Nonlinear Measures: A New Approach to Exponential Stability Analysis for Hopfield-Type Neural Networks

Hong Qiao, Jigen Peng, and Zong-Ben Xu

***Abstract*—In this paper, a new concept called nonlinear measure is introduced to quantify stability of nonlinear systems in the way similar to the matrix measure for stability of linear systems. Based on the new concept, a novel approach for stability analysis of neural networks is developed. With this approach, a series of new sufficient conditions for global and local exponential stability of Hopfield type neural networks is presented, which generalizes those existing results. By means of the introduced nonlinear measure, the exponential convergence rate of the neural networks to stable equilibrium point is estimated, and, for local stability, the attraction region of the stable equilibrium point is characterized. The developed approach can be generalized to stability analysis of other general nonlinear systems.**

***Index Terms*—Global exponential stability, Hopfield-type neural networks, local exponential stability, matrix measure, nonlinear measures.**

I. Introduction

IN APPLICATION of neural networks either as associative memories (or pattern recognition) or as optimization solvers, the stability of networks is prerequisite. Particularly, when neural networks are employed as associative memories, the equilibrium points represent the stored patterns, and, the stability of each equilibrium point means that each stored pattern can be retrieved even in the presence of noise. While when employed as an optimization solver, the equilibrium points of neural networks correspond to possible optimal solutions, and the stability of networks then ensures the convergence to optimal solutions. Also, stability of neural networks is fundamental for network designs. Due to these, stability analysis of neural networks has received extensive attentions in recent past years (see, for example, [1]–[11]).

To the best of authors' knowledge, the approaches extensively used in the existing investigation into stability of neural networks are mainly those based on Lyapunov' direct method, that is, based on construction of Lyapunov functions. It is known, however, that no general rule can guide how a proper Lyapunov function should be constructed for a given system. Therefore, the construction of Lyapunov function becomes very skillful, and consequently, there is little compatibility among the existing results. In addition, the techniques based on Lyapunov's direct method can neither be used to estimate the convergence rate nor be used to determine the attraction region of stable equilibrium points. (It should be noted that when neural networks applied as associative memories, the attraction region of a stable equilibrium point characterizes the error-correction capability of the corresponding stored pattern, and hence, the identification of such attraction region is fundamental.)

In this paper, our purpose is to quantify the stability of nonlinear systems by introducing several novel qualities similar to the matrix measure for linear systems, and hence, to develop a new approach to stability analysis for nonlinear systems. The developed approach can be immediately applied to the stability analysis (particularly, the exponential stability analysis) of neural networks. We will show that based on the new approach, not only can a series of new sufficient conditions for global and local exponential stability derived, but also the exponential convergence rate of the neural networks to equilibrium points and the attraction region of a stable equilibrium point can be derived.

The model we consider in the present paper is the neural networks modeled by the equations

$$\frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + I_i \qquad i = 1, 2, \ldots, n \tag{1}$$

where

$u_i(t)$	neural voltages;
R_i	resistances;
$W = (w_{ij})$	connection weight matrix;
f_i	transfer functions;
I_i	external inputs.

The model (1) was suggested by Hopfield in [12] and therefore referred to as Hopfield-type neural networks henceforth. The stability of Hopfield-type neural networks has received extensive attentions, due to the fact that some other neural network models can be regarded either as direct generalizations or as extensions of Hopfied-type neural networks (see, e.g., [21]).

The main difficulty for stability analysis of the model (1) comes from the nonlinearity of the transfer functions f_i. Almost all stability analysis of (1) is conducted under some special assumptions on f_i. These assumptions frequently include those such as differentiability, boundedness and/or the monotonic increasing property [1]–[11]. In the early studies (say, e.g., [2], [8], [9], and [11]), the transfer functions f_i are usually assumed to be sigmoidal, that is, each f_i is differentiable, $f_i'(0) \geq f_i'(r) > 0$ for any $r \in R$ and $\lim_{r\to\pm\infty} f_i(r) = \pm 1$. Such a typical sigmoid function, for instance, is given by $f(u) = \tanh(\beta u)$ with some $\beta > 0$. However, in the studies on the neural networks

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(see, e.g., [22] and [23]), the transfer functions are assumed to be neither smooth nor strictly monotonic. This is because it has been shown [14], [15] that when applied as associative memories, the network (1)'s absolute capacity can be remarkably improved by replacing the usual sigmoid transfer functions with nonmonotonic transfer functions. Motivated by these, in recent investigations, the differentiability and/or the monotonicity of f_i are abandoned. Instead, the Lipschitz conditions on f_i are supposed (see, e.g., [1]–[3]).

In the present investigation, we will only assume the following properties of f_i:

(**H**) For each $i = 1, 2, \ldots, n$, f_i is Lipschitz continuous, that is, there is a constant $M_i > 0$ such that

$$|f_i(s) - f_i(t)| \le M_i|s - t| \quad \text{for any } s, t \in R.$$

It should be noted that unlike the most previous investigations that assume the global boundedness of f_i to imply existence of equilibrium point of the networks by degree theory (see, e.g., [4] and [7]), we do not assume the boundedness of f_i at all. This may lead to difficulty in proving existence of equilibrium points. In addition, we do not make any assumptions on the connection matrix $W = (w_{ij})$ which are limited in the previous works to the classes of, such as, symmetry (i.e., for any $i, j = 1, 2, \ldots, n$, $w_{ij} = w_{ji}$), nonself-connection (i.e., $w_{ii} = 0$) or co-operative properties (i.e., $w_{ij} > 0$), see, for example, [8] and [13].

For convenience, we denote by m_i, in what follows, the minimum Lipschitz constant of the transfer function f_i, that is,

$$m_i = \sup_{s,t\in R, s\ne t} \frac{|f_i(t) - f_i(s)|}{|t - s|}. \tag{2}$$

It is easy to see that $m_i = \sup_{s\in R} |f_i'(s)|$ whenever f_i is differentiable on R.

This paper is organized as follows. In Section II, we introduce the nonlinear measure, as a novel generalization of the matrix measure in linear system, to characterize the stability of nonlinear system. Based on the introduced notion, a new useful stability analysis approach for nonlinear systems is developed. In Sections III and IV, the new approach is applied, respectively, to analyze the global and local exponential stability of Hopfield-type neural networks (1). A series of new general sufficient conditions for global and local exponential stability of (1) is presented. The exponential decay estimate and the attraction region are also clearly characterized. This paper then concludes in Section V.

II. NONLINEAR MEASURES

In this section, aiming to quantify the stability of nonlinear system, we introduce the generalization of the matrix measure concept in linear system, and develop a novel approach to stability analysis of nonlinear system. In the subsequent sections, this new approach will be applied to the detailed stability analysis of the networks (1).

Consider the system of the type

$$\frac{dx(t)}{dt} = F(x(t)), \qquad t \ge t_0 \tag{3}$$

where F is a Lipschitz (locally) continuous operators defined on an open subset Ω of R^n, and $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \Omega$.

