{j=1}^{n}\left(\frac{d_j}{d_i}|w_{ji}| + \frac{l_j}{l_i}|w_{ij}|\right) < 2\frac{\lambda_i}{m_i}, \quad i = 1, 2, \ldots, n, \tag{12}$$

*then for each set of external input $I_i$, (1) has a unique equilibrium point $u^*$.*

**Proof.** Define $G:\mathbb{R}^n \to \mathbb{R}^n$ by $G(u) = (G_1(u), G_2(u), \ldots, G_n(u))^T$, $\forall u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ and

$$G_i(u) = -\left[b_i(u_i) - \sum_{j=1}^{n} w_{ij} f_j(u_j) + I_i\right], \quad i = 1, 2, \ldots, n.$$

Therefore, $u^*$ is an equilibrium of (1) if and only if $G(u^*) = 0$. Let $Q = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $P = \mathrm{diag}(\frac{l_1}{m_1}, \frac{l_2}{m_2}, \ldots, \frac{l_n}{m_n})$. In the following, we will prove that $m_{\mathbb{R}^n}(QGP) < 0$. For all $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\left\langle QGP(x) - QGP(y), x - y\right\rangle \\
&\quad = \sum_{i=1}^{n}\left\{-d_i\left\{b_i\left(\frac{l_i}{m_i}x_i\right) - b_i\left(\frac{l_i}{m_i}y_i\right) - \sum_{j=1}^{n} w_{ij}\left[f_j\left(\frac{l_j}{m_j}x_j\right) - f_j\left(\frac{l_j}{m_j}y_j\right)\right]\right\}(x_i - y_i)\right\} \\
&\quad \leqslant \sum_{i=1}^{n}\left\{-d_i\lambda_i\frac{l_i}{m_i}(x_i - y_i)^2 + d_i\sum_{j=1}^{n}\left(|w_{ij}| \cdot l_j|x_j - y_j||x_i - y_i|\right)\right\}
\end{aligned}$$

$$\leqslant -\sum_{i=1}^{n} \frac{\lambda_i}{m_i} d_i l_i (x_i - y_i)^2 + \sum_{j=1}^{n}\sum_{i=1}^{n} \left\{ d_i l_j |w_{ij}| \cdot \frac{(x_j - y_j)^2 + (x_i - y_i)^2}{2} \right\}$$
$$= -\sum_{i=1}^{n} \left\{ \frac{\lambda_i}{m_i} d_i l_i - \frac{1}{2}\sum_{j=1}^{n} d_j l_i |w_{ji}| - \frac{1}{2}\sum_{j=1}^{n} d_i l_j |w_{ij}| \right\} (x_i - y_i)^2.$$

Therefore, by means of (12), we derive

$$m_{\mathbb{R}^n}(QGP) \leqslant -\min_{1\leqslant i\leqslant n} \left\{ \frac{\lambda_i}{m_i} d_i l_i - \frac{1}{2}\sum_{j=1}^{n} \bigl( d_j l_i |w_{ji}| + d_i l_j |w_{ij}| \bigr) \right\} < 0.$$

By virtue of Lemma 1, we infer that $QGP$ is a homeomorphism of $\mathbb{R}^n$, which indicates that $QGP(u^*) = 0$ has a unique solution $u^*$. Since $Q$, $P$ are invertible, we derive $G(u^*) = 0$, and therefore (1) has a unique equilibrium point $u^*$. □

**Remark 1.** Wang et al. [12, Theorem 3.1] and Liao et al. [13, Lemma 1] only proved the existence of an equilibrium, but they additionally assumed that each $f_i$ is bounded. Lu et al. [14, Theorem 1] also proved the existence of a unique equilibrium, however they required that each $f_i$ be monotonically increasing.

In the following, let $r^+ := \max\{r, 0\}$ for any $r \in \mathbb{R}$. When in particular each $f_i$ is monotonically nondecreasing, we can further improve Theorem 1 as follows.

**Theorem 2.** *Suppose that* $(H_1)$, $(H_2)$, $(H_3)$ *hold. If each* $f_i$ *is monotonically nondecreasing, and there exist two sets of real numbers* $d_i, l_i > 0$ *such that*

$$2w_{ii} + \sum_{j=1, j\neq i}^{n} \left( \frac{d_j}{d_i} |w_{ji}| + \frac{l_j}{l_i} |w_{ij}| \right) < 2\frac{\lambda_i}{m_i}, \quad i = 1, 2, \ldots, n, \tag{13}$$

*then for each set of external input* $I_i$, (1) *has a unique equilibrium point* $u^*$.

