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Chaos, Solitons and Fractals 31 (2007) 424–436

CHAOS
SOLITONS & FRACTALS

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A new stability criterion for discrete-time neural networks: Nonlinear spectral radius

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Accepted 30 September 2005

Abstract

In this paper, the exponential stability of nonlinear discrete-time systems is studied. A novel notion of nonlinear spectral radius is defined. Under the assumption of Lipschitz continuity for the activation function, the developed approach is applied to stability analysis of discrete-time neural networks. A series of sufficient conditions for global exponential stability of the neural networks are established and an estimate of the exponential decay rate is also derived for each case.

1. Introduction

The stability of recurrent neural networks remains to be an important issue in the design of such networks for engineering applications. When neural networks are employed as associative memories [1,2], the equilibrium points of the networks correspond to the stored patterns, and the stability of each equilibrium point represents the ability of the networks to recall the corresponding stored pattern in the presence of noise. Therefore, when the storing capacity is fixed, the local stability and the magnitude of the corresponding attraction basin should be considered carefully by system designers. When neural networks are employed as an optimization solver [3–5], the equilibrium points of the networks correspond to the possible optimal solutions, and the stability of the networks, thus, ensures that the optimization process converges to the optimal solution. In order to avoid the presence of "spurious" attractors, the networks should be designed in such a way that the process is globally stable. Hence, stability analysis of neural networks has received extensive attentions during the past decades for various disciplines [6–9].

While stability analysis of continuous-time neural networks can employ the stability theory of differential equations [10–13], it is much harder to study the stability of discrete-time (analog) neural networks [14–16] and with time delays [17–21] or impulses [22–24]. The techniques currently available in the literature for discrete-time systems are mostly based on the contraction mapping method [25] or Liapunov's direct method [26]. The main idea of the first method

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2005.09.075

is to estimate a contraction constant for the activation dynamics with respect to some metrics endowed to the state space. Hence, stability conditions obtained with this method usually vary with the topology of the state space. For Liapunov's direct method, it is well known that no general rule exists to guide the construction of a proper Liapunov function for a given system. In fact, the construction of the Liapunov function becomes a very difficult task (e.g., constructing the Liapunov function for networks with asymmetric weight matrix). In addition, the techniques based on Liapunov's direct method can neither be used to estimate the convergence rate nor to determine the attraction basin of stable equilibrium points.

The discrete-time neural networks considered in this paper can be conveniently modeled by the difference equations:

$$x_i^{k+1} = f_i\left(\sum_{j=1}^{n} w_{ij}x_j^k + I_i\right), \qquad i = 1,2,\ldots,n, \quad k = 1,2,\ldots, \tag{1}$$

where x_i^k are the neural states at time k, f_i the activation functions, w_{ij} the synaptic weights, and I_i the threshold parameters. All the activation functions are continuous and the updating is conducted in parallel. Set $x^k = (x_1^k, x_2^k, \ldots, x_n^k)^t$, $I = (I_1, I_2, \ldots, I_n)^t$, $W = (w_{ij})_{n\times n}$, and $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^t$, where the superscript t represents the transpose. Eq. (1) can be reduced to the compact form:

$$x^{k+1} = f(Wx^k + I), \quad k = 1,2,\ldots. \tag{2}$$

In previous studies, the activation functions f_i are often assumed to be either saturation functions or sigmoidal functions (i.e., f_i is bounded and differentiable, $f_i'(0) > f_i'(t)$ for all $t \in \hbar$, and $\lim_{t\to\pm\infty} f_i(t) = \pm 1$), and the weight matrix W is frequently assumed to be symmetric [1]. These assumptions may reduce the performance and capability of the neural networks. Indeed, Yoshizawa et al. [2] had shown that when an associative memory model is considered, the absolute capacity of (1) is remarkably improved if the activation functions f_i are properly chosen as nonmonotonic functions instead of the conventional sigmoid functions. It should be pointed out that it is difficult to prove the existence of equilibrium points of neural dynamics without the boundedness assumption on the activation functions; it is because the Brouwer theorem on the existence of fixed points is no longer available. In addition, it is very difficult to construct a proper Liapunov function for the network model described by Eq. (1) without the symmetry assumption on the weight matrix.

In this paper, a novel spectral radius criterion is developed for stability analysis of nonlinear discrete-time systems. The proposed approach is applied to analyze the stability of discrete-time neural networks. A series of sufficient conditions to ensure global exponential stability of discrete-time neural networks are established. In the proposed approach, we impose no restriction on the weight matrix and we assume only that the activation function satisfies the following general conditions:

(**H**) Each activation function f_i is Lipschitz continuous. Hence, there exists a positive constant l_i such that

$$|f_i(x) - f_i(y)| \leqslant l_i \cdot |x - y|, \qquad \forall x, \quad y \in \hbar. \tag{3}$$

The constant l_i in Eq. (3) is commonly called the Lipschitz constant of f_i. It is evident that the Lipschitz constant of a Lipschitz function is not unique, and that the smallest Lipschitz constant of f_i, denoted by β_i, can be evaluated by the equation

$$\beta_i = \sup_{s,t\in\hbar, s\neq t} \frac{|f_i(s) - f_i(t)|}{|s - t|}. \tag{4}$$

It follows that if f_i is continuously differentiable, then $\beta_i = \sup_{r\in\hbar} |f_i'(r)|$. Hence, the smallest Lipschitz constant can be easily computed for most frequently used activation functions. For example, the smallest Lipschitz constant of a saturation function equals to 1, and the smallest Lipschitz constant of the sigmoidal function $f(t) = \tanh(\beta t)$ equals to β.

The remainder of the paper is organized as follows. In Section 2, the concept of nonlinear spectral radius is introduced for establishing the exponential stability of nonlinear discrete-time systems. This approach is then applied to the stability analysis of discrete-time neural networks in Section 3. A series of results on the global exponential stability of discrete-time neural networks are established, and an estimate of the exponential decay rate is also formulated for each case. The conclusion is given in Section 4.

2. Nonlinear spectral radius criterion

An equilibrium point of the discrete-time system governed by the following difference equations:

$$x^{k+1} = F(x^k), \qquad x^0 \in \Omega, \quad k = 0,1,2,\ldots \tag{5}$$

is the state vector $x^* \in \Omega$ such that $F(x^*) = x^*$ (i.e., x^* is a fixed point of F), where Ω is a closed convex set of the real n-dimensional space $\hbar^n$ endowed with the vector norm $\|\cdot\|$, F is a general mapping from Ω into itself, and $x^k \in \Omega$ is the state vector at time k. An equilibrium point x^* is said to be exponentially asymptotic stable with attraction basin $H(x^*) \subset \Omega$ if there exist two positive constants r and M such that every solution x^k of system (5) initiated from $x^0 \in H(x^*)$ evolves to x^* in the manner that $\|x^k - x^*\| \leqslant M\mathrm{e}^{-rk}\|x^0 - x^*\|$ for all $k = 1, 2, \ldots$. The constant r is commonly called the exponential decay rate. For convenience, system (5) is defined as *globally exponential stable in* Ω if it possesses a unique equilibrium point that is exponentially asymptotic stable with the attraction basin Ω. This section is devoted to the analysis of global exponential stability of system (5) under the general assumption:

(H′) *F is Lipschitz continuous in* Ω. *Hence, there exists a positive constant* l_F *called the Lipschitz constant of F, such that*

$$\|F(x) - F(y)\| \leqslant l_F \cdot \|x - y\|, \quad \forall x, \ y, \in \Omega. \tag{6}$$

It can be clearly seen that F becomes a contraction mapping with respect to the metric $d(x, y) = \|x - y\|$ as long as $l_F < 1$. In view of the famous Banach's theorem on contraction mapping [27], this implies that system (5) is globally exponential stable if the Lipschitz constant satisfies $l_F < 1$. However, it is not rigorous enough to employ the Lipschitz constant to characterize the stability of system (5) since the Lipschitz constant depends on the vector norm $\|\cdot\|$ of $\hbar^n$. This can be demonstrated easily by using the linear case, in which $\Omega = \hbar^n$ and F is governed by a certain $n \times n$ matrix A (i.e., $F(x) = Ax$). Then, F is Lipschitz continuous and with $\|A\|$, the matrix norm of A, being the smallest Lipschitz constant of F. This "smallest" seems to indicate that nothing can be said about the stability of system (5) provided that $\|A\| \geqslant 1$. However, it is well known that the global exponential stability of a linear system can be characterized by the spectral radius of its coefficient matrix. Unlike matrix norm, spectral radius is independent of the linear topology endowed to $\hbar^n$, and in most cases is strictly less than the matrix norm. Therefore, it is important to develop a new way to characterize the stability of nonlinear system (5) more precisely than just using the Lipschitz constant.