When Ω is the whole space and F is linear (an $n \times n$ matrix), it is well known that the stability of (3) can be characterized by a quality called the matrix measure $\mu(F)$ of F: If $\mu(F) < 0$, then system (3) is (globally) exponential stable [16], [17].

Our aim here is to extend the matrix-measure criteria for linear system to the nonlinear case. For this purpose, let R^n be the n-dimension real vector space with vector norm $\|\cdot\|$, and recall that the l^1-norm $\|\cdot\|_1$ of R^n is defined, for each vector $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, by

$$\|x\|_1 = \sum_{i=1}^n |x_i|.$$

There are also other kinds of norms defined in R^n. We will always use, however, the l^1-norm throughout the present research.

Given an $n \times n$ matrix A, denote, respectively, by $\|A\|$ and $\mu(A)$ its matrix norm and matrix measure induced by the given vector norm $\|\cdot\|$ of R^n. Then, by definitions

$$\|A\| = \sup_{x\ne 0} \frac{\|Ax\|}{\|x\|}, \quad \mu(A) = \lim_{s\to 0^+} \frac{\|I + sA\| - 1}{s} \tag{4}$$

where I represents the identity matrix on R^n. It should be noted that the matrix norm depends on the endowed norm of R^n, and thereby, so is the matrix measure. When l^1-norm is used, we can particularly show that the norm $\|A\|_1$ and the measure $\mu_1(A)$ of $A = (a_{ij})_{n\times n}$ are given, respectively, by

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$$

and

$$\mu_1(A) = \max_{1\le j\le n} \left(a_{jj} + \sum_{i\ne j}^n |a_{ij}| \right).$$

In order to motivate a nonlinear generalization of the matrix measure, we now make the following important observation: The matrix measure of A can be equivalently calculated by

$$\mu_1(A) = \sup_{x\in R^n, x\ne 0} \frac{\langle Ax, \text{sign}(x)\rangle}{\|x\|_1} \tag{5}$$

where $\langle u, v\rangle$ denotes the inner product of vectors u and v in R^n, and $\text{sign}(x) = (\text{sign}(x_1), \text{sign}(x_2), \ldots, \text{sign}(x_n))^T$ is the sign vector of x, with $\text{sign}(t)$ defined by

$$\text{sign}(t) = \begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -1, & \text{for } t < 0. \end{cases} \tag{6}$$

In fact, observing that, for each $x \in R^n$

$$\begin{cases} \|x\|_1 = \langle x, \text{sign}(x)\rangle & \text{and} \\ \|x\|_1 \ge \langle x, \text{sign}(y)\rangle & \text{for all } y \in R^n \end{cases} \tag{7}$$

we find from (4) that

$$\begin{aligned}\mu_1(A) &= \lim_{s\to 0^+} \frac{\|I+sA\|-1}{s}\\ &= \lim_{s\to 0^+}\sup_{x\neq 0} \frac{\|x+sAx\|_1-\|x\|_1}{s\|x\|_1}\\ &\geq \lim_{s\to 0^+}\sup_{x\neq 0} \frac{\langle x+sAx,\ \text{sign}(x)\rangle-\langle x,\ \text{sign}(x)\rangle}{s\|x\|_1}\\ &\geq \sup_{x\neq 0}\frac{\langle Ax,\ \text{sign}(x)\rangle}{\|x\|_1}. \end{aligned}\tag{8}$$

To derive the inverse inequality, we let

$$r=\sup_{x\neq 0}\frac{\langle Ax,\ \text{sign}(x)\rangle}{\|x\|_1}$$

and notice that, for any $s>0$ and $x\in R^n$

$$\begin{aligned}\|(I-sA)x\|_1 &\geq \langle (I-sA)x,\ \text{sign}(x)\rangle\\ &= \langle x,\ \text{sign}(x)\rangle - s\langle Ax,\ \text{sign}(x)\rangle\\ &\geq (1-sr)\|x\|_1\end{aligned}$$

this implies that whenever $s<r^{-1}$, $(I-sA)$ will be nonsingular and its inverse $(I-sA)^{-1}$ satisfies

$$\left\|(I-sA)^{-1}\right\|_1 \leq \frac{1}{1-sr}.$$

Hence, by (4), we obtain

$$\begin{aligned}&\mu_1(A)\\ &= \lim_{s\to 0^+}\frac{\left\|(I-sA)^{-1}\right\|_1-1+\|I+sA\|_1-\left\|(I-sA)^{-1}\right\|_1}{s}\\ &\leq \lim_{s\to 0^+}\frac{1}{s}\left(\frac{1}{1-sr}-1\right)+\lim_{s\to 0^+}\frac{1}{s}\ \left\|I+sA-(I-sA)^{-1}\right\|_1\\ &\leq r+\lim_{s\to 0^+}\frac{1}{s}\ \left\|\left((I-s^2A^2)-I\right)(I-sA)^{-1}\right\|_1 \leq r,\end{aligned}$$

as expected.

In light of the above observation, we now introduce the following concept.

Definition 1: Suppose Ω is an open set of R^n, F is an operator from Ω into R^n, and $x^0\in\Omega$ is any fixed vector.

1) The constant

$$m_\Omega(F)=\sup_{x,y\in\Omega,x\neq y}\frac{\langle F(x)-F(y),\ \text{sign}(x-y)\rangle}{\|x-y\|_1}\tag{9}$$

is called the nonlinear measure of F on Ω.

2) The constant

$$m_\Omega(F,\,x^0)=\sup_{x\in\Omega,x\neq x^0}\frac{\langle F(x)-F(x^0),\ \text{sign}(x-x^0)\rangle}{\|x-x^0\|_1}\tag{10}$$

is called the relative nonlinear measure of F at x^0.

By this definition, it is clear that $m_\Omega(F,\,x^0)\leq m_\Omega(F)$ for any x^0 and all open set Ω. However, when Ω is specially the whole R^n and F is linear, $\mu_1(A)=m_\Omega(F)=m_\Omega(F,\,x^0)$ for any x^0. Here important is to note that the inequality $m_\Omega(F,\,x^0)<m_\Omega(F)$ may strictly hold for nonlinear operators F (such an example will be presented in Example 1 below).

Similar to the matrix measure, the following properties of the nonlinear measure defined by Definition 1 can readily be proved:

1) $m_\Omega(F+G)\leq m_\Omega(F)+m_\Omega(G)$ for any two functions F, G from Ω into R^n;
2) $m_\Omega(\beta F)=\beta m_\Omega(F)$ for any function F from Ω into R^n and any $\beta\geq 0$;
3) $m_\Omega(\alpha I+F)=\alpha+m_\Omega(F)$ for any function F from Ω into R^n and any real number α, where $\alpha I+F$ denotes the function mapping each vector x onto $\alpha x+F(x)$.

We further prove several useful properties of the nonlinear measure, which will be needed in subsequent applications. In the following, we always assume that F is a nonlinear operator defined on an open set Ω of R^n.

