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Existence and stability of periodic solution of a predator–prey model with state-dependent impulsive effects

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Abstract

According to integrated pest management for pests, we investigate the dynamic behavior of a class predator–prey system with state-dependent impulsive effects by releasing natural enemies and spraying pesticide at different thresholds. Using the Poincaré map and the properties of the Lambert *W* function, we prove the sufficient conditions for the existence and stability of semi-trivial solutions and positive periodic solutions. Numerical results are carried out to illustrate the feasibility of our main results.

Keywords: Impulsive state feedback control; Lambert *W* function; Stability

1. Introduction

The effect of pest control in population dynamics has become an increasingly complex issue over the past two decades. In pest management, insecticides are useful because they quickly kill a significant proportion of an insect population and they sometimes provide the only feasible method for preventing economic loss. However, there are many deleterious effects associated with the use of chemicals that need to be reduced or eliminated. Another important method for pest control is biological control, which is a manipulation of existing natural enemies to increase their effectiveness. This can be achieved by mass production and periodic release of natural enemies of the pest, and by genetic enhancement of the enemies to increase their effectiveness at control. However, research on augmentation as a biological control method has shown that some practices are cost-effective and others are not. Integrated pest management (involves combining biological, mechanical, and chemical tactics) has been proved to be more effective than the classic methods (such as biological control or chemical control) both experimentally (e.g. [8,22,23,21]) and theoretically (e.g. [4,25]).

Recently, in order to consider the consequences of spraying pesticide and introducing additional predators into a natural pest-predator system, many authors have suggested impulsive differential equations (IDEs) (see [2,12]) to investigate the dynamics of a pest control model. IDEs are found in almost every domain of applied science. The theory

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and applications of IDEs are emerging as an important area of investigation, since it is far richer than the corresponding theory of nonimpulsive differential equations.

In the last decades, there has been a significant development in theory of IDEs, especially in the area in which impulses are at fixed times, and investigations are mainly focused on the basic theories (see [7,6,17,18,3,14,13,9,1,27,19,16]). In some ecological systems, however, we note that the control measures (by catching, poisoning or releasing the natural enemy, etc) are taken only when the amount of species reaches a threshold, rather than the usual impulsive fixed time control strategy. Recently,IDEs with state-dependent impulsive effects are studied in [10,11,20,26,15]. However, this work considers the impulsive effects (poisoning the prey and releasing the predator) at the same threshold at the same time. They ignored the side effects of pesticide on natural enemies, and they assumed the time of spraying pesticide and the releasing of natural enemies is at the same. This approach is unreasonable.

In this paper, according to biological and chemical control strategy for pest, we construct a predator–prey state-dependent impulsive system by releasing natural enemies and spraying pesticide at different thresholds. The system can be written as

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)[a-by(t)] \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y(t)\left[\frac{\lambda bx(t)}{1+bhx(t)}-d\right] \end{aligned}\right\} & x \neq h_1, h_2, \\ \left.\begin{aligned} \triangle x(t) &= 0 \\ \triangle y(t) &= \tau \end{aligned}\right\} x = h_1, & \\ \left.\begin{aligned} \triangle x(t) &= -px(t) \\ \triangle y(t) &= -qy(t) \end{aligned}\right\} x = h_2, & \end{cases} \tag{1.1}$$

where x and y represent the population densities at time t; a, b, λ, h and d are positive constants, $\tau \geq 0$, $p, q \in (0, \ 1)$, $h_1 > 0$, $h_2 > 0$ and $(1-p)h_2 < h_1$. When the amount of the prey reaches the threshold h_1 at time $t_k(h_1)$, controlling measures are taken (releasing natural enemies, that is the predator) and the amount of predator abruptly turn to $y(t_k(h_1)) + \tau$. Further, when the amount of the prey reaches the threshold h_2 at time $t_k(h_2)$, spraying pesticide and the amount of prey and predator abruptly turn to $(1-p)x(t_k(h_2))$ and $(1-q)y(t_k(h_2))$, respectively.