In view of assumption (H′) and the Rademacher theorem [28], F is absolutely continuous and hence differentiable almost everywhere in Ω. Set

$$\Pi_\Omega(F) = \{F'(x) : x \in \Gamma_\Omega(F)\}, \tag{7}$$

where $\Gamma_\Omega(F) = \{x \in \Omega : F \text{ is differentiable at } x\}$. Hence, $\|A\| \leqslant l_F$ for all $A \in \Pi_\Omega(F)$.

It is clear that if F is induced by an $n \times n$ matrix A, then $\Pi_\Omega(F) = \{A\}$, a single-point set. The formula for computing the spectral radius $\rho(A)$ of the matrix A is [29]

$$\rho(A) = \limsup_{k \to +\infty} \|A^k\|^{\frac{1}{k}} \tag{8}$$

which can be reduced to

$$\rho(A) = \limsup_{k \to +\infty} \sup_{A_i \in \Pi_{\hbar^n}(A)} \|A_1 A_2 \cdots A_k\|^{\frac{1}{k}}. \tag{9}$$

Definition 1. Let $\Pi_\omega(F)$ be defined as in Eq. (7). The nonlinear spectral radius of F with respect to Ω is defined as follows:

$$\rho_\Omega(F) = \lim \sup_{k \to +\infty} \sup_{A_i \in \Pi_\Omega(F)} \|A_1 A_2 \cdots A_k\|^{\frac{1}{k}}. \tag{10}$$

In particular, when $\Omega = \hbar^n$, $\rho_\Omega(F)$ is represented by $\rho(F)$.

It can easily be shown that the nonlinear spectral radius possesses the following two basic properties: (1) $\rho_\Omega(F) \leqslant l_F$ (since $\|A\| \leqslant l_f$ for all $A \in \Pi_\Omega(F)$), and (2) it does not depend on the linear topology of $\hbar^n$, i.e., for any vector norm $\|\cdot\|_*$ of $\hbar^n$ the equation $\rho_\Omega(F) = \limsup_{k \to +\infty} \sup_{A_i \in \Pi_\Omega(F)} \|A_1 A_2 \cdots A_k\|_*^{\frac{1}{k}}$ always holds, where $\|A\|_*$ represents the matrix norm of A with respect to the vector norm $\|\cdot\|_*$.

We prove in the following that the nonlinear spectral radius of F can be used to characterize the global exponential stability of system (5):

Lemma 1. *For any pair of* $x, y \in \Omega$, *there exists an* $n \times n$ *matrix* $A_{xy} \in \overline{Co}^{\text{cl.}} \Pi_\Omega(F)$ *such that* $F(x) - F(y) = A_{xy}(x - y)$, *where* $\overline{Co}^{\text{cl.}} G$ *represents the closed convex hull of set G.*

Proof. Let $f(t) = F(tx + (1 - t)y)$, $0 \leqslant t \leqslant 1$. Then, as a Lipschitz continuous real function in the interval [0,1], f is differentiable almost everywhere in [0, 1], and satisfies that $f'(t) = F'(tx + (1 - t)y)(x - y)$. Integrating both sides from 0 to 1 thus yields that $F(x) - F(y) = f(1) - f(0) = \int_0^1 F'(tx + (1 - t)y)\,\mathrm{d}t(x - y)$. Therefore, with $A = \int_0^1 F'(tx + (1 - t)y)\,\mathrm{d}t$, the proof is completed. □

Theorem 1. *Let* $\varepsilon > 0$ *and* $r_\varepsilon = \rho_\Omega(F) + \varepsilon$. *There exists a positive constant* M_ε *such that*

$$\|x^k - y^k\| \leqslant M_\varepsilon \cdot r_\varepsilon^k \cdot \|x^0 - y^0\|, \quad k = 1, 2, \ldots, \tag{11}$$

where x^k *and* y^k *are any two solutions of Eq. (5) with initial values equal to* x^0 *and* y^0, *respectively. Particularly, if* $\rho_\Omega(F) < 1$, *the system governed by Eq. (5) is globally exponential stable in* Ω, *and the exponential decay rate*, r, *satisfies that* $0 < r < -\ln\rho_\Omega(F)$ *and that*

$$\|x^k - x^*\| \leqslant M \cdot \mathrm{e}^{-rk} \cdot \|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{12}$$

where x^k *is any solution of Eq. (5) initiated from* x^0, x^* *is the unique equilibrium point of system (5), and* M *is a positive constant related to* r *and the vector norm* $\|g\|$.

Proof. Let $\Pi^0 = \{I\}$ and $\Pi^k = \{A_1A_2\cdots A_k : A_i \in \Pi_\Omega(F)\}$ for all $k = 1, 2, \ldots$, where I is the identity matrix. In view of Eq. (10), an integer k_ε can be found such that $\|B_k\| \leqslant (\rho_\Omega(F) + \varepsilon)^k$ for all $B_k \in \Pi^k$ whenever $k \geqslant k_\varepsilon$. Hence, the following positive function

$$\|x\|_\varepsilon = \sup_{B_k \in \Pi^k, k=0,1,\ldots} \frac{\|B_k x\|}{(\rho_\Omega(F) + \varepsilon)^k} \tag{13}$$

is defined well everywhere in $\hbar^n$. Indeed, the relation

$$\|x\| \leqslant \|x\|_\varepsilon \leqslant \max_{k=1,2,\ldots,k_\varepsilon} \left\{1, \frac{l_F^k}{(\rho_\Omega(F) + \varepsilon)^k}\right\} \cdot \|x\| \tag{14}$$

always holds for all $x \in \hbar^n$. Moreover, it can easily be shown that $\|\cdot\|_\varepsilon$ is really a vector norm of $\hbar^n$, i.e., $\|\cdot\|_\varepsilon$ possesses the following three properties: (1) $\|x\|_\varepsilon = 0$ if and only if $x = 0$, (2) $\|\lambda x\|_\varepsilon = |\lambda|\|x\|_\varepsilon$ for all $\lambda \in \hbar$ and $x \in \hbar^n$, and (3) $\|x + y\|_\varepsilon \leqslant \|x\|_\varepsilon + \|y\|_\varepsilon$ for all pairs of $x, y \in \hbar^n$.