Lemma 1: If $m_\Omega(F)<0$, then F is a one-to-one mapping on Ω [i.e., $F(x)=F(y)$ implies $x=y$]. In addition, if $\Omega=R^n$, then F is a homeomorphism of R^n (that is, F is an one-to-one and onto mapping).

Proof: Suppose $u,\,v\in\Omega$ satisfy $F(u)=F(v)$ but $u\neq v$. Then, by (9), we find

$$\begin{aligned}m_\Omega(F)&=\sup_{x,y\in\Omega,x\neq y}\frac{\langle F(x)-F(y),\ \text{sign}(x-y)\rangle}{\|x-y\|_1}\\ &\geq\frac{\langle F(u)-F(v),\ \text{sign}(u-v)\rangle}{\|u-v\|_1}=0\end{aligned}$$

which contradicts to $m_\Omega(F)<0$. Therefore, the one-to-one property of F follows.

Suppose, furthermore, that Ω is the whole space. Then $m_\Omega(F)<0$ implies that, for all $x,\,y\in\Omega$

$$\begin{aligned}\|F(x)-F(y)\|_1&\geq|\langle F(x)-F(y),\ \text{sign}(x-y)\rangle|\\ &\geq -\langle F(x)-F(y),\ \text{sign}(x-y)\rangle\\ &\geq -m_\Omega(F)\|x-y\|_1.\end{aligned}$$

This shows that, for any fixed y, $\|F(x)\|_1\to\infty$ whenever $\|x\|_1\to\infty$, that is, F is norm-coercive. By norm-coercive theorem [18], thus, F is a homeomorphism of R^n. This implies Lemma 1. □

Lemma 1 particularly implies that $F(x)=0$ will have and only have one solution whenever $\Omega=R^n$ and $m_\Omega(F)<0$, that is, the solution set $F^{-1}(0)$ contains exactly one point in this case.

Lemma 2: If $x^*\in\Omega$ is an equilibrium point of the system (3) and $m_\Omega(F,\,x^*)<0$, then, there is no equilibrium point in Ω other than x^* [i.e., the equilibrium point of (3) is unique in Ω].

Proof: Suppose $u\in\Omega$ is any other equilibrium point of (3) different from x^*, i.e., $F(u)=0$, $u\neq x^*$. Then, by (10)

$$\begin{aligned}m_\Omega(F,\,x^*)&=\sup_{x\in\Omega,x\neq x^*}\frac{\langle F(x)-F(x^*),\ \text{sign}(x-x^*)\rangle}{\|x-x^*\|_1}\\ &\geq\frac{\langle F(u)-F(x^*),\ \text{sign}(u-x^*)\rangle}{\|u-x^*\|_1}=0.\end{aligned}$$

Since $m_\Omega(F, x^*) < 0$, this implies $x^* = u$, contradicting to the assumption. This contradiction shows the uniqueness of equilibrium point of (3) in Ω □.

Lemmas 1 and 2 show that, exactly as the matrix measure that can characterize the uniqueness of equilibrium point of linear system, the nonlinear measure and relative nonlinear measure defined here can also characterize the uniqueness of equilibrium of nonlinear system. In the following, we will further justify that the nonlinear measures can actually characterize the stability of nonlinear systems.

Theorem 1: If $m_\Omega(F) < 0$, then there is at most one equilibrium point of (3) in Ω. Moreover, any two solutions $x(t)$ and $y(t)$ in Ω initiated, respectively, from x^0 and $y^0 \in \Omega$ at t_0 satisfy

$$\|x(t) - y(t)\|_1 \le e^{m_\Omega(F)\cdot(t-t_0)}\|x^0 - y^0\|_1, \quad \text{for all } t \ge t_0. \tag{11}$$

Proof: Since $m_\Omega(F) < 0$, Lemma 1 implies that F is one-to-one in Ω, that is, there is at most one $u \in \Omega$ such that $F(u) = 0$. Accordingly, there is at most one equilibrium point of (3) in Ω.

By (7), we can show that, for all $s > 0$ and $t > s + t_0$

$$\begin{aligned}&\frac{\|x(t) - y(t)\|_1 - \|x(t-s) - y(t-s)\|_1}{s}\\ &\quad\le \frac{\langle (x(t)-y(t)) - (x(t-s) - y(t-s)),\ \text{sign}(x(t)-y(t))\rangle}{s}.\end{aligned}$$

Hence, the derivative of $\|x(t) - y(t)\|_1$ satisfies almost everywhere in interval (t_0, ∞) that (see, for example, [19, Th. 6.9, p. 129])

$$\begin{aligned}\frac{d\|x(t) - y(t)\|_1}{dt} &\le \left\langle \frac{d(x(t) - y(t))}{dt},\ \text{sign}(x(t) - y(t))\right\rangle\\ &= \langle F(x(t)) - F(y(t)),\ \text{sign}(x(t) - y(t))\rangle\\ &\le m_\Omega(F) \cdot \|x(t) - y(t)\|_1. \end{aligned} \tag{12}$$

Integrating the above differential inequality, we then obtain the exponential decay estimation (11). This finishes the proof of Theorem 1. □

Given any two operators F and G we denote by $F \cdot G$ the composition of F and G (whenever it is possible) in the sense that $F \cdot G(x) = F(G(x))$ for all x in the domain of G. By Definition 1, it is easy to verify that $m_\Omega(F) < 0$ if and only if $m_{A(\Omega)}(F \cdot A^{-1}) < 0$ for some diagonal strictly positive matrix A. From this, a corollary immediately follows.

Corollary 1: If $m_{A(\Omega)}(F \cdot A^{-1}) < 0$ for a diagonal matrix $A = \text{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0$, then there is at most one equilibrium point of (3) in Ω. Moreover, any two solutions $x(t)$ and $y(t)$ in Ω initiated, respectively, from x^0 and $y^0 \in \Omega$ at t_0 satisfy

$$\begin{aligned}\|x(t) - y(t)\|_1 \le\, & e^{m_{A(\Omega)}(F\cdot A^{-1}) \min_{l\le i\le n}(a_i)\cdot(t-t_0)}\\ &\cdot \|x^0 - y^0\|_1 \quad \text{for all } t \ge t_0. \end{aligned} \tag{13}$$

Proof: We only need to verify the exponential estimate (13). Similar to (12), we have

$$\begin{aligned}&\frac{d\|x(t) - y(t)\|_1}{dt}\\ &\quad\le \left\langle \frac{d(x(t) - y(t))}{dt},\ \text{sign}(x(t) - y(t))\right\rangle\\ &\quad= \langle F(x(t)) - F(y(t)),\ \text{sign}(A(x(t) - y(t)))\rangle\\ &\quad= \langle F\cdot A^{-1}(Ax(t)) - F\cdot A^{-1}(Ay(t)),\ \text{sign}(Ax(t) - Ay(t))\\ &\quad\le m_{A(\Omega)}(F \cdot A^{-1}) \cdot \|Ax(t) - Ay(t)\|_1\\ &\quad\le m_{A(\Omega)}(F \cdot A^{-1}) \min_{1\le i\le n}(a_i)\ \|x(t) - y(t)\|_1.\end{aligned}$$

Hence, integrating the both sides yields the expected estimate (13). □

Remark 1: In Theorem 1 (Corollary 1), if Ω is the whole space, then, (3) has a unique equilibrium point, and, the exponential decay estimation (13) shows that any trajectory of (3) will eventually evolve to the unique equilibrium point, that is, the system will be globally exponentially stable.