This paper is organized as follows. In the next section, as preliminaries we present some basic definitions, two Poincaré maps, Lambert W function and a important lemma. In Section 3, we state and prove a general criterion for a positive periodic solution of system (1.1). The sufficient conditions for the existence and stability of positive periodic solutions of system (1.1) is obtained in Section 4. In the last section, some specific examples are given to illustrate our results.

2. Preliminaries

In system (1.1), when parameter $p = q = \tau = 0$ we obtain the following system without impulsive effect:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)[a-by(t)], \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)\left[\dfrac{\lambda bx(t)}{1+bhx(t)}-d\right]. \end{cases} \tag{2.1}$$

The dynamic behaviors for system (2.1) are studied by considerable investigators. It has a saddle $(0, 0)$ and stable nod $(d/(\lambda b - bdh), a/b)$ under the condition $\lambda > dh$.

Throughout this paper, we assume that the condition $\lambda > dh$ holds. By the biological background of system (1.1), we only consider system (1.1) in the biological meaningful region $D = \{(x, y) : x \geq 0, y \geq 0\}$. Obviously, the global existence and uniqueness of solutions of system (1.1) are guaranteed by the smoothness properties of the right-side of system (1.1); for details see Lakshmikantham et al. [12] and Bainov and Simeonov [2].

To discuss the dynamics of system (1.1), we define three cross-sections to the vector field (1.1) by $\sum_1 = \{(x, y) : x = (1-p)h_2, y \geq 0\}$, $\sum_2 = \{(x, y) : x = h_1, y \geq 0\}$ and $\sum_3 = \{(x, y) : x = h_2, y \geq 0\}$. Now, we construct two Poincaré maps of $\sum_3$. First, we define the Poincaré maps of $\sum_3$ with $\tau = 0$. Suppose system (1.1) has a positive T periodic solution $z(t) = (\phi(t), \varphi(t))$ with the initial condition $z_0 = z(0) = ((1-p)h_2, y_0)$, where $y_0 > 0$. Let the periodic trajectory $O^+(z_0, 0)$ intersects the sections $\sum_1$ and $\sum_3$ at the points $A^+((1-p)h_2, y_0)$ and $A(h_2, y_1)$, respectively. At the point A, the trajectory of (1.1) is subjected by impulsive effects to jumps to the point A^+ again. Thus

$$\phi(0) = (1-p)h_2, \quad \varphi(0) = y_0, \quad \phi(T) = h_2, \quad \text{and} \quad \varphi(T) = y_1 = \frac{y_0}{1-q}.$$

Now, we consider another solution $\tilde{z}(t) = (\tilde{\phi}(t), \tilde{\varphi}(t))$ of small-amplitude perturbation of the periodic solution $z(t)$ with initial condition $\tilde{z}_0 = \tilde{z}(0) = ((1-p)h_2, \tilde{y}_0)$. Suppose the trajectory $O^+(\tilde{z}_0, 0)$ which starting from $A_0((1-p)h_2, \tilde{y}_0)$ first intersects the section $\sum_3$ at the point $A_1(h_2, \tilde{y}_1)$ when $t = T + \delta t$ and then jumps to the point $A_1^+((1-p)h_2, \tilde{y}_2)$ on the section $\sum^p$. Then, we have

$$\tilde{\phi}(0) = (1-p)h_2, \quad \tilde{\varphi}(0) = \tilde{y}_0, \quad \tilde{\phi}(T+\delta t) = h_2 \quad \text{and} \quad \tilde{\varphi}(T+\delta t) = \tilde{y}_1.$$