For any given pair of $x, y \in \Omega$, it follows from Lemma 1 that there exists an $n \times n$ matrix $A_{xy} \in \overline{Co}^{\mathrm{cl}}\Pi_\Omega(F)$ such that $F(x) - F(y) = A_{xy}(x - y)$. Let $A_i \in \Pi_\Omega(F)$ and $\alpha_i > 0$, $i = 1, 2, \ldots, l$, such that $\sum_{i=1}^L \alpha_i = 1$ and $\sum_{i=1}^l \alpha_i A_i \to A_{xy}$ as $l \to +\infty$. Then, it can be shown by using Eq. (13) that

$$\begin{aligned} \|F(x) - F(y)\|_\varepsilon &= \|A_{xy}(x - y)\|_\varepsilon = \sup_{B_k \in \Pi^k,\ k=0,1,\ldots} \frac{\|B_k A_{xy}(x - y)\|}{(\rho_\Omega(F) + \varepsilon)^k} = \sup_{B_k \in \Pi^k,\ k=0,1,\ldots} \lim_{l \to +\infty} \frac{\|\sum_{i=1}^l \alpha_i B_k A_i(x - y)\|}{(\rho_\Omega(F) + \varepsilon)^k} \\ &\leqslant \lim_{l \to +\infty} \sup_{B_k \in \Pi^k,\ k=0,1,\ldots} \frac{\sum_{i=1}^l \alpha_i \|B_k A_i(x - y)\|}{(\rho_\Omega(F) + \varepsilon)^k} \leqslant \lim_{l \to +\infty} \sum_{i=1}^l \alpha_i \sup_{B_{k+1} \in \Pi^{k+1},\ k=0,1,\ldots} \frac{\|B_{k+1}(x - y)\|}{(\rho_\Omega(F) + \varepsilon)^k} \\ &\leqslant (\rho_\Omega(F) + \varepsilon) \cdot \|x - y\|_\varepsilon. \end{aligned} \tag{15}$$

Hence, if x^k and y^k are the solutions of Eq. (5) with initial values equal to x^0 and y^0, respectively, the relation $\|x^k - y^k\|_\varepsilon = \|F^k(x^0) - F^k(y^0)\|_\varepsilon \leqslant (\rho_\Omega(F) + \varepsilon)^k \cdot \|x^0 - y^0\|_\varepsilon$ holds for all $k = 0, 1, 2, \ldots$. This relation together with relation (14) yields relation (11).

Let $0 < \varepsilon < 1 - \rho_\Omega(F)$ if $\rho_\Omega(F) < 1$, then $r_\varepsilon = \rho_\Omega(F) + \varepsilon < 1$. In view of relation (15), F is strictly contractive with respect to the metric $d(x, y) = \|x - y\|_\varepsilon$. Based on Banach's theorem on contraction mapping, F has a unique fixed point x^*. Therefore, substituting x^* for y^k in relation (11) yields relation (12). □

Remark 1. Although Theorem 1 shows that the nonlinear spectral radius introduced in this paper is an effective means to characterize the global exponential stability of nonlinear systems, it is not easy to compute, especially when F is not smooth. It is therefore necessary to establish appropriate estimates for the nonlinear spectral radius. The following theorems are presented and proved for such a purpose.

Theorem 2. *If there exists a positive constant such that, α for all $A_1, A_2, \ldots, A_k \in \Pi_\Omega(F)$, $k = 1, 2, \ldots$, and*

$$\rho(A_1A_2\cdots A_k) \leqslant \alpha\cdot\rho(A_1)\rho(A_2)\cdots\rho(A_k), \tag{16}$$

then $\rho_\Omega(F) \leqslant sup\{\rho(A): A \in \Pi_\Omega(F)\}$.

Proof. Let $r = \sup\{\rho(A): A \in \Pi_\Omega(F)\}$, and Π^k be defined as in the proof of Theorem 1. Relation (16) clearly indicates that $\rho(B) \leqslant \alpha\cdot r^k$ for all $B \in \Pi^k$ and $k = 0, 1, 2, \ldots$. It follows that

$$\rho_* =: \limsup_{k\to+\infty} \sup_{B\in\Pi^k} \rho(B)^{\frac{1}{k}} \leqslant r. \tag{17}$$

Berger and Wang [30] showed that

$$\limsup_{k\to+\infty} \sup_{A_i\in\Gamma} \rho(A_1A_2\cdots A_k)^{\frac{1}{k}} = \limsup_{k\to+\infty} \sup_{A_i\in\Gamma} \|A_1A_2\cdots A_k\|^{\frac{1}{k}} \tag{18}$$

if Γ is a bounded set of matrices. Therefore, it follows from Eqs. (17) and (18) that $\rho_\Omega(F) \leqslant r$ since $\Pi_\Omega(F)$ is bounded. □

Remark 2. It follows from Theorems 1 and 2 that the spectral radius of every Jacobian matrix of F can characterize the exponential stability of system (5) provided that relation (16) is satisfied. This partially proves the famous LaSalle' conjecture [31] that system (5) is globally asymptotic stable if F possesses a unique fixed point and is continuously differentiable with $\rho(F'(x)) < 1$ for all $x \in \hbar^n$. However, it should be noted that Theorems 1 and 2 do not (1) require F to have a unique fixed point nor (2) assume that F is continuously differentiable. Indeed, relation (16) can be tested easily, although it does not always hold. The following corollary lists two practical cases in which relation (16) holds.

Corollary 1. *If either condition* (*a*) *or condition* (*b*) *shown below is satisfied, then $\rho_\Omega(F) \leqslant \sup\{\rho(A): A \in \Pi_\Omega(F)\}$.*

(a) *$\Pi_\Omega(F)$ is commutative, i.e., $AB = BA$ for all pairs of A, $B \in \Pi_\Omega(F)$;*
(b) *F is triangular, i.e., $\forall x - (x_1, x_2, \ldots, x_n)^t \in \Omega$,*

$$F(x) = (F_1(x_1), F_2(x_1, x_2), \ldots, F_n(x_1, x_2, \ldots, x_n))^t, \text{ or}$$

$$F(x) = (F_1(x_1, x_2, \ldots, x_n), F_2(x_2, x_3, \ldots, x_n), \ldots, F_n(x_n))^t.$$

Proof. In view of Theorem 2, it suffices to show that both conditions (a) and (b) can yield inequality (16).

(a) If $\Pi_\Omega(F)$ is commutative, then the property of matrix spectrum [29] indicates that $\rho(AB) \leqslant \rho(A)\rho(B)$ for all A, $B \in \Pi_\Omega(F)$. This clearly implies that, for all $k = 1, 2, \ldots$ and A_1, $A_2, \ldots, A_k \in \Pi_\Omega(F)$, $\rho(A_1A_2\cdots A_k) \leqslant \rho(A_1)\rho(A_2)\cdots\rho(A_k)$ Hence, inequality (16) follows with $\alpha = 1$.
(b) If F is triangular, then each Jacobian matrix $F'(x)$ is also triangular. Since the product of any two triangular matrices is still triangular, and the spectral radius of a triangular matrix equals to the maximum modulus of its diagonal entries, it can easily be verified that inequality (16) holds for the triangular function F. □

It can be further proved that if F is triangular, system (5) will be globally exponential stable in Ω as long as $r := \sup\{|\frac{\partial F_i(x)}{\partial x_i}| : x \in \Omega, 1 \leqslant i \leqslant n\} < 1$. In fact, if F is triangular, the spectral radius of every Jacobian matrix $F'(x)$ can be calculated by the formula $\rho(F'(x)) = \sup_{1\leqslant i\leqslant n}|\frac{\partial F_i(x)}{\partial x_i}|$. That is, $\sup\{\rho(A) : A \in \Pi_\Omega(F)\} = \sup\{|\frac{\partial F_i(x)}{\partial x_i}| : x \in \Omega, 1 \leqslant i \leqslant n\}$. When $r < 1$, we can ensure the global exponential stability of system (5) by using Theorems 1 and 2.

In the subsequent discussions, a comparative method to estimate the nonlinear spectral radius will be established. For an $n \times n$ matrix $A = (a_{ij})$, let $|A|$ denote the nonnegative matrix $(|a_{ij}|)_{n\times n}$, and $A \leqslant B$ imply $a_{ij} \leqslant b_{ij}$ for all i, $j = 1, 2, \ldots, n$.