As an immediate consequence, we can claim that (3) is globally exponentially stable if $I + \alpha F$ is contractive with respect to l^1-norm for some positive number α. Indeed, for any positive number α,

$$\begin{aligned}&\|(I + \alpha F)(x) - (I + \alpha F)(y)\|_1\\ &\quad\ge \langle (I + \alpha F)(x) - (I + \alpha F)(u),\ \text{sign}(x - y)\rangle\\ &\quad= \|x - y\|_1 + \alpha\langle F(x) - F(y),\ \text{sign}(x - y)\rangle\end{aligned}$$

holds for all $x, y \in R^n$. Hence, $m_{R^n}(F) < 0$ if $I + \alpha F$ is contractive for some $\alpha > 0$. This shows that Theorem 1 (Corollary 1) is sharper than the well-known contraction mapping approach for stability analysis. Furthermore, we can show that the present nonlinear measure approach can apply to the systems that the contraction mapping approach can not apply to. Such a system is, for instance, given by

$$x'(t) = -x^3(t) - x(t), \qquad t \ge 0.$$

[In fact, let $F(x) = -(1/3)x^3 - x$ and $\Omega = R$, it is then easily seen that the minimal Lipschitz constant of $I + \alpha F$ is infinitely large for any positive number α while $m_R(F) \le -1$.]

Theorem 2: Suppose x^* is an equilibrium of system (3) and Γ is an open l^1-ball centered at x^*. If $m_\Gamma(F, x^*) < 0$, then,

1) any solution $x(t)$ of (3) initiated from $x^0 \in \Gamma$ will be kept in Γ for any $t \ge t_0$;
2) x^* is exponentially stable, with its attraction region containing Γ, and, furthermore, the exponential decay estimation of $x(t)$ is governed by

$$\|x(t) - x^*\|_1 \le \|x^0 - x^*\|_1 \cdot e^{m_\Gamma(F,x^*)(t-t_0)}. \tag{14}$$

Proof: Denote by $x(t)$ the solution of (3) initiated from the vector $x_0 \in \Gamma$. We first prove that $x(t) \in \Gamma$ for all $t > t_0$. Let t_∞ denote the maximum time such that $x(t) \in \Gamma$ for all $t \in (t_0, t_\infty)$, then we only need to prove that t_∞ is infinite.

Suppose this is not, i.e., $t_\infty < +\infty$. The point $x(t_\infty)$ then lies on the boundary of Γ due to the extension theorem of the solution. Similar to (12), we can justify that

$$\frac{d\|x(t)-x^*\|_1}{dt} \le m_\Gamma(F, x^*) \cdot \|x(t)-x^*\|_1$$

for all $t \in (t_0, t_\infty)$. Integrating this differential inequality then yields

$$\|x(t)-x^*\|_1 \le e^{m_\Gamma(F,x^*)(t-t_0)} \cdot \|x^0-x^*\|_1 \qquad (15)$$

for all $t \in (t_0, t_\infty)$. Since $m_\Gamma(F, x^*) < 0$, this immediately implies

$$\|x(t_\infty)-x^*\|_1 < \|x_0-x^*\|_1.$$

So $x(t_\infty)$ can not be on the boundary of Γ because of the openness of Γ, which contradicts to that $x(t_\infty)$ is on the boundary of Γ. Thus, $t_\infty = +\infty$, and the whole trajectory $x(t)$ will be kept in Γ, as claimed.

The rest of Theorem 2 is immediate from (15). This completes the proof. □

Corresponding to Corollary 1, we likewise have the following corollary.

Corollary 2: Suppose x^* is an equilibrium of system (3) and Γ is an open l^1-ball centered at x^*. If $m_{A(\Gamma)}(F\cdot A^{-1}, Ax^*) < 0$ for some $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0$, then,

1) any solution $x(t)$ of (3) initiated from $x^0 \in \Gamma$ will be kept in Γ for any $t \ge t_0$;
2) $x^*$is exponentially stable, with its attraction region containing Γ, and, furthermore, the exponential decay estimation of $x(t)$ is governed by

$$\begin{aligned}&\|x(t)-x^*\|_1\\ &\quad \le \|x^0-x^*\|_1 \cdot e^{m_{A(\Gamma)}(F\cdot A^{-1}, Ax^*)\min_{1\le i\le n}(a_i)(t-t_0)}.\end{aligned} \qquad (16)$$

Remark 2: Theorems 1 and 2 show that if the (relative) nonlinear measure is less than zero, then the system (3) is exponential stable. This conclusion is exactly a direct generalization of matrix criterion of linear system stability.

It should be also noted, however, that we have incorporated into Corollaries 1 and 2 an adjustable parameter matrix A, which makes Theorems 1 and 2 more versatile in concluding stability of nonlinear system in general, and much more facilitating in stability analysis of neural network system in particular. This will be substantiated in the next section.

Remark 3: Since $m_\Gamma(F, x^*) \le m_\Gamma(F)$ for any operator F and open ball Γ centered at x^*, the condition $m_\Gamma(F) < 0$ can sufficiently guarantee the exponential stability of system (3). But, it should be remarked that the relative nonlinear measure $m_\Omega(F, x^*)$ may be strictly less than the nonlinear measure $m_\Omega(F)$, and hence the relative nonlinear measure is sometimes more precise than the nonlinear measure for characterizing the exponential stability of equilibrium point under consideration. The following example serves to clarify this statement.

Example 1: Consider the nonlinear system (3) with the function F specified by

$$F(x) = \sin^2 x - \frac{4\pi x}{\pi^2+4}, \qquad x \in R.$$

Obviously, $x^* = 0$ is an equilibrium point. By Definition 1, we can show that

$$\begin{aligned} m_R(F) &= \sup_{x\ne y} \frac{[F(x)-F(y)]\cdot \mathrm{sign}(x-y)}{|x-y|}\\ &= \sup_{x\ne y} \frac{[\sin^2 x - \sin^2 y]\cdot \mathrm{sign}(x-y)}{|x-y|} - \frac{4\pi}{4+\pi^2}\\ &= 1 - \frac{4\pi}{4+\pi^2} > 0\end{aligned}$$

and

$$\begin{aligned} m_R(F, x^*) &= \sup_{x\ne 0} \frac{F(x)\cdot \mathrm{sign}(x)}{|x|}\\ &= \sup_{x\ne 0} \frac{\sin^2(x)\cdot \mathrm{sign}(x)}{|x|} - \frac{4\pi}{4+\pi^2} < 0.\end{aligned}$$

That is, $m_R(F, x^*) < m_R(F)$ holds. So, by Theorem 2, the equilibrium point $x^* = 0$ will be globally exponentially stable (with $\Gamma = R$). However, the nonlinear measure of F [namely, $m_R(F)$] is strictly larger than $m_R(F, x^*)$, the relative nonlinear measure.