Let $u(t) = \tilde{\phi}(t) - \phi(t)$ and $v(t) = \tilde{\varphi}(t) - \varphi(t)$, then $u_0 = u(0) = \tilde{\phi}(0) - \phi(0) = 0$ and $v_0 = v(0) = \tilde{\varphi}(0) - \varphi(0)$. Let $v_1 = \tilde{y}_2 - y_0$ and $v_0^* = \tilde{y}_1 - y_1$. It is known that for $0 < t < T$, the variables $u(t)$ and $v(t)$ are described by the relation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Psi(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + o(u_0^2 + v_0^2) = \Psi(t) \begin{pmatrix} 0 \\ v_0 \end{pmatrix} + o \begin{pmatrix} 0 \\ v_0^2 \end{pmatrix}, \tag{2.2}$$

where the fundamental solution matrix $\Psi(t)$ satisfies the matrix equation

$$\frac{d\Psi(t)}{dt} = \begin{pmatrix} a - b\varphi(t) & -b\phi(t) \\ \dfrac{\lambda b\varphi(t)}{(1+bh\phi(t))^2} & \dfrac{\lambda b\phi(t)}{1+bh\phi(t)} - d \end{pmatrix} \Psi(t) \tag{2.3}$$

with $\Psi(0) = I$ (the unit matrix). Set $f_1(t) = \phi(t)[a - b\varphi(t)]$ and $f_2(t) = -d + \lambda b\phi(t)\varphi(t)/[1 + bh\phi(t)]$. We can express the perturbed trajectory in a first-order Taylor expansion

$$\begin{cases} \tilde{\phi}(T+\delta t) \approx \phi(T) + u(T) + f_1(T)\delta t, \\ \tilde{\varphi}(T+\delta t) \approx \varphi(T) + v(T) + f_2(T)\delta t. \end{cases}$$

From $\tilde{\phi}(T+\delta t) = \phi(T) = h$, we have

$$\delta t = -\frac{u(T)}{f_1(T)} \quad \text{and} \quad v_0^* = \tilde{y}_1 - y_1 = v(T) - \frac{f_2(T)u(T)}{f_1(T)}.$$

In view of $\tilde{y}_2 = (1-q)\tilde{y}_1$ and $\tilde{y}_2 - y_0 = (1-q)(\tilde{y}_1 - y_1)$, thus $v_1 = (1-q)v_0^*$. So, we defined the Poincaré map of $\sum_1$ as follows

$$v_1 = P_1(q, v_0) = (1-q)\left[v(T) - \frac{f_2(T)u(T)}{f_1(T)}\right], \tag{2.4}$$

where $u(T)$ and $v(T)$ are calculated according to (2.2).

Now, we consider another Poincaré map with $\tau \in (0, \tau^*)$, where τ^* is a positive constant. Suppose that the trajectory $O^+(G_n, t_n)$ starting from the point $G_n(h_2, y_n)$ on the section $\sum_3$, then it jumps to the point $E_{n+1}((1-p)h_2, (1-q)y_n)$ on $\sum_1$ due to the impulsive effects $\triangle x(t) = -px(t)$ and $\triangle y(t) = -qy(t)$, and then reaches the point $F_{n+1}(h_1, \tilde{y}_{n+1})$ on the section $\sum_2$. Further, the point $F_{n+1}(h_1, \tilde{y}_{n+1})$ jumps to the point $F_{n+1}^+(h_1, \tilde{y}_{n+1} + \tau)$ on $\sum_2$ due to the impulsive effects $\triangle x(t) = 0$ and $\triangle y(t) = \tau$, and then reaches the point $G_{n+1}(h_2, y_{n+1})$ on the section $\sum_3$, where y_{n+1} is decided by the parameters q, τ and y_n. Therefore, we defined the Poincaré map of $\sum_3$ as follows