Theorem 3. *If there exists a positive $n \times n$ matrix A_0 such that $|A| \leqslant A_0$ for all $A \in \Pi_\Omega(F)$, then $\rho_\Omega(F) \leqslant \rho(A_0)$.*

Proof. Wielandt's lemma [32] indicates that for any pair of matrices A and B, with A being nonnegative, if $|B| \leqslant A$, then $\rho(B) \leqslant \rho(A)$. Since $|A| \leqslant A_0$ for all $A \in \Pi_\Omega(F)$, it is not difficult to verify that $|A_1A_2\cdots A_k| \leqslant A_0^k$ for all A_1, $A_2, \ldots, A_k \in \Pi_\Omega(F)$ and $k = 1, 2, \ldots$. Hence, it follows from Wielandt's lemma that $\rho_* =: \limsup_{k\to+\infty}\sup_{A_i\in\Pi_\Omega(F)}\rho(A_1A_2\cdots A_k)^{\frac{1}{k}} \leqslant \rho(A_0)$. In view of Eq. (18), it can be shown that $\rho_\Omega F \leqslant \rho(A_0)$. □

Let $a_{ij} = \sup_{x\in\Gamma_\Omega(F)}|\frac{\partial F_i}{\partial x_j}(x)|$, and $A_0 = (a_{ij})_{n\times n}$, where $\Gamma_\Omega(F)$ is defined in relation (7). Then, $|A| \leqslant A_0$ for all $A \in \Pi_\Omega(F)$. Hence, it can be shown that $\rho_\Omega(F) \leqslant \rho(A_0)$ due to Theorem 3. This shows that the global exponential stability of system (5) in Ω can be determined by the quantity $\rho(A_0)$. Therefore, a simple and practical criterion for the exponential stability of nonlinear system (5) is derived.

3. Global exponential stability of neural networks

In this section, the novel nonlinear spectral radius approach is applied to analyze the global exponential stability of discrete-time neural networks (1). A series of sufficient conditions for achieving global exponential stability of the system governed by Eq. (1) are provided.

Theorem 4. *Let* $\Delta = \{\mathrm{diag}(r_1, r_2, \ldots, r_n): |r_i| \leqslant \beta_i\}$, *where* β_i *is given in Eq.* (4). *If*

$$\alpha = \limsup_{k\to+\infty} \sup_{D_i\in\Delta} \|D_1WD_2W\cdots D_kW\|^{\frac{1}{k}} < 1 \tag{19}$$

then, for any threshold parameter I_i, $i = 1, 2, \ldots, n$, *the system governed by Eq.* (1) *is globally exponential stable, and the corresponding exponential decay rate,* r, *satisfies* $0 < r < -\ln\alpha$ *and*

$$\|x^k - x^*\| \leqslant M\cdot e^{-rk}\cdot\|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{20}$$

where x^k *is any solution of Eq.* (1) *starting from* x^0, x^* *is the unique equilibrium point, and M is a positive constant related to r and the vector norm* $\|\hbar\|$.

Proof. Let $\Omega = \hbar^n$ and $F:\Omega\to\Omega$ be defined as $F_i(x) = f_i(\sum_{j=1}^n w_{ij}x_j + I_i)$, $i = 1, 2, \ldots, n$, where $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^t$, $x = (x_1, x_2, \ldots, x_n)^t \in \Omega$. Then Eq. (1) can be reduced to Eq. (5). Since each f_i is Lipschitz continuous, it is easy to show that F is also Lipschitz continuous in Ω, i.e., F satisfies assumption (H′). Moreover, it can be easily verified that if F is differentiable at x, then the corresponding Jacobian matrix $F'(x)$ satisfies the equation

$$F'(x) = \mathrm{diag}(f_1'(v_1), f_2'(v_2), \ldots, f_n'(v_n))\cdot W, \tag{21}$$

where $v_i = \sum_{j=1}^n w_{ij}x_j + I_i$, $i = 1, 2, \ldots, n$. Since $|f_i'(v_i)| \leqslant \beta_i$ for each i, it follows from Eq. (21) that $F'(x) \in \{DW: D \in \Delta\}$ for all $x \in \Omega$ at which F is differentiable, i.e., $\Pi_\Omega(F) \subseteq \{DW: D\in\Delta\}$ using the notation of Theorem 1. Hence, it can easily be proved that $\rho_\Omega(F) \leqslant \alpha$ Therefore, if $\alpha < 1$, it can be concluded by using Theorem 1 that the system governed by Eq. (1) is globally exponential stable. □

It is important to note that the constant α given in Eq. (19) is independent of the vector norm endowed to $\hbar^n$. A significant advantage of this feature is that, through endowing $\hbar^n$ with various vector norms (such as l^1-norm, l^2-norm, and the norms induced by a nonsingular matrix and a given norm, etc.), it is possible to derive a number of estimates of α and the various sufficient conditions for the global exponential stability of the system governed by Eq. (1). The following corollaries present some major generalizations of existing results. To facilitate the presentation, let Δ be defined as in Theorem 4.

Corollary 2. *If there exists a positive definite matrix H such that* $H - W^tDHDW$ *is positive definite for all* $D \in \Delta$, *then, for any threshold parameter* I_i, $i = 1, 2, \ldots, n$, *the system governed by Eq.* (1) *is globally exponential stable, and the corresponding exponential decay rate, r, satisfies that* $0 < r < -\frac{1}{2}\ln(1-\delta)$ *and*

$$\|x^k - x^*\| \leqslant M\cdot e^{-rk}\cdot\|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{22}$$

where $\delta = \min\{1, \inf_{D\in\Delta}\frac{\lambda_m(H-W^tDHDW)}{\lambda_M(H)}\}$, $\lambda_m(A)$ *and* $\lambda_M(B)$ *denote the smallest real eigenvalue of matrix A and the largest real eigenvalue of matrix B, respectively.*

Proof. Since H is positive definite, there exists a positive definite matrix H_0 such that $H = H_0^tH_0$ [33, Theorem 1, p. 181]. Let $\|x\|^* = \|H_0x\|$, where $\|\cdot\|$ denotes the l^2-norm of $\hbar^n$ (i.e., $\|x\|^2 = x^tx$ for all $x \in \hbar^n$). The nonnegative function $\|\cdot\|^*$ is actually a vector norm of $\hbar^n$. Indeed, it is evident that $\|x\|^* \geqslant 0$ for all $x\in\hbar^n$, and $\|x\|^* = 0 \iff H_0x = 0 \iff x = 0$ (because H_0 is positive definite). Moreover, $\|\cdot\|^*$ has the positive homogeneous property (i.e., $\|\lambda x\|^* = |\lambda|\cdot\|x\|^*$ for all $x\in\hbar$ and $\lambda\in\hbar$), as a result of the positive homogeneousness of the l^2-norm $\|\cdot\|$ and the homogeneousness of H_0. Finally, since $\|x+y\|^* = \|H_0(x+y)\| \leqslant \|H_0x\| + \|H_0y\| = \|x\|^* + \|y\|^*$ for any pair of x, y in $\hbar^n$, the triangular property of $\|\cdot\|^*$ is, thus, assured.

For any given $D \in \Delta$, since $H - W^tDHDW$ is positive definite, $\lambda_m(H - W^tDHDW) > 0$ and $x^tHx - x^tW^tDHDW x \geqslant \lambda_m(H - W^tDHDW)x^tx$ for all $x \in \hbar^n$. Let $\delta = \min\{1, \inf_{D\in\Delta} \frac{\lambda_m(H-W^tDHDW)}{\lambda_M(H)}\}$, where $\lambda_M(H)$ represents the largest real eigenvalue of H, which is positive and satisfies that $x^tHx \leqslant \lambda_M(H)x^tx$ for all $x \in \hbar^n$ because H is positive definite. Hence, $\delta > 0$. Otherwise, a sequence of $D_k \in \Delta$ must exist such $\lambda_m(H - W^tD_kHD_kW) \to 0$ as $k \to +\infty$. Without loss of generality, let $D_k \to D_0$ as $k \to +\infty$. Then $D_0 \in \Delta$ because of the compactness of Δ, and $\lambda_m(H - W^tD_0HD_0W) = 0$. This contradicts the positive definiteness of $H - W^tD_0HD_0W$.