Theorems 1 and 2 show that the (relative) nonlinear measure do have characterized the stability of nonlinear system in the same way as the matrix measure for linear system. This leads to a new approach for stability analysis of nonlinear system. In subsequent sections, we will apply these theorems to conduct stability analysis of Hopfield-type neural networks (1), demonstrating the power of the developed new approach here.

III. Global Exponential Stability of Neural Networks

In this section, we apply the nonlinear measure approach introduced in the last section to analyze the global stability of neural-network system (1). A series of general criteria for the global exponential stability as well as the exponential decay estimation on the solutions of (1) will be provided.

Theorem 3: If there exists a set of real numbers $d_i > 0$ such that

$$\sum_{i=1}^n \frac{d_j}{d_i}|w_{ij}| < (R_j m_j)^{-1}, \qquad j = 1, 2, \ldots, n \qquad (17)$$

then, corresponding to each set of external inputs I_i, the neural network (1) is globally exponential stable, and the exponential decay estimate is governed by

$$\begin{aligned}\|u(t)-x^*\|_1 \le \frac{\max(d_i)}{\min(d_i)}\, &e^{(-b/\max(R_i))(t-t_0)}\\ &\cdot \|u^0-x^*\|_1, \quad \text{for all } t \ge t_0\end{aligned} \qquad (18)$$

where $u(t)$ is any solution of (1) initiated from u^0, and $b = 1 - \max_{1\le j\le n}(m_j R_j \sum_{i=1}^n (d_j/d_i)|w_{ij}|)$.

Proof: Define the operator $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$ on R^n by

$$F_i(x) = -\frac{x_i}{R_i} + \sum_{j=1}^{n} w_{ij} f_j(x_j) + I_i, \qquad i = 1, 2, \ldots, n$$

for any $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, $P = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $A^{-1} = \mathrm{diag}(R_1, R_2, \ldots, R_n)$. It is noted that, for any $x, y \in R^n$, (7) implies

$$\begin{aligned} &\langle P^{-1} \cdot F(PA^{-1}x) - P^{-1} \cdot F(PA^{-1}y), \text{sign}(x-y)\rangle \\ &= -\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} d_i^{-1} \sum_{j=1}^{n} w_{ij} \\ &\quad \cdot (f_j(d_j R_j x_j) - f_j(d_j R_j x_j))\text{sign}(x_i - y_i) \\ &\le -\sum_{i=1}^{n} |x_i - y_i| + \sum_{j=1}^{n} m_j\, d_j R_j |x_j - y_j| \\ &\quad \cdot \sum_{i=1}^{n} d_i^{-1} |w_{ij}| \le -b\|x-y\|_1 \end{aligned} \tag{19}$$

which shows $m_{R^n}(P^{-1}F \cdot P \cdot A^{-1}) \le -b < 0$ because of assumption (17). Therefore, by Lemma 1 and Corollary 1, we conclude that the system

$$\frac{dx(t)}{dt} = P^{-1} \cdot F \cdot P(x(t)), \qquad t \ge t_0 \tag{20}$$

has an unique equilibrium point x^* which is globally exponentially stable, and the exponential decay estimation of the solution $x(t)$ of (20) obeys to

$$\|x(t) - x^*\|_1 \le e^{(-b/\max(R_j))(t-t_0)} \|x^0 - x^*\|_1, \qquad t \ge t_0$$

where $x(t)$ is assumed to be initiated from the vector x^0. Observing that $x(t) = Px(t)$ is clearly the unique solution of (20) whenever $u(t)$ is a solution of (1). Theorem 3 therefore follows. □

Theorem 4: If each transfer function f_i of (1) is monotonically increasing and there exists a set of real numbers $d_i > 0$ such that

$$w_{jj} + \sum_{i \ne j}^{n} \frac{d_j}{d_i} |w_{ij}| < (m_j R_j)^{-1}, \qquad j = 1, 2, \ldots, n \tag{21}$$

then, corresponding to each set of external inputs I_i, the network (1) has a unique equilibrium point x^* which is globally exponential stable, and the exponential decay estimation is given by

$$\|u(t) - u^*\|_1 \le \frac{\max(d_i)}{\min(d_i)} e^{(-b/\max(R_i))(t-t_0)} \|u^0 - u^*\|_1 \qquad t \ge t_0 \tag{22}$$

where $b = 1 - \max(0, \mu)$ with

$$\mu = \max_{1 \le j \le n} \left\{ m_j R_j \left(w_{jj} + \sum_{i \ne j}^{n} \frac{d_j}{d_i} |w_{ij}| \right) \right\}$$

and $u(t)$ is any solution of (1) initiated from u^0.

Proof: Define the operator F and the matrices P, A as in the proof of Theorem 3. Since each f_i is monotonically increasing, we have

$$|f_i(t) - f_i(s)| = (f_i(t) - f_i(s))\text{sign}(a(t-s))$$

for any $t, s \in R$ and $a > 0$. We calculate that, for any $x, y \in R^n$,

$$\begin{aligned} &\langle P^{-1}F(PA^{-1}x) - P^{-1}F(PA^{-1}y), \text{sign}(x-y)\rangle \\ &= -\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} d_i^{-1} \sum_{j=1}^{n} w_{ij} \\ &\quad \cdot (f_j(d_j R_j x_j) - f_j(d_j R_j y_j))\, \text{sign}(x_i - y_i) \\ &\le -\|x-y\|_1 + \sum_{j=1}^{n} [f_j(d_j R_j x_j) - f_j(d_j R_j y_j)] \\ &\quad \cdot \sum_{i=1}^{n} d_i^{-1} w_{ij}\, \text{sign}(x_i - y_i) \\ &\le -\|x-y\|_1 \\ &\quad + \sum_{j=1}^{n} \Bigg\{ w_{jj}\, d_j^{-1} |f_j(d_j R_j x_j) - f_j(d_j R_j y_j)| \\ &\qquad + \sum_{i \ne j}^{n} d_i^{-1} |w_{ij}| \cdot |f_j(d_j R_j x_j) - f_j(d_j R_j y_j)| \Bigg\} \\ &\le -\|x-y\|_1 + \sum_{j=1}^{n} \left[w_{jj} + \sum_{i \ne j}^{n} \frac{d_j}{d_i} |w_{ij}| \right] \\ &\quad \cdot d_j^{-1} |f_j(d_j R_j x_j) - f_j(d_j R_j y_j)|\} \\ &\le -(1 - \max(0, \mu))\|x-y\|_1. \end{aligned}$$

Hence, by using the condition (21), we get $m_{R^n}(P^{-1} \cdot F \cdot P \cdot A^{-1}) \le -b < 0$. Thus, the rest of the proof can be completed similar to that of Theorem 3. □

Remark 4: It is worth emphasizing the significance of incorporation of the adjustable parameters d_i into (17) and (21) in Theorems 3 and 4. This not only makes it possible that many known global exponential stability criteria in the literature can be deduced from Theorems 3 and 4 (through specific choice of the parameters d_i), but also can directly yield some new criteria for global exponential stability of the networks (1). The following corollary provides us with such examples.