**Proof.** Define $G$ and $Q$, $P$ the same as in the proof of Theorem 1. Analogously, we only need to prove $m_{\mathbb{R}^n}(QGP) < 0$. For all $x, y \in \mathbb{R}^n$, we have

$$\bigl\langle QGP(x) - QGP(y), x - y \bigr\rangle$$
$$\leqslant -\sum_{i=1}^{n} d_i \lambda_i \frac{l_i}{m_i} (x_i - y_i)^2 + \sum_{j=1}^{n} \left\{ d_j w_{jj} \left| f_j\left(\frac{l_j}{m_j} x_j\right) - f_j\left(\frac{l_j}{m_j} y_j\right) \right| |x_j - y_j| \right.$$
$$\left. + \sum_{i=1, i\neq j}^{n} \left( d_i |w_{ij}| \left| f_j\left(\frac{l_j}{m_j} x_j\right) - f_j\left(\frac{l_j}{m_j} y_j\right) \right| |x_i - y_i| \right) \right\}$$
$$\leqslant -\sum_{i=1}^{n} \frac{d_i l_i}{m_i} \lambda_i (x_i - y_i)^2 + \sum_{j=1}^{n} \left\{ l_j |x_j - y_j| \left[ d_j w_{jj} |x_j - y_j| + \sum_{i=1, i\neq j}^{n} d_i |w_{ij}| |x_i - y_i| \right]^+ \right\}.$$

When $d_j w_{jj} |x_j - y_j| + \sum_{i=1, i\neq j}^{n} d_i |w_{ij}| |x_i - y_i| > 0$, we have

$$\bigl\langle QGP(x) - QGP(y), x - y \bigr\rangle$$
$$\leqslant -\sum_{i=1}^{n} \frac{d_i l_i}{m_i} (\lambda_i - m_i w_{ii})(x_i - y_i)^2 + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} \bigl\{ d_i l_j |w_{ij}| |x_j - y_j| |x_i - y_i| \bigr\}$$
$$\leqslant -\sum_{i=1}^{n} \frac{d_i l_i}{m_i} (\lambda_i - m_i w_{ii})(x_i - y_i)^2 + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} \left\{ d_i l_j |w_{ij}| \frac{(x_j - y_j)^2 + (x_i - y_i)^2}{2} \right\}$$
$$= -\sum_{i=1}^{n} \left\{ d_i l_i \frac{\lambda_i}{m_i} - d_i l_i w_{ii} - \frac{1}{2} l_i \sum_{j=1, j\neq i}^{n} d_j |w_{ji}| - \frac{1}{2} d_i \sum_{j=1, j\neq i}^{n} l_j |w_{ij}| \right\} (x_i - y_i)^2.$$

When $d_j w_{jj} |x_j - y_j| + \sum_{i=1, i\neq j}^{n} d_i |w_{ij}| |x_i - y_i| \leqslant 0$, we have

$$\bigl\langle QGP(x) - QGP(y), x - y \bigr\rangle \leqslant -\sum_{i=1}^{n} d_i l_i \frac{\lambda_i}{m_i} (x_i - y_i)^2.$$

Therefore, for all $x, y \in \mathbb{R}^n$, we deduce

$$\langle QGP(x) - QGP(y), x - y\rangle \leqslant -\sum_{i=1}^{n}\left\{d_i l_i \frac{\lambda_i}{m_i} - \left[d_i l_i w_{ii} + \frac{1}{2}\sum_{j=1, j\neq i}^{n}\big(d_j l_i |w_{ji}| + d_i l_j |w_{ij}|\big)\right]^{+}\right\}(x_i - y_i)^2.$$

In view of (13), we derive $m_{\mathbb{R}^n}(QGP) < 0$. This completes the proof. □

In the following, we will employ Lemma 2 to investigate the stability of neural networks (1).