For every $D \in \Delta$, the matrix norm of $\|DW\|_*$ of DW, with respect to the vector norm $\|\cdot\|^*$ defined above, can be evaluated by

$$\|DW\|_* = \sup_{x\in\hbar^n,\|x\|^*=1} \sqrt{(H_0DWx)^tH_0DWx} = \sup_{x\in\hbar^n,\|x\|^*=1} \sqrt{x^tHx - (x^tHx - x^tW^tDHDWx)} \leqslant \sqrt{1 - \lambda_m(H - W^tDHDW)x^tx}$$
$$\leqslant \sqrt{1-\delta} < 1.$$

Thus, in view of relation (20), it can readily be shown that

$$\alpha = \limsup_{k\to+\infty} \sup_{D_i\in\Delta} \|D_1WD_2W\cdots D_kW\|_*^{\frac{1}{k}} \leqslant \sqrt{1-\delta} < 1.$$

Therefore, this corollary is deduced from Theorem 4. □

Remark 3. If H is diagonally dominant, i.e., $\sum_{j\neq i}^n |h_{ij}| \leqslant h_{ii}$ for all $i = 1,2,\ldots,n$, then it can be shown easily that $x^tDHDx \leqslant \beta^2x^tHx$ for all $x \in \imath^n$, where $\beta = \max\{\beta_i : i = 1,2,\ldots,n\}$ and β_i is defined as in (4) (see, [26, Lemma 1]). Hence, the positive definiteness of $H - \beta^2W^tHW$ implies the positive definiteness of $H - W^tDHDW$ for all $D \in \Delta$, and hence ensures that the system governed by Eq. (1) is globally exponential stable. In the special case when all activation functions are of the saturation type (in the case, $\beta_i = 1$, $i = 1,2,\ldots,n$), Liu and Michel [26] proved that if a positive definite and diagonally dominant matrix H exists such that $H - W^tHW$ is positive definite, then the system governed by Eq. (1) is globally asymptotical stable. However, exponential stability was not discussed.

Remark 4. In the special case when all the activation functions have the form $f_i(t) = \tanh(t)$, and all threshold parameters I_i, $i = 1,2,\ldots,n$, equal to zero, Barabanov and Prokhorov [16] proposed a criterion to ensure that the system governed by Eq. (1) is globally exponential stable. It can be shown that the proposed criterion is a particular case of Theorem 4 of this paper. The proposed criterion requires that a positive definite matrix H exists such that

$$\xi^tH\xi - x^tHx + \xi^t\Gamma(Wx - \xi) < 0, \quad \forall (x,\xi) \neq 0, \tag{23}$$

where $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) > 0$. Indeed, let $\xi = DWx$ for all $x \in \hbar^n$ and $D \in \Delta$. Then $\xi^t\Gamma(Wx - \xi) = x^tW^tD\Gamma(I - D)Wx$. Since the modulus of each element of D is less than 1 because $\beta_i = 1$ for $f_i(t) = \tanh(t)$, the diagonal matrix $D\Gamma(I - D)$ is nonnegative. Hence, $\xi^t\Gamma(Wx - \xi) \geqslant 0$. It then follows from relation (23) that $x^tW^tDHDWx - x^tHx < 0$, $\forall x \in \hbar^n$, $D \in \Delta$, which indicates that $H - W^tDHDW$ is positive definite for all $D \in \Delta$. Therefore, criterion (23) can be regarded as a special case of Theorem 4.

Corollary 3. *Let $W = I + CT$, with C being a diagonal matrix of positive entries. If there exists a set of $\delta_i > 0$ such that either condition (i) or condition (ii) shown below is fulfilled,*

$$\text{(i)} \quad \varsigma_i = \beta_i\left(|1 + c_iT_{ii}| + \delta_ic_i\sum_{j\neq i}^n |T_{ij}|\delta_j^{-1}\right) < 1, \quad i = 1,2,\ldots,n \tag{24}$$

$$\text{(ii)} \quad \tau_i = \beta_i|1 + c_iT_{ii}| + \delta_ic_i\sum_{j\neq i}^n \delta_j^{-1}\beta_j|T_{ji}| < 1, \quad i = 1,2,\ldots,n. \tag{25}$$

Then, for any threshold parameters I_i, $i = 1,2,\ldots,n$, the system governed by Eq. (1) is globally exponential stable, and the corresponding exponential decay rate, r, satisfies either $0 < r < -\ln(\max_{1\leqslant i\leqslant n}\varsigma_i)$ in case (i), or $0 < r < -\ln(\max_{1\leqslant i\leqslant n}\tau_i)$ in case (ii), and

$$\|x^k - x^*\| \leqslant M \cdot \mathrm{e}^{-rk} \cdot \|x^0 - x^*\|, \quad k = 1,2,\ldots, \tag{26}$$

where x^k is any solution of Eq. (1) starting from x^0, x^* is the unique equilibrium point, and M is a positive constant related to r and the vector norm $\|\hbar\|$.

Proof. Case (i) Let $\hbar^n$ be endowed with the vector norm $\|x\|_{1^*} = \max_{1\leqslant i\leqslant n}|\delta_i x_i|$. Then, for all $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \Delta$ and $x = (x_1, x_2, \ldots, x_n)^t \in \hbar^n$,

$$\|DWx\|_{1*} = \max_{1\leqslant i\leqslant n}\left|\delta_i \sum_{j=1}^{n} d_i w_{ij} x_j\right| \leqslant \max_{1\leqslant i\leqslant n}|\delta_i d_i| \sum_{j=1}^{n}|w_{ij}x_j| \leqslant \max_{1\leqslant i\leqslant n}\delta_i\beta_i \sum_{j=1}^{n}|w_{ij}\delta_j^{-1}| \cdot \|x\|_{1^*}$$

$$= \max_{1\leqslant i\leqslant n}\delta_i\beta_i\left(\delta_i^{-1}|1 + c_iT_{ii}| + \sum_{j\neq i} c_i\delta_j^{-1}|T_{ij}|\right)\cdot\|x\|_{1^*} = \max_{1\leqslant i\leqslant n}\beta_i\left(|1 + c_iT_{ii}| + \delta_i\sum_{j\neq i}c_i\delta_j^{-1}|T_{ij}|\right)\cdot\|x\|_{1^*}$$

which implies that, with respect to the vector norm $\|\cdot\|_{1*}$, the matrix norm, $\|DW\|_1$, of DW is bounded above by $\max\limits_{1\leqslant i\leqslant n}\varsigma_i$. It can be shown by using inequality (24) and the notation of Theorem 4 that

$$\alpha = \limsup_{k\to\infty}\sup_{D_i\in\Delta}\|D_1W, D_2W, \ldots, D_kW\|_1^{\frac{1}{k}} \leqslant \limsup_{k\to\infty}\sup_{D_i\in\Delta}\left(\|D_1W\|_1\cdot\|D_2W\|_1\cdots\|D_kW\|_1\right)^{\frac{1}{k}} \leqslant \max_{1\leqslant i\leqslant n}\varsigma_i < 1,$$

which follows the results of Theorem 4. The proof is therefore completed for case (i).