Corollary 3: If each transfer function f_i is differentiable and satisfies $0 \le f_i'(r) \le \beta_i$ for all $r \in R$ and some $\beta_i \in R$, then, corresponding to each set of external inputs I_i, the neural network (1) is globally exponential stable provided

1) $w_{jj}\beta_j R_j + \beta_j R_j \sum_{i\neq j}^{n} |w_{ij}| < 1$, or
2) $w_{jj}\beta_j R_j + \sum_{i\neq j}^{n} \beta_i R_i |w_{ij}| < 1$, or
3) $w_{jj}\beta_j R_j + \beta_j \sum_{i\neq j}^{n} R_i |w_{ij}| < 1$, or
4) $w_{jj}\beta_j R_j + R_j \sum_{i\neq j}^{n} \beta_i |w_{ij}| < 1$.

for all $j = 1, 2, \ldots, n$. In any case when one of these conditions is satisfied, the following exponential decay estimation holds:

$$\|u(t) - u^*\|_1 \le \alpha \cdot e^{(-b/\max(R_i))(t-t_0)} \|u^0 - u^*\|_1 \tag{23}$$

where $u(t)$ is any solution of (1) with initial point u^0, u^* is the unique equilibrium point, and

a)
$$\alpha = 1,$$
$$b = 1 - \left[\max_{1\le j\le n}\left\{m_j R_j \left(w_{jj} + \sum_{i\neq j}^{n} |w_{ij}|\right)\right\}\right]^+$$
in case 1)

b)
$$\alpha = \frac{\max(\beta_i R_i)}{\min(\beta_i R_i)}$$
$$b = 1 - \left[\max_{1\le j\le n}\left\{m_j R_j \left(w_{jj} + \sum_{i\neq j}^{n} \frac{\beta_i R_i}{\beta_j R_j} |w_{ij}|\right)\right\}\right]^+$$
in case 2)

c)
$$\alpha = \frac{\max(R_i)}{\min(R_i)},$$
$$b = 1 - \left[\max_{1\le j\le n}\left\{m_j R_j \left(w_{jj} + \sum_{i\neq j}^{n} \frac{R_i}{R_j} |w_{ij}|\right)\right\}\right]^+$$
in case 3)

d)
$$\alpha = \frac{\max(\beta_i)}{\min(\beta_i)}$$
$$b = 1 - \left[\max_{1\le j\le n}\left\{m_j R_j \left(w_{jj} + \sum_{i\neq j}^{n} \frac{\beta_i}{\beta_j} |w_{ij}|\right)\right\}\right]^+$$
in case 4).

Proof: It is easily known that, in this case, each f_i is monotonically increasing and its minimum Lipschitz constant m_i can be upper bounded by β_i (i.e., $m_i \le \beta_i$) because $0 \le f_i'(r) \le \beta_i$ for all $r \in R$. We also can readily check that the condition (21) will hold with $d_i = 1$, $d_i = \beta_i R_i$, $d_i = R_i$ and $d_i = \beta_i$ respectively corresponding to the case 1), 2), 3), and 4). Consequently, the stability and the exponential decay estimation (23) immediately follow from Theorem 4. □

Remark 5: Driessche and Zou [2] recently proved the global stability of (1) under the condition 2) of Corollary 3. But, in their theorem ([2, Th. 3.1]), they additionally require that each transfer function f_i should be bounded, and even so, no convergence rate of the solution of (1) to the unique equilibrium point has been obtained.

Remark 6: Under the same assumptions on the transfer functions with Corollary 3, Fang and Kincaid [3] have ever proved the global exponential stability of (1) but required that

$$\left[w_{jj} + \sum_{i\neq j}^{n} |w_{ij}|\right]^+ \beta_j R_j < 1, \qquad j = 1, 2, \ldots, n \tag{24}$$

where $a^+ = \max\{0, a\}$ for real a. It is seen that these requirements naturally implies the condition 1) in Corollary 3. So their result is a special case of Theorem 4.

Under the assumption that each transfer function f_i satisfies $\max_{1\le i\le n}\sup_{r\in R} f_i'(r) = 1$, Matsuoka [20] has also shown the global stability of the network (1) with $R_i = 1$ but provided

$$\max_{1\le j\le n}\left\{w_{jj} + \sum_{i\neq j}^{n} |w_{ij}|\right\} < 1. \tag{25}$$

It is clear that (25) implies condition 1) of Corollary 3. So his result is also a special case of Theorem 4.

Moreover, it should be noted that in Fang and Kincaid [3] and Matsuoka [20], no explicit estimation on solution decay like (23) is given.

We further present an example to show that Theorems 3 and 4 can yield completely new criteria for stability of (1).

Example 2: Consider the networks

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + 0.9\, f_2(u_2(t)) + I_1 \\ \dfrac{du_2(t)}{dt} = -u_2(t) + 0.55\, f_1(u_1(t)) + I_2 \end{cases} \tag{26}$$

where $f_1(r) = \tanh(2r)$ and $f_2(r) = \tanh(r)$ for all real number r. It is calculated that $0 \le f_1'(r) \le 2$ and $0 \le f_2'(r) \le 1$ for all real number r. Through setting $R_1 = R_2 = 1$; $w_{11} = w_{22} = 0$, $w_{12} = 0.9$, $w_{21} = 0.55$; $\beta_1 = 2$, $\beta_2 = 1$ and $d_1 = 1$, $d_2 = 1.11$, we easily see that the criteria (17) and (21) both are met. Therefore, corresponding to any external inputs I_1 and I_2, the network (26) will be globally exponential stable according to Theorems 3 and 4, and, furthermore, the exponential decay estimation is given by

$$\begin{aligned} &|u_1(t) - u_1^0| + |u_2(t) - u_2^0| \\ &\quad \le 1.11\, e^{-0.001(t-t_0)} \left\{|u_1^0 - u_1^*| + |u_2^0 - u_2^*|\right\} \end{aligned}$$

where $(u_1(t), u_2(t))^T$ is any solution of (26) with initial point $(u_1^0, u_2^0)^T$ at t_0, and $(u_1^*, u_2^*)^T$ is the unique equilibrium point. However, we can check that in this example the conditions (24)