**Theorem 3.** *Suppose that* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $(\mathrm{H}_3)$ *hold. If there exist two sets of real numbers* $e_i > 0$ *and* $l_i > 0$ *such that*

$$\sum_{j=1}^{n}\left(\frac{e_j}{e_i}\acute{\alpha}_j |w_{ji}| + \frac{l_j}{l_i}\acute{\alpha}_i |w_{ij}|\right) < 2\frac{\grave{\alpha}_i \lambda_i}{m_i}, \quad i = 1, 2, \ldots, n, \tag{14}$$

*then for each set of external input* $I_i$, *neural networks* (1) *is globally exponentially stable, and the exponential decay estimate of any solution initiated from* $u_0 = u(t_0)$ *is governed by*

$$\big\|u(t) - u^*\big\|_2 \leqslant \sqrt{c}e^{-b_1(t-t_0)}\big\|u_0 - u^*\big\|_2, \quad t \geqslant t_0, \tag{15}$$

*where* $c = \frac{\max_{1\leqslant i\leqslant n} e_i m_i / l_i}{\min_{1\leqslant i\leqslant n} e_i m_i / l_i}$ *and* $b_1 = \min_{1\leqslant i\leqslant n}\{\grave{\alpha}_i\lambda_i - \frac{m_i}{2}\sum_{j=1}^{n}(\frac{e_j}{e_i}\acute{\alpha}_j|w_{ji}| + \frac{l_j}{l_i}\acute{\alpha}_i|w_{ij}|)\}$.

**Proof.** It is easy to verify that (14) implies that (12) holds for $d_i = e_i\acute{\alpha}_i$ $(i = 1, 2, \ldots, n)$, and thus it follows from Theorem 1 that there exists a unique equilibrium $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of (1). Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $x_i(t) = \sqrt{\frac{m_i}{l_i}}(u_i(t) - u_i^*)$. Substitution of $u_i(t) = \sqrt{\frac{l_i}{m_i}}x_i(t) + u_i^*$ into (1) leads to

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -\sqrt{\frac{m_i}{l_i}}a_i\left(\sqrt{\frac{l_i}{m_i}}x_i(t) + u_i^*\right)\left\{b_i\left(\sqrt{\frac{l_i}{m_i}}x_i(t) + u_i^*\right) - b_i\big(u_i^*\big) - \sum_{j=1}^{n}w_{ij}\left[f_j\left(\sqrt{\frac{l_j}{m_j}}x_j(t) + u_j^*\right) - f_j\big(u_j^*\big)\right]\right\},\\ &\quad i = 1, 2, \ldots, n.\end{aligned} \tag{16}$$

Let $p_i(x_i(t)) = a_i(\sqrt{l_i/m_i}x_i(t) + u_i^*)$, $q_i(x_i(t)) = b_i(\sqrt{l_i/m_i}x_i(t) + u_i^*) - b_i(u_i^*)$ and $s_j(x_j(t)) = f_j(\sqrt{l_j/m_j}x_j(t) + u_j^*) - f_j(u_j^*)$. Define $F:\mathbb{R}^n \to \mathbb{R}^n$ by $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$, $F_i(x) = -\sqrt{m_i/l_i}\,p_i(x_i)[q_i(x_i) - \sum_{j=1}^{n}w_{ij}s_j(x_j)]$, $\forall x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Then system (16) is reformulated as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F\big(x(t)\big), \quad t \geqslant t_0. \tag{17}$$

Let $P = \mathrm{diag}(\sqrt{e_1}, \sqrt{e_2}, \ldots, \sqrt{e_n})$. Consider the following system

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = PFP^{-1}\big(y(t)\big), \quad t \geqslant t_0. \tag{18}$$

Clearly, 0 is an equilibrium of (17) and (18). In the following, we will prove that $m_{\mathbb{R}^n}(PFP^{-1}, 0) < 0$.