Case (ii) Endow $\hbar^n$ with the vector norm $\|x\|_{2*} = \sum_{i=1}^{n}|\delta_i^{-1}x_i|$. Then, for all $D = \mathrm{diag}(d_1,\ d_2, \ldots, d_n) \in \Delta$ and $x = (x_1, x_2, \ldots, x_n)^t \in \hbar^n$,

$$\|DWx\|_{2*} = \sum_{i=1}^{n}\left|\delta_i^{-1}d_i\sum_{j=1}^{n}w_{ij}x_j\right| \leqslant \sum_{i=1}^{n}\delta_i^{-1}|d_i|\sum_{j=1}^{n}|w_{ij}x_j| = \sum_{i=1}^{n}|x_i|\sum_{j=1}^{n}\delta_j^{-1}\delta_j|w_{ji}| \leqslant \sum_{i=1}^{n}\delta_i^{-1}|x_i|\left(\delta_i\sum_{j=1}^{n}\delta_j^{-1}\beta_j|w_{ji}|\right)$$

$$\leqslant \sum_{i=1}^{n}\delta_i^{-1}|x_i|\left(\beta_i|1 + c_iT_{ii}| + \delta_i c_i\sum_{j\neq i}^{n}\delta_j^{-1}\beta_j|T_{ij}|\right) \leqslant \max_{1\leqslant i\leqslant n}\tau_i\cdot\|x\|_{2^*}$$

which indicates that, with respect to the vector norm $\|\cdot\|_2^*$, the matrix norm, $\|DW\|_2$, of DW satisfies the inequality $\|DW\|_2 \leqslant \max_{1\leqslant i\leqslant n}\tau_i$. Hence, similar to case (i), it is easily to verify that the corresponding α is less than $\max_{1\leqslant i\leqslant n}\tau_i$. In view of relation (25), the proof is therefore completed for case (ii). □

Remark 5. It should be pointed out that conditions (24) and (25) become more general due to the incorporation of a set of parameters δ_i, $i = 1, 2, \ldots, n$. Indeed, a series of sufficient conditions for ensuring global exponential stability for the system governed by Eq. (1) can be obtained by adjusting the values of these parameters. For examples, the following series of inequalities can be derived by assigning $\delta_i \equiv 1$, $\delta_i = \beta_i$, and $\delta_i = c_i^{-1}$ in relations (24) and (25), respectively:

(a) $\beta_i(|1 + c_iT_{ii}| + c_i\sum_{j\neq i}^{n}|T_{ij}|) < 1, \quad i = 1, 2, \ldots, n;$
(b) $\beta_i|1 + c_iT_{ii}| + c_i\sum_{j\neq i}^{n}|T_{ij}|\beta_j^1 < 1, \quad i = 1, 2, \ldots, n;$
(c) $\beta_i(|1 + c_iT_{ii}| + \sum_{j\neq i}^{n}|T_{ij}|c_j) < 1, \quad i = 1, 2, \ldots, n;$
(d) $\beta_i|1 + c_iT_{ii}| + c_i\sum_{j\neq i}^{n}\beta_j|T_{ji}| < 1, \quad i = 1, 2, \ldots, n;$
(e) $\beta_i|1 + c_iT_{ii}| + \beta_ic_i\sum_{j\neq i}^{n}|T_{ji}| < 1, \quad i = 1, 2, \ldots, n;$
(f) $\beta_i|1 + c_iT_{ii}| + \sum_{j\neq i}^{n}c_i\beta_j|T_{ji}| < 1, \quad i = 1, 2, \ldots, n;$

Hence, the system governed by Eq. (1) is globally exponential stable if anyone of the above six sufficient condition is satisfied.

Remark 6. Pérez-Ilzarbe [4] employed networks (1) to solve a quadratic optimization problem with bound constraints. In the study, the activation functions were represented by saturation functions and the weight matrix is of the form $W = I + CT$. The condition that T is negative definite and fulfills the relation

$$|T_{ii}| - \sum_{j\neq i}^{n}|T_{ij}| > 0, \qquad 0 < c_i < \frac{2}{\sum\limits_{j=1}^{n}|T_{ij}|}, \quad i = 1, 2, \ldots, n, \tag{27}$$

was proposed to ensure global exponential convergence of the networks. However, it can be verified easily that relation (27) suffices to ensure that inequality (a) (where $\beta_i = 1$) of Remark 5 is satisfied, and is thus a special case of relation (24). The following example further demonstrates that relation (27) is not necessary for the system governed by Eq. (1) to be globally exponential stable.

Example 1. Consider the following neural networks:

$$\begin{cases} x_1^{k+1} = f_1(x_1^k + 0.8(-x_1^k - 9x_2^k + I_1)), \\ x_2^{k+1} = f_2(x_2^k + 4(-0.2x_2^k + 0.01x_1^k + I_2)), \end{cases} \tag{28}$$

where f_1 and f_2 are saturation functions defined in the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, and both I_1 and I_2 are any given real constants.

Using the notation of [4], it can be clearly seen that $T = \begin{pmatrix} -1 & -9 \\ 0.01 & -0.2 \end{pmatrix}$ and $C = \mathrm{diag}(0.8,4)$ in the networks. Since $|T_{11}| = 1 < 9 = |T_{12}|$ and $c_1 = 0.8 > \frac{2}{|T_{11}|+|T_{12}|} = 0.2$, inequalities (27) do not hold. However, if $\delta_1 = 1$ and $\delta_2 = 10$, then

$$\begin{cases} |1 + c_1T_{11}| + c_1\delta_1|T_{12}|\delta_2^{-1} = 0.92 < 1, \\ |1 + c_2T_{22}| + c_2\delta_2|T_{21}|\delta_1^{-1} = 0.6 < 1. \end{cases}$$

Hence, relation (24) given in Corollary 4 is fulfilled. Therefore, the networks governed by Eq. (28) is globally exponential stable due to Corollary 4, with $r = -\ln(0.92)$ being the smallest exponential decay rate. Fig. 1 shows that the system governed by Eq. (28) is globally exponential stable, with $x^* = (0,0)^t$ being the unique equilibrium point, even though conditions (27) proposed by Pérez-Ilzarbe [4] are not fulfilled.

When the weight matrix W is triangular, another simple criterion for the system governed by Eq. (1) to be globally exponential stable can be easily derived by using the results of Theorem 4. The analysis is stated in the following corollary.

Corollary 4. *Suppose the weight matrix W is triangular. If $r_0 := \max_{1\leqslant i\leqslant n} \beta_i|w_{ii}| < 1$, then, for any threshold parameters I_i, $i = 1, 2, \ldots, n$, the system governed by Eq. (1) is globally exponential stable, and the exponential decay rate, r, satisfies that $0 < r < -\ln r_0$ and*

$$\|x^k - x^*\| \leqslant M \cdot \mathrm{e}^{-rk} \cdot \|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{29}$$

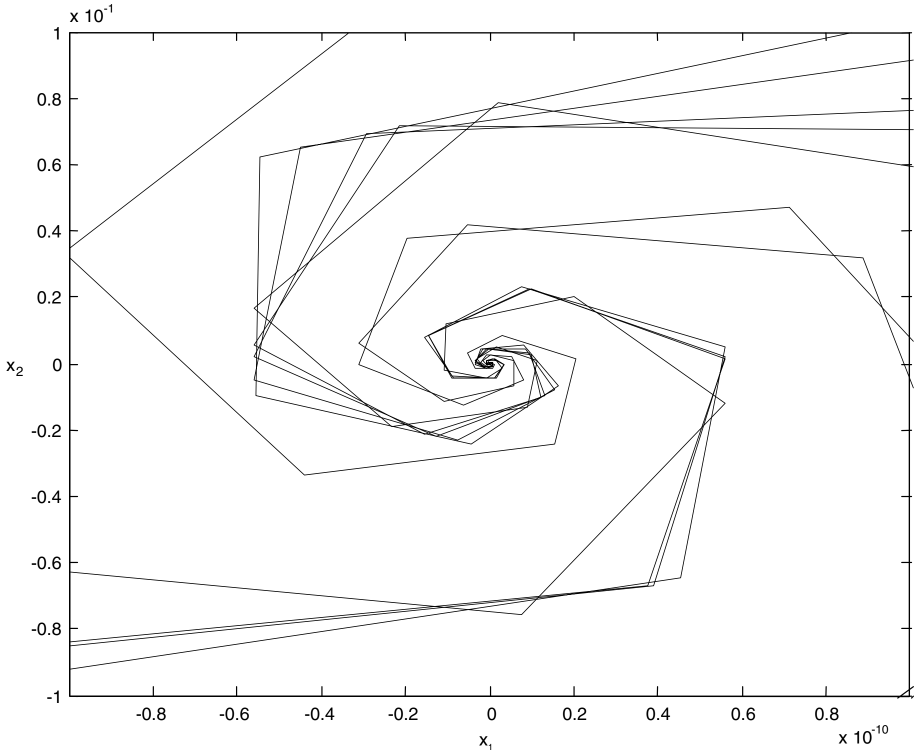

Fig. 1. A phase plane portrait of the trajectories of the neural networks defined in Example 1, where $I_1 = I_2 = 0$, $a_1 = -2.5$, $b = 2.5$, and $a_2 = -3$, $b_2 = 3$.

where x^k is any solution starting at x^0, x^* is the unique equilibrium point, and M is a positive constant related to r and the vector norm $\|\hbar\|$.