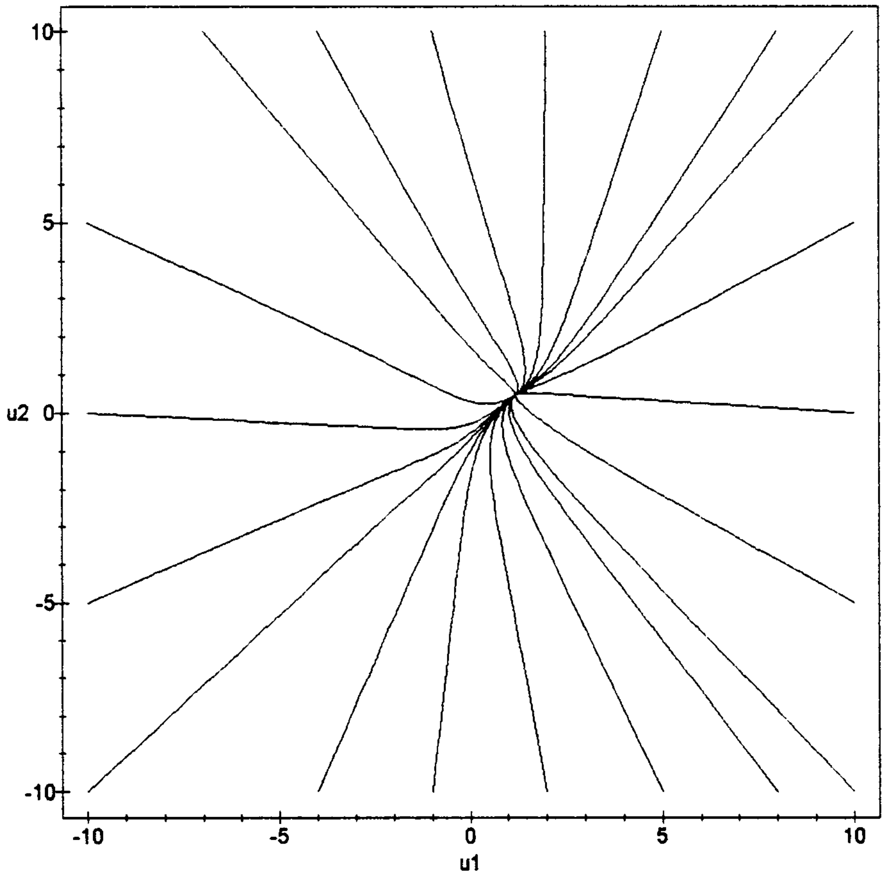

Fig. 1. A phase plane portrait of the trajectories of the neural network defined in Example 2.

and (25) are all not satisfied. See Fig. 1 for the phase plots of (26) with initial points $(-10, -10)$, $(-1, -10)$, $(8, 10)$, etc.

IV. Local Exponential Stability of Neural Networks

As mentioned previously, when neural networks (1) are applied as associative memories, their equilibrium points represent the stored patterns and the stability of any equilibrium point will imply that the corresponding stored pattern can be recalled from some noises. So, the number of stable (locally) equilibrium points characterizes the capacity of the neural networks (1), and, the attraction region of each locally stable equilibrium points characterizes the error-correction capability of neural networks (1) (see, e.g., [2] and [14]). Thus, the local stability analysis of (1) is extremely important (see, e.g., [3]–[6], etc.). In this section, we apply the relative nonlinear measure approach developed in Section II to analyze the local exponential stability of (1).

For convenience, we denote by Ω_i the projection of a subset Ω on the ith axis of R^n in this section.

Theorem 5: An equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of (1) is exponentially stable and its attraction region contains Γ (where Γ is any open l^1-ball centered at u^*), if

$$R_j m_j(\Gamma) \sum_{i=1}^{n} |w_{ij}| < 1, \qquad j = 1, 2, \ldots, n \tag{27}$$

where $m_j(\Gamma)$ is defined by

$$m_j(\Gamma) = \sup_{r \in \Gamma_j, r \neq u_j^*} \frac{|f_j(r) - f_j(u_j^*)|}{|r - u_j^*|}. \tag{28}$$

In this case, the exponential decay estimation of solution of (1) is given by

$$\|u(t) - u^*\|_1 \le e^{(-b/\max(R_i))(t-t_0)} \|u^0 - u^*\|_1, \qquad t \ge t_0 \tag{29}$$

where $b = 1 - \max_{1 \le j \le n} \{R_j m_j(\Gamma) \sum_{i=1}^{n} |w_{ij}|\}$, and $u(t)$ is the solution of (1) initiated from $u^0 \in \Gamma$.

Proof: With F, A defined as in the proof of Theorem 3, we can verify that, for any $x \in A(\Gamma)$,

$$\begin{aligned}&\langle F(A^{-1}x) - F(A^{-1}Au^*),\ \text{sign}(x - Au^*)\rangle \\ &\quad \le -b\|x - Au^*\|_1.\end{aligned}$$

This, together with (27), implies that $m_{A(\Gamma)}(F \cdot A^{-1}, Au^*) \le -b < 0$. Therefore, according to Theorem 2, we conclude that u^* is exponentially stable with Γ as a part of attraction region. The exponential decay estimation (29) is an immediate consequence of (14) with $\min_{1 \le i \le n}(a_i) = \{\max_{1 \le i \le n}(R_i)\}^{-1}$. The proof is then completed. $\square$

Morita [14], and Yoshizawa *et al.* [15] have shown that as an associative memory model the absolute capacity of (1) may be remarkably improved by replacing conventional sigmoid transfer function with the nonmonotonic transfer functions like

$$f(u) = \frac{1-e^{-cu}}{1+e^{-cu}} \cdot \frac{1+ke^{c'(|u|-h)}}{1+e^{c'(|u|-h)}} \tag{30}$$

where c, c' and h are positive constants and k is a negative constant (see, also [2]). Now, we present an example to show how Theorem 5 can be successfully applied to conduct exponential stability analysis of the neural networks with nonmonotonic transfer function defined by (30).

Example 3: Consider the networks

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^{4} w_{ij}\, f(u_j(t)), \qquad i = 1, 2, 3, 4 \tag{31}$$

where f_i are as defined in (30) with the positive constant $c = c' > 1$ and $k = -1$, and the connection matrix W is defined as follows:

$$W = \begin{pmatrix} 0.5 & -0.2 & 0.95 & 0.8 \\ 2 & -3 & 0.1 & 1.2 \\ 0.6 & 0.4 & -0.3 & 0.1 \\ -0.9 & -0.4 & 0.15 & 0.7 \end{pmatrix}.$$

Clearly, $(0, \ldots, 0)^T$ is one of the equilibrium points of (31). With Theorem 5, we consider the stability of this equilibrium point.