$$y_{n+1} = P_2(q, \tau, y_n). \tag{2.5}$$

Let $z(t) = (x(t), y(t))$ be a solution of system (1.1) with initial conditions $z_0 = z(t_0) = ((1-p)h_2, y_0) \in R_+^2$. This trajectory $O^+(z_0, t_0)$ starts from the point $A_0((1-p)h_2, y_0)$ first intersects the section $\sum_2$ at the point $B_0(h_1, \hat{y}_0)$, next jumps to the point $B_0^+(h_1, \hat{y}_0 + \tau)$ on the section $\sum_2$ due to the impulsive effects, and then reaches the point $C_1(h_2, \tilde{y}_0)$ on the section $\sum_3$. At the state C_1, the trajectory of (1.1) is subjected by impulsive effects to jumps to the point $A_1((1-p)h_2, y_1)$ on the section $\sum_1$ again. Repeating the above process, we have two impulsive points' sequences $\{A_k((1-p)h_2, y_k)\}$ and $\{C_k(h_2, \tilde{y}_k)\}(k = 0, 1, 2, \ldots)$. We notice that the coordinates satisfy the relation $y_k = (1-q)\tilde{y}_{k-1}(k = 1, 2, \ldots)$.

Definition 1. A trajectory $O^+(z_0, t_0)$ of system (1.1) is said to be order-k periodic if there exists a positive integer $k \geq 1$ such that k is the smallest integer for $y_0 = y_k$.

Definition 2. A solution $z(t) = (x(t), y(t))$ of system (1.1) is called a semi-trivial solution if one of its components is zero and the other is nonzero.

Definition 3 *(Corless et al. [5])*. The Lambert W function is defined to be a multiple valued inverse of the function $f : Z \mapsto ze^z$ satisfying

$$W(z)\exp(W(z)) = z$$

The Lambert W function $W(z)$ has two branches for $z \geq -1/e$ where we define the inverse function of $W(z)$ restricted to the interval $[-1, \infty)$ to be $W_0(z)$ and the inverse function of $W(z)$ restricted to the interval $(-\infty, -1]$ to be $W_{-1}(z)$. It is clear that the branch $W_0(z)$ satisfies $-1 < W_0(z) < 0$ for $z \in (-\exp(-1), 0)$ and its derivative satisfies

$$W_0'(z) = \frac{W_0(z)}{z(1 + W_0(z))}. \tag{2.6}$$

This follows from the Lagrange inversion theorem (see e.g. [5]), which gives the series expansion below for $W_0(z)$

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n.$$

For more details of the concepts and properties of the Lambert W function, see Corless et al. [5] and Waldvogel [24].

Next, we consider the autonomous system with impulsive effects

$$\begin{cases} \dfrac{dx}{dt} = P(x, y), \quad \dfrac{dy}{dt} = Q(x, y), \quad \varphi(x, y) \neq 0, \\ \triangle x = \xi(x, y), \quad \triangle y = \eta(x, y), \quad \varphi(x, y) = 0, \end{cases} \tag{2.7}$$

where $P(x, y)$ and $Q(x, y)$ are continuous differential functions defined on R^2, and $\varphi(x, y)$ is a sufficiently smooth function with $grad\varphi(x, y) \neq 0$. Let $(\mu(t), \nu(t))$ be a positive T-periodic solution of system (2.6). By Corollary 2 of Theorem 1 given in Simeonov and Bainov [18], there is the following Lemma.

Lemma 1 ((Analogue of Poincaré's criterion)). *If the Floquet multiplier* μ *satisfies the condition* $|\mu| < 1$, *where*

$$\mu = \prod_{j=1}^{n} \kappa_j \exp\left[\int_0^T \left(\frac{\partial P(\mu(t), \nu(t))}{\partial x} + \frac{\partial Q(\mu(t), \nu(t))}{\partial y}\right) dt\right]$$

with

$$\kappa_j = \frac{((\partial\eta/\partial y)(\partial\varphi/\partial x) - (\partial\eta/\partial x)(\partial\varphi/\partial y) + (\partial\varphi/\partial x))P_+ + ((\partial\xi/\partial x)(\partial\varphi/\partial y) - (\partial\xi/\partial y)(\partial\varphi/\partial x) + (\partial\varphi/\partial y))Q_+}{(\partial\varphi/\partial x)P + (\partial\varphi/\partial y)Q}$$