Proof. Since W is triangular, DW for every $D \in \Delta$ is triangular. Indeed, any product $D_1W\cdot D_2W\cdots D_kW$ is also triangular. Because Δ is a bounded matrix set, Eq. (18) holds for $\Gamma = \{DW : D \in \Delta\}$. Therefore, in view of Theorem 4,

$$\alpha = \limsup_{k\to+\infty} \sup_{D_i\in\Delta} \|D_1W\cdot D_2W\cdots D_kW\|^{\frac{1}{k}} = \limsup_{k\to+\infty} \sup_{D_i\in\Delta} \rho(D_1W\cdot D_2W\cdots D_kW)^{\frac{1}{k}}$$
$$\leqslant \limsup_{k\to+\infty} \sup_{D_i\in\Delta} \max_{1\leqslant j\leqslant n} |w_{jj}^k d_{1j}d_{2j}\cdots d_{kj}|^{\frac{1}{k}} \leqslant \max_{1\leqslant j\leqslant n} \beta_j|w_{jj}|,$$

where $D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in})$. If $r_0 =: \max_{1\leqslant j\leqslant n} \beta_j|w_{jj}| < 1$, the global exponential stability of the system governed by Eq. (1) is ensured by Theorem 4. □

Theorem 5. *If $r_0 = \rho(\Lambda\cdot|W|) < 1$, where $\Lambda = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ is a diagonal matrix with entries β_i, then, for any threshold parameters I_i, $i = 1, 2, \ldots, n$, the system governed by Eq. (1) is globally exponential stable in $\hbar^n$, and the exponential decay rate, r, satisfies that $0 < r < -\ln r_0$ and*

$$\|x^k - x^*\| \leqslant M\cdot e^{-rk}\cdot\|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{30}$$

where x^k is any solution starting at x^0, x^* is the unique equilibrium point, and M is a positive constant related to r and the vector norm $\|\hbar\|$.

Proof. Let F be defined as in the proof of Theorem 4. In view of Theorem 4, if F is differentiable at x, then the corresponding Jacobian matrix $F'(x)$ is given by

$$F'(x) = \mathrm{diag}(f_1'(v_1), f_2'(v_2), \ldots, f_n'(v_n))\cdot W,$$

where $v_i = \sum_{j=1}^n w_{ij}x_j + I_i$, $i = 1, 2, \ldots, n$. Since $|f_i'(t)| \leqslant \beta_i$ for all $i = 1, 2, \ldots, n$ and all $t \in \hbar$ it follows that $|F'(x)| \leqslant \Lambda\cdot|W|$. Hence, by Definition 1, it can be shown that $|A| \leqslant \Lambda\cdot|W|$ for all $A \in \Pi_\Omega(F)$. In view of Theorems 1 and 3, Theorem 5 is therefore established. □

When the networks are connected co-operatively, every weight w_{ij} from neuron i to neuron j is positive. Thus, the corresponding weight matrix W is nonnegative, i.e., $|W| = W$. In this case, Theorem 5 becomes the following corollary.

Corollary 5. *If $\rho(\Lambda W) < 1$, then, for any threshold parameters I_i, $i = 1, 2, \ldots, n$, the co-operatively connected networks governed by Eq. (1) is globally exponential stable in $\hbar^n$ and the exponential decay rate, r, satisfies that $0 < r < -\ln < \rho(\Lambda W)$ and*

$$\|x^k - x^*\| \leqslant M\cdot e^{-rk}\cdot\|x^0 - x^*\|, \quad k = 1, 2, \ldots, \tag{31}$$

where x^k is any solution starting at x^0, x^* is the unique equilibrium point,, and M is a positive constant related to r and the vector norm $\|\hbar\|$.

Remark 7. It should be noted that, in general, condition "$\rho(\Lambda|W|) < 1$" in Theorem 5 cannot be simplified to "$\rho(\Lambda W) < 1$". Hence, the system governed by Eq. (1) may not be stable even if $\rho(\Lambda W) < 1$. The following example demonstrates this aspect:

Example 2. Consider the following discrete-time neural networks:

$$\begin{cases} x_1^{k+1} = f_1(-x_1^k + x_2^k + a), \\ x_2^{k+1} = f_2(-4x_1^k + 4x_2^k + b), \end{cases} \tag{32}$$

where $f_1(t) = \tanh(4t)$ and $f_2(t) = \tanh(t)$, $a = \frac{1}{4}\ln(2+\sqrt{3}) + f_1(\frac{1}{4}\ln(2+\sqrt{3}))$, and $b = 4f_1(\frac{1}{4}\ln(2+\sqrt{3}))$.

It can be shown that, in this case, the weight matrix W is $\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}$ and inequality (3) holds for each of the two activation functions f_1 and f_2 with $\beta_1 = 4$ and $\beta_2 = 1$, respectively. Hence, $\Lambda = \mathrm{diag}(4,1)$. Also, it can be shown that $x^* = (\frac{3+2\sqrt{3}}{4+2\sqrt{3}}, 0)^t$ is an equilibrium point of the system governed by Eq. (32). The corresponding Jacobian matrix is given by the equation

$$F'(x^*) = \begin{pmatrix} f_1'(\frac{1}{4}\ln(2+\sqrt{3})) & 0 \\ 0 & f_2'(0) \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}.$$

Hence, $\rho(F'(x^*)) = 3$ since $\lambda = 3$ is an eigenvalue of $F'(x^*)$. This implies that the equilibrium point x^* is not stable, and thus the system (32) is not globally exponential stable. However, since

$$\Lambda W = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix}$$

the inequality $\rho(\Lambda W) = 0 < 1$ holds.

Remark 8. Gégout [8] stated that if a constant v exists such that $|f_i'(t)| < v$ for all $i = 1,2,\ldots,n$ and $t \in \hbar$ the activation dynamics $K(x) = f(Wx + T)$ is convergent, stable and robust if $\rho(W) < \frac{1}{v}$. However, the following example clearly shows that this statement needs to be modified:

Example 3. Let $f_1(t) = \tanh(\beta t)$, $f_2(t) = \tanh(t)$, and $\beta > 1$ be an adjustable parameter. The neural dynamics is described by the equations

$$\begin{cases} x_1^{k+1} = f_1(-x_1^k + x_2^k), \\ x_2^{k+1} = f_2(-x_1^k + x_2^k). \end{cases} \tag{33}$$

The corresponding activation dynamics are governed by the equation $K(x) = f(Wx + T)$, with $f(x) = (f_1(x_1), f_2(x_2))^t$ and $W = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. The following cases can be noted: (a) $\rho(W) = 0$, which implies that $\rho(W) < \frac{1}{v}$ for any $v > 0$, and (b) $\rho(K'(x^*)) = \beta - 1$, since the Jacobian matrix becomes $K'(x^*) = \begin{pmatrix} -\beta & \beta \\ -1 & 1 \end{pmatrix}$ at the equilibrium point $x^* = (0,0)^t$. Therefore, the activation dynamic $K(x)$ cannot be stable whenever $\beta > 2$. Indeed, in view of Theorem 4, Gégout's statement given in [8] may be modified to clarify that the activation dynamics $K(x) = f(Wx + T)$ of the system is convergent, stable and robust if $\rho(|W|) < \frac{1}{v}$.