Let us define the open ball $\Gamma = \{(x_1, \ldots, x_4)^T \in R^4\colon \sum_{i=1}^4 |x_i| < \delta\}$ with an adjustable parameter $\delta > 0$. Obviously, the projection Γ_i of Γ to the ith axis is the interval $(-\delta, \delta)$ for $i = 1, \ldots, 4$. From (27), we then calculate that

$$\begin{aligned} m_i(\Gamma) &= \sup_{u\in\Gamma_i, u\neq 0} \frac{|f_i(u)|}{|u|} \\ &= \sup_{u\in\Gamma_i, u\neq 0} \frac{1}{|u|} \left| \frac{1-e^{-cu}}{1+e^{-cu}} \cdot \frac{1-e^{c(u-h)}}{1+e^{c(u-h)}} \right| \\ &= \frac{1}{\delta} \max_{u=-\delta,\, \mathrm{or},\, u=\delta} \left| 1 - \frac{2(1+e^{-ch})}{1+e^{-ch}+e^{-cu}+e^{c(u-h)}} \right|. \end{aligned} \tag{32}$$

If δ is chosen such that $m_1(\Gamma) = m_2(\Gamma) < 1/4$, $m_3(\Gamma) < 1/1.5$ and $m_4(\Gamma) < 1/2.8$, then we see that (27) holds for $R_1 = R_2 = R_3 = R_4 = 1$. Consequently, by Theorem 5, $(0, \ldots, 0)^T$ is exponentially stable, its attraction region contains Γ, and, furthermore, any solution $(u_1(t), \ldots, u_4(t))^T$ of (31) initiated from $u^0 = (u_1^0, \ldots, u_4^0)^T \in \Gamma$ satisfies, for all $t \geq t_0$

$$-\delta \leq u_i(t) \leq \delta, \qquad i = 1, 2, 3, 4$$

and

$$\sum_{i=1}^{4} |u_i(t)| \leq e^{-b(t-t_0)} \sum_{i=1}^{4} |u_i^0|$$

where $b = 1 - \max\{m_1(\Gamma), \ldots, m_4(\Gamma)\}$.

The above example demonstrate how Theorem 5 can be applied to provide sufficient conditions for local exponential stability of the neural networks even when all transfer functions are nonmonotonic, and, how the decay rate of the solutions and the attraction region of a stable equilibrium point can be estimated. However, it should be noted that Theorem 5 could be sharpened and improved if all the transfer functions are monotonic. We state such an improvement as the following theorem.

Theorem 6: Suppose u^* is equilibrium point of (1) and Γ is an open l^1-ball centered at u^*. If each transfer function is monotonically increasing on Γ, and

$$m_j(\Gamma) R_j \left(w_{jj} + \sum_{i\neq j}^{n} |w_{ij}| \right) < 1, \qquad j = 1, 2, \ldots, n \tag{33}$$

then, 1) each solution $u(t)$ of (1) is kept in Γ whenever it initialized from a point u^0 in Γ; 2) u^* is an exponential stable equilibrium point with Γ as a portion of the attraction region; 3) the exponential decay estimation of the solution $u(t)$ is governed by (29) with $b = 1 - \max\{0, \mu\}$ and $\mu = \max_{1\leq j\leq n} \{m_j(\Gamma) R_j (w_{jj} + \sum_{i\neq j}^n |w_{ij}|)\}$.

Proof: Define the operator F and the matrix A as in the proof of Theorem 3. Then, similar to the proof of Theorem 4, we can show that, for any $x \in A(\Gamma)$

$$\left\langle F(A^{-1}x) - F(A^{-1}Au^*),\ \mathrm{sign}(x-u^*) \right\rangle \leq -b\|x - Au^*\|_1.$$

So, by (10) and (32), $m_{A(\Gamma)}(FA^{-1}, Au^*) \leq -b < 0$. From this estimation, all the conclusions are readily followed from Corollary 2. The proof is completed. □

We remark here that, in the most previous investigations, the differentiability of each transfer functions is necessarily supposed in order that either Lyapunov approach or the classical linearization method could be adopted. As compared, no any differentiability of the transfer functions is assumed in our Theorems 5 and 6. This reveals the promising of the new approach developed. However, we notice that whenever f_i is differentiable, Theorem 5 can be further improved and simplified.

Corollary 4: Suppose $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ is an equilibrium point of (1) and each transfer function f_i is differentiable in the interval $(u_i^* - \delta, u_i^* + \delta)$ for some $\delta > 0$, and $0 \leq f_i'(r) \leq \beta_i$ for all $r \in (u_i^* - \delta, u_i^* + \delta)$ and some $\beta_i > 0$. If

$$\beta_i R_i \left(w_{jj} + \sum_{i\neq j}^{n} |w_{ij}| \right) < 1, \qquad j = 1, 2, \ldots, n \tag{34}$$

then, 1) any solution $u(t)$ of (1) lies in the region $\Gamma = \{u \in R^n\colon \sum_{i=1}^n |u_i - u_i^*| < \delta\}$ whenever it is initiated from $u^0 \in \Gamma$; 2) u^* is exponentially stable, with Γ as a part of its attraction region; and 3) the exponential decay estimation of the solution is given by

$$\|u(t) - u^*\| \leq e^{(-b/\max(R_i))(t-t_0)} \|u^0 - u^*\|_1, \qquad t \geq t_0 \tag{35}$$

where $b = 1 - \max\{0, \mu\}$ and $\mu = \max_{1\leq j\leq n} \{\beta_j R_j (w_{jj} + \sum_{i\neq j}^n |w_{ij}|\}$.

Proof: It follows from $0 \leq f_i'(r) \leq \beta_i$ and the monotonic increasing property of f_i that $m_i(\Gamma) \leq \beta_i$ [where $m_i(\Gamma)$ is defined as (27) with $\Gamma_i = (\mu_i^* - \delta, u_i^* + \delta)$]. So Theorem 5 directly implies Corollary 3. □

Remark 7: The above theorems not only characterize the local stability of equilibrium point under consideration, but also tell us whether Γ is a part of attraction region of stable equilibrium point or not. So, when the neural network (1) are applied as associative memories, their error-correction capacity can be roughly characterized.

V. Conclusion

Through generalizing the matrix measure concept to nonlinear operator case, we have defined two important qualities called the nonlinear measure and the relative nonlinear measure of a nonlinear operator. We have shown that these two qualities can be applied to quantify stability of nonlinear systems in a way similar to the matrix measure for stability of linear systems, leading to a novel and powerful approach to the stability analysis of neural networks. With the new approach, we have formulated a series of new generic sufficient conditions for global and local exponential stability of Hopfield type neural networks, which generalize most existing criteria. Moreover, we have employed the new approach to estimate the decay rate of the solution of neural networks to its stable equilibrium points, and to characterize the attraction region of the local stable equilibrium point. In most previous investigations, no such convergence rate and attraction region identification was made for Hopfield type neural networks.

The significance of the new developed approach can be summarized as follows. 1) It can be applied to characterize both global and local stability of Hopfield-type continuous neural networks even if the transfer functions are neither differentiable nor monotonic. 2) It can yield very generic global stability conditions with a set of adjustable parameters (cf. Theorem 3 and Theorem 4) which then often offer much more generic stability criteria than any other approaches. 3) It provides the useful means of quantifying the error-correction capability of the stored pattern by directly evaluating the related minimum Lipschitz constants of the transfer functions when the networks are applied as associative memories. 4) It is not only valid for the neural network system of Hopfield-type (1), but also for other general nonlinear system (see, for the generalized Hopfield-type neural networks

$$\frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij}\, f_j\left(\sum_{k=1}^{n} \beta_k u_k(t)\right) + I_i, \quad i = 1, 2, \ldots, n).$$

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