and P, Q, $\partial\xi/\partial x$, $\partial\xi/\partial y$, $\partial\eta/\partial x$, $\partial\eta/\partial y$, $\partial\varphi/\partial x$ *and* $\partial\varphi/\partial y$ *are calculated at the point* $(\mu(\tau_j), \nu(\tau_j))$, $P_+ = P(\mu(\tau_j^+), \nu(\tau_j^+))$, $Q_+ = Q(\mu(\tau_j^+), \nu(\tau_j^+))$, *and* $\tau_j (j \in N)$ *is the time of the j-th jump. Then,* $(\mu(t), \nu(t))$ *is orbitally asymptotically stable.*

For any two points (x_1, y_1) and (x_2, y_2) on a trajectory of continuous system (2.1), we have

$$a \ln\left(\frac{y_2}{y_1}\right) - b(y_2 - y_1) = \frac{\lambda}{h} \ln\left(\frac{1 + bhx_2}{1 + bhx_1}\right) - d \ln\left(\frac{x_2}{x_1}\right). \tag{2.8}$$

For convenience in this paper, we give the following notations

$$L_1 := \frac{\lambda}{h} \ln\left(\frac{1+bhh_1}{1+(1-p)bhh_2}\right) - d\ln\left(\frac{h_1}{(1-p)h_2}\right)$$

and

$$L_2 := \frac{\lambda}{h} \ln\left(\frac{1+bhh_2}{1+bhh_1}\right) - d\ln\left(\frac{h_2}{h_1}\right).$$

3. Existence and stability of positive periodic solution with $\tau = 0$

Let $y(t) = 0$ for $t \in [0, \infty)$, then from system (1.1) we have

$$\begin{cases} \dfrac{dx(t)}{dt} = ax(t), & x \neq h_2, \\ \triangle x = x(t^+) - x(t) = -px, & x = h_2. \end{cases}$$

Set $x_0 = x(0) = (1-p)h_2$, then the solution of equation $dx(t)/dt = ax(t)$ is $x(t) = c\exp(at)$, where $c = (1-p)h_2$. Let $T = -a^{-1}\ln(1-p)$, then $x(T) = h_2$ and $x(T^+) = x_0$. This means that system (1.1) with $\tau = 0$ has the following semi-trivial periodic solution for $(k-1)T < t \leq kT (k = 1, 2, \ldots)$,

$$\begin{cases} \phi(t) = (1-p)h_2 \exp[a(t-(k-1)T)], \\ \varphi(t) = 0. \end{cases} \tag{3.1}$$

On the stability of this semi-trivial periodic solution, we have the following result.

Theorem 1. *Suppose that the following condition*

$$0 < \mu = (1-q)(1-p)^{\frac{d}{a}} \left(\frac{1+bhh_2}{1+(1-p)bhh_2}\right)^{\lambda/ah} < 1 \tag{3.2}$$

holds. Then, (3.1) be a stable semi-trivial periodic solution of system (1.1) with $\tau = 0$.

Proof. In fact, from $\varphi(t) = 0$ and (2.3), we have

$$\frac{d\Psi(t)}{dt} = \begin{pmatrix} a & -b\phi(t) \\ 0 & \dfrac{\lambda b\phi(t)}{1+bh\phi(t)} - d \end{pmatrix} \Psi(t), \quad M(0) = I. \tag{3.3}$$

Let

$$\Psi(t) = \begin{pmatrix} \psi_{11}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{22}(t) \end{pmatrix},$$

then, by (3.3) we can obtain that

$$\begin{cases} \dfrac{d\psi_{11}(t)}{dt} = a\psi_{11}(t) - b\phi(t)\psi_{21}(t), & \psi_{11}(0) = 1, \\ \dfrac{d\psi_{12}(t)}{dt} = a\psi_{12}(t) - b\phi(t)\psi_{22}(t), & \psi_{12}(0) = 0, \\ \dfrac{d\psi_{21}(t)}{dt} = [-d + \dfrac{\lambda b\phi(t)}{1+bh\phi(t)}]\psi_{21}(t), & \psi_{21}(0) = 0, \\ \dfrac{d\psi_{22}(t)}{dt} = [-d + \dfrac{\lambda b\phi(t)}{1+bh\phi(t)}]\psi_{22}(t), & \psi_{22}(0) = 1, \end{cases} \tag{3.4}$$

for $0 < t < T$.