Corollary 6. *If a constant $\sigma \in (0,1)$ exists such that $(\sigma I - \Lambda|W|)v \geqslant 0$ for some positive vector v, then, for any threshold parameters I_i, $i = 1,2,\ldots,n$, the system governed by Eq. (1) is globally exponential stable in $\hbar^n$, and the corresponding exponential decay rate, r, satisfies that $0 < r < \ln\sigma$ and*

$$\|x^k - x^*\| \leqslant M \cdot e^{-rk} \cdot \|x^0 - x^*\|, \quad k = 1,2,\ldots, \tag{34}$$

where Λ is defined as in Theorem 4, and M is a positive constant related to r and the vector norm $\|\hbar\|$.

Proof. Since $\sigma I - \Lambda|W|$ is a Z-matrix and $(\sigma I - \Lambda|W|)v \geqslant 0$ with v being positive, $\sigma I - \Lambda|W|$ is an M-matrix. Hence, each real eigenvalue of $\sigma I - \Lambda|W|$ is nonnegative. For the nonnegative matrix $\Lambda|W|$, it is well known that $\rho(\Lambda|W|)$ is a real eigenvalue. Since $\sigma - \rho(\Lambda|W|)$ is a real eigenvalue of $\sigma I - \Lambda|W|$, $\sigma - \rho(\Lambda|W|) \geqslant 0$, i.e., $\rho(\Lambda|W|) \leqslant \sigma < 1$. In view of Theorem 5, Corollary 6 is therefore established. □

Remark 9. This corollary essentially comes from Chu [7]. However, it should be pointed out that all the activation functions f_i considered in [7] are assumed to be sigmoidal, i.e., each f_i is Lipschitz continuous, bounded and monotonic increasing.

4. Conclusion

In this paper, a novel approach based on the notion of the nonlinear spectral radius has been proposed for analyzing the global exponential stability of discrete-time neural networks. A series of sufficient conditions for ensuring global exponential stability of the networks have been established. There are several advantages of the developed approach, including (i) it does not only establish sufficient conditions for ensuring global exponential stability for discrete-time neural networks, but also describes the exponential decay of any solution to the equilibrium point; (ii) the invariability of the nonlinear spectral radius with the linear topology of the state space $\hbar^n$, enables the development of some new sufficient conditions for ensuring global exponential stability of neural networks by adjusting the vector norm of $\hbar^n$, and (iii) it does not require any special properties of the activation functions and the weight matrix, e.g., the smoothness, boundedness, monotonicity and sigmoidalness, on the activation functions, and the symmetry, co-operativity, and non-self-connection on the weight matrix.

References

[1] Marcus CM, Waugh FR, Westervelt RM. Associative memory in an analog iterated-map neural network. Phys Rev A 1990;41(6):3355–64.

[2] Yoshizawa S, Morita M, Amari SI. Capacity of associative memory using a nonmonotonic neural model. Neural Networks 1993;6:167–76.

[3] Forti M, Nistri P, Quincampoix M. Generalized neural network for nonsmooth nonlinear programming problems. IEEE Trans Circ Syst—I: Regular Pap 2004;51(9):1741–54.

[4] Pérez-Ilzarbe MJ. Convergence analysis of a discrete-time recurrent neural network to perform quadratic real optimization with bound constraints. IEEE Trans Neural Networks 1998;9(6):1344–51.

[5] Xia Y, Wang J. A recurrent neural network for nonlinear convex optimization subject to nonlinear inequality constraints. IEEE Trans Circ Syst—I: Regular Pap 2004;51(7):1385–94.

[6] Chen LN, Aihara K. Chaos and asymptotical stability in discrete-time neural networks. Physica D 1997;104:286–325.

[7] Chu TG. Convergence in discrete-time neural networks with specific performance. Phys Rev E 2001;63:1–5.

[8] Gégout C. Stable and convergent dynamics for discrete-time recurrent networks. Nonlinear Anal: Theory Meth Appl 1997;30(3):1663–8.

[9] Tiňo P, Horne BG, Giles CL. Attractive periodic sets in discrete-time recurrent neural networks (with emphasis on fixed-point stability and bifurcations in two-neuron networks). Neural Comput 2001;13:1379–414.

[10] Liang XB, Wang J. Absolute exponential stability of neural networks with a general class of activation functions. IEEE Trans Circ Syst I 2000;47(8):1258–63.

[11] Liang XB, Si J. Global exponential stability of neural networks with globally Lipschitz continuous activations and its application to linear variational inequality problem. IEEE Trans Neural Networks 2001;12(2):349–59.

[12] Zhang Q, Wei XP, Xu J. New stability conditions for neural networks with constant and variable delays. Chaos, Solitons & Fractals 2005;26(5):1391–8.

[13] Yuan K, Cao J. Periodic oscillatory solution in delayed competitive–cooperative neural networks: a decomposition approach. Chaos, Solitons & Fractals 2006;27(1):223–31.

[14] Jin L, Nikiforuk PN, Gupta MM. Absolute stability conditions for discrete-time recurrent neural networks. IEEE Trans Neural Networks 1994;6:945–55.

[15] Jin L, Gupta MM. Globally asymptotical stability of discrete-time analog neural networks. IEEE Trans Neural Networks 1996;7(4):1024–31.

[16] Barabanov NE, Prokhorov DV. Stability analysis of discrete-time recurrent neural networks. IEEE Trans Neural Networks 2002;13(2):292–303.

[17] Liao X, Chen G, Sanchez EN. LMI-based approach for asymptotically stability analysis of delayed neural networks. IEEE Trans Circ Syst—I 2002;49:1033–9.

[18] Cao J, Wang J. Global Asymptotic and robust stability of recurrent neural networks with time delays. IEEE Trans Circ Syst—I 2005;52(2):417–26.

[19] Arik S. Global robust stability analysis of neural networks with discrete time delays. Chaos, Solitons & Fractals 2005;26(5):1407–14.

[20] He Y, Wang QG, Zheng WX. Global robust stability for delayed neural networks with polytopic type uncertainties. Chaos, Solitons & Fractals 2005;26(5):1349–54.

[21] Xu B, Teo KL, Liao X, Liu XZ. Exponential stability of high order Hopfield type neural networks with time delays. Chaos, Solitons & Fractals, submitted for publication.

[22] Liu ZG, Chen AP, Huang LH. Periodic oscillatory solution to delayed BAM neural networks with impulses. Int J Nonlinear Sci Numer 2004;5(4):355–62.

[23] Li YK, Xing WY, Lu LH. Existence and global exponential stability of periodic solution of a class of neural networks with impulses. Chaos, Solitons & Fractals 2006;27(2):437–45.

[24] Liu XZ, Liu Y, Teo KL. Stability analysis of impulsive control systems. Comput Math Model 2003;37:1357–70.

[25] Kelly DG. Stability in contractive nonlinear neural networks. IEEE Trans Biomed Eng 1990;37(3):231–42.

[26] Liu DR, Michel AN. Asymptotic stability of discrete-time systems with saturation non-linearities with applications to digital filters. IEEE Trans Circ Syst—I 1992;39(10):798–807.

[27] Khalil HK. Nonlinear systems. Upper Saddle River, NJ: Prentice Hall; 1996.

[28] Clarke FK. Optimization and nonsmooth analysis. New York: Wiley; 1983.

[29] Horn RA, Johnson CR. Topics in matrix analysis. Cambridge UK: Cambridge University Press; 1991.

[30] Berger MA, Wang YY. Bounded semigroups of matrices. Linear Algebra Appl 1992;166:21–7.
[31] LaSalle J. The stability of dynamical system, regional conference series in applied mathematics. Philadephia: SIAM; 1976.
[32] Wielandt H. Unzerlegbare, nicht negative matrizen. Math Z 1950;52:642–8.
[33] Lancaster P, Tismenetsky M. The theory of matrices. 2nd ed. London: Academic Press; 1985.