For all $y \in \mathbb{R}^n$, we have

$$\begin{aligned}&\big\langle PFP^{-1}(y), y\big\rangle\\ &\quad = -\sum_{i=1}^{n}\left\{e_i^{1/2}\left(\frac{m_i}{l_i}\right)^{1/2}p_i\big(e_i^{-1/2}y_i\big)\left[q_i\big(e_i^{-1/2}y_i\big) - \sum_{j=1}^{n}w_{ij}s_j\big(e_j^{-1/2}y_j\big)\right]y_i\right\}\\ &\quad \leqslant -\sum_{i=1}^{n}\left(e_i^{1/2}\left(\frac{m_i}{l_i}\right)^{1/2}\grave{\alpha}_i\lambda_i\left|\left(\frac{l_i}{m_i}\right)^{1/2}e_i^{-1/2}y_i\right||y_i|\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\acute{\alpha}_i|w_{ij}|e_i^{1/2}\left(\frac{m_i}{l_i}\right)^{1/2}m_j\left|\left(\frac{l_j}{m_j}\right)^{1/2}e_j^{-1/2}y_j\right||y_i|\right\}\\ &\quad \leqslant -\sum_{i=1}^{n}\grave{\alpha}_i\lambda_i y_i^2 + \sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\acute{\alpha}_i|w_{ij}|\cdot\frac{1}{2}\left(\frac{m_j e_i}{e_j}y_j^2 + \frac{m_i l_j}{l_i}y_i^2\right)\right\}\\ &\quad = -\sum_{i=1}^{n}\left\{\grave{\alpha}_i\lambda_i - \frac{m_i}{2}\sum_{j=1}^{n}\left(\frac{e_j}{e_i}\acute{\alpha}_j|w_{ji}| + \frac{l_j}{l_i}\acute{\alpha}_i|w_{ij}|\right)\right\}y_i^2,\end{aligned}$$

therefore, $m_{\mathbb{R}^n}(PFP^{-1},0) \leqslant -b_1 < 0$. By Lemma 2, we derive that system (18) is globally exponentially stable, and the exponential decay estimate of any solution $y(t)$ of (18) obeys to

$$\|y(t)\|_2 \leqslant \|y(t_0)\|_2 \cdot e^{-b_1(t-t_0)}. \tag{19}$$

It is clear that $y(t) = Px(t)$ is the unique solution of (18) whenever $x(t)$ is a solution of (17). Then, substituting $y(t) = Px(t)$ and $x_i(t) = \sqrt{m_i/l_i}(u_i(t) - u_i^*)$ into (19), we derive

$$\left[\sum_{i=1}^{n}\left(\sqrt{e_i}\sqrt{\frac{m_i}{l_i}}(u_i(t)-u_i^*)\right)^2\right]^{1/2} \leqslant e^{-b_1(t-t_0)}\left[\sum_{i=1}^{n}\left(\sqrt{e_i}\sqrt{\frac{m_i}{l_i}}(u_i(t_0)-u_i^*)\right)^2\right]^{1/2},$$

which directly yields the exponential decay estimation (15). □

**Theorem 4.** *Suppose that* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $(\mathrm{H}_3)$ *hold. If each* $f_i$ *is monotonically nondecreasing, and there exist two sets of real numbers* $e_i, l_i > 0$ *such that*

$$2\grave{\alpha}_i w_{ii} + \sum_{j=1, j\neq i}^{n}\left(\frac{e_j}{e_i}\acute{\alpha}_j|w_{ji}| + \frac{l_j}{l_i}\acute{\alpha}_i|w_{ij}|\right) < 2\frac{\grave{\alpha}_i\lambda_i}{m_i}, \quad i = 1,2,\ldots,n, \tag{20}$$

*then for each set of external input* $I_i$, *neural networks* (1) *is globally exponentially stable, and the exponential decay estimate of any solution* $u(t)$ *of* (1) *initiated from* $u_0 = u(t_0)$ *is governed by*

$$\|u(t)-u^*\|_2 \leqslant \sqrt{c}e^{-b_2(t-t_0)}\|u_0-u^*\|_2, \quad t \geqslant t_0, \tag{21}$$

*where*

$$b_2 = \min_{1\leqslant i\leqslant n}\left\{\grave{\alpha}_i\lambda_i - \frac{m_i}{2}\left[2\grave{\alpha}_i w_{ii} + \sum_{j=1, j\neq i}^{n}\left(\frac{e_j}{e_i}\acute{\alpha}_j|w_{ji}| + \frac{l_j}{l_i}\acute{\alpha}_i|w_{ij}|\right)\right]^+\right\}.$$

**Proof.** Analogous to the proof of Theorem 3, the existence of a unique equilibrium $u^*$ can be readily deduced by Theorem 2. Define $x_i(t)$, $F$ and $P$ the same as in the proof of Theorem 3. Therefore, it suffices to prove that $m_{\mathbb{R}^n}(PFP^{-1},0) < -b_2 < 0$.