Let $\tilde{z}(t) = (\tilde{\phi}(t), \tilde{\varphi}(t))$ be any positive solution of system (1.1) with the initial condition $\tilde{z}(0) = ((1-p)h_2, y_0)(y_0 > 0)$. Note that $u_0 = 0$ and $g_2(T) = 0$, from (2.2) and (2.4), we have

$$v_1 = (1-q)\left[v(T) - \frac{f_2(T)u(T)}{f_1(T)}\right] = (1-q)v(T)$$
$$= (1-q)[\psi_{21}(T)u_0 + \psi_{22}(T)v_0] = (1-q)\psi_{22}(T)v_0$$

Now it is only necessary to calculate $\psi_{22}(t)$. From the fourth equation of (3.4), we obtain that

$$\psi_{22}(t) = c_1 \exp\left\{\int \left[-d + \frac{(1-p)\lambda bh_2 \exp(at)}{1+(1-p)bhh_2\exp(at)}\right] \mathrm{d}t\right\}$$
$$= c_1[1+(1-p)bhh_2\exp(at)]^{\lambda/ah}\exp(-\mathrm{d}t),$$

where $c_1 = [1+(1-p)bhh_2]^{-\lambda/ah}$. Further, from $T = -a^{-1}\ln(1-p)$, it follows that

$$\psi_{22}(T) = (1-p)^{\frac{d}{a}}\left(\frac{1+bhh_2}{1+(1-p)bhh_2}\right)^{\lambda/ah}.$$

Therefore,

$$v_1 = (1-q)\psi_{22}(T)v_0$$
$$= (1-q)(1-p)^{\frac{d}{a}}\left(\frac{1+bhh_2}{1+(1-p)bhh_2}\right)^{\lambda/ah} v_0.$$

Note that $y_0 = 0$ is a fixed point of $P_1(q, y_k)$ and

$$D_{v_0}P_1(q,0) = (1-q)(1-p)^{\frac{d}{a}}\left(\frac{1+bhh_2}{1+(1-p)bhh_2}\right)^{\lambda/ah}.$$

If (3.2) holds, then $0 < D_{v_0}P_1(q,0) < 1$. Thus, (3.1) is a stable semi-trivial periodic solution of system (1.1) with $\tau = 0$. This completes the proof of this theorem. □

Theorem 2. *For any $p, q \in (0, 1)$, if $h_2 < [(1-p)^{-dh/\lambda}(1-q)^{-ah/\lambda} - 1]/pbh$. Then system (1.1) has a unique positive order-1 periodic solution $(\phi(t), \varphi(t))$ which starts from (h_2, η_0), and the period T of the periodic solution is*

$$T = \int_{(1-p)h_2}^{h_2} \frac{\mathrm{d}x}{x(1-by_0(x))},$$

where,

$$y_0(x) = -\frac{a}{b}W_0\left(-(1-q)\frac{b\eta_0}{a}\exp\left(\frac{A(x)-(1-q)b\eta_0}{a}\right)\right)$$

and

$$A(x) = \frac{\lambda}{h}\ln\left(\frac{1+bhx}{1+(1-p)bhh_2}\right) - d\ln\left(\frac{x}{(1-p)h_2}\right).$$

The proof of Theorem 2 is similar to that of Theorems 4.1 and 4.3 of [21], so we omit it here.

Next, we state and