For all $y \in \mathbb{R}^n$, we deduce

$$\begin{aligned}
&\langle PFP^{-1}(y), y\rangle \\
&\leqslant -\sum_{i=1}^{n} p_i(e_i^{-1/2}y_i)\lambda_i y_i^2 + \sum_{j=1}^{n}\left\{e_j^{1/2}\left(\frac{m_j}{l_j}\right)^{1/2} p_j(e_j^{-1/2}y_j)w_{jj}|s_j(e_j^{-1/2}y_j)||y_j|\right. \\
&\quad \left. + |s_j(e_j^{-1/2}y_j)|\sum_{i=1,i\neq j}^{n}\left[e_i^{1/2}\left(\frac{m_i}{l_i}\right)^{1/2} p_i(e_i^{-1/2}y_i)|w_{ij}||y_i|\right]\right\} \\
&\leqslant -\sum_{i=1}^{n} p_i(e_i^{-1/2}y_i)\lambda_i y_i^2 + \sum_{j=1}^{n}\left\{m_j\left(\frac{l_j}{m_j}\right)^{1/2} e_j^{-1/2}|y_j|\left[e_j^{1/2}\left(\frac{m_j}{l_j}\right)^{1/2} p_j(e_j^{-1/2}y_j)w_{jj}|y_j|\right.\right. \\
&\quad \left.\left. + \sum_{i=1,i\neq j}^{n}\left(e_i^{1/2}\left(\frac{m_i}{l_i}\right)^{1/2} p_i(e_i^{-1/2}y_i)|w_{ij}||y_i|\right)\right]^+\right\}.
\end{aligned}$$

When $(e_j\frac{m_j}{l_j})^{1/2}p_j(e_j^{-1/2}y_j)w_{jj}|y_j| + \sum_{i=1,i\neq j}^{n}(e_i\frac{m_i}{l_i})^{1/2}p_i(e_i^{-1/2}y_i)|w_{ij}||y_i| > 0$, we have

$$\begin{aligned}
&\langle PFP^{-1}(y), y\rangle \\
&\leqslant -\sum_{i=1}^{n} p_i(e_i^{-1/2}y_i)(\lambda_i - m_i w_{ii})y_i^2 + \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n}\left\{\acute{\alpha}_i|w_{ij}|\left(\frac{e_i}{e_j}m_j\right)^{1/2}\left(m_i\frac{l_j}{l_i}\right)^{1/2}|y_i||y_j|\right\} \\
&\leqslant -\sum_{i=1}^{n}\left\{\grave{\alpha}_i(\lambda_i - m_i w_{ii}) - \frac{1}{2}m_i\left[\sum_{j=1,j\neq i}^{n}\frac{e_j}{e_i}\acute{\alpha}_j|w_{ji}| + \acute{\alpha}_i\sum_{j=1,j\neq i}^{n}\frac{l_j}{l_i}|w_{ij}|\right]\right\}y_i^2.
\end{aligned}$$

When $(e_j\frac{m_j}{l_j})^{1/2}p_j(e_j^{-1/2}y_j)w_{jj}|y_j| + \sum_{i=1,i\neq j}^{n}(e_i\frac{m_i}{l_i})^{1/2}p_i(e_i^{-1/2}y_i)|w_{ij}||y_i| \leqslant 0$, we obtain

$$\langle PFP^{-1}(y), y\rangle \leqslant -\sum_{i=1}^{n}\grave{\alpha}_i\lambda_i y_i^2.$$

Therefore, for all $y \in \mathbb{R}^n$, we derive

$$\langle PFP^{-1}(y), y \rangle \leqslant -\sum_{i=1}^{n} \left\{ \grave{\alpha}_i \lambda_i - m_i \left[ \grave{\alpha}_i w_{ii} + \frac{1}{2} \sum_{j=1, j \neq i}^{n} \left( \frac{e_j}{e_i} \acute{\alpha}_j |w_{ji}| + \frac{l_j}{l_i} \acute{\alpha}_i |w_{ij}| \right) \right]^+ \right\} y_i^2,$$

and thus it follows from (20) that $m_{\mathbb{R}^n}(PFP^{-1}, 0) \leqslant -b_2 < 0$. The rest of the proof is analogous to that of Theorem 3. □

## 4. Conclusions

In this Letter, the exponential stability of Cohen–Grossberg neural networks is investigated. By means of the extended nonlinear measure approach, new mild criteria for the existence of a unique equilibrium and exponential stability of the neural networks are presented. Moreover, the exponential convergence rate of the neural networks to stable equilibrium point is precisely estimated. Compared with those existing results, our criteria require neither the boundedness, differentiability and monotonicity restriction on the activation functions, nor any symmetry assumption of the connection matrix. Therefore, a more extensive application domain for the neural networks is provided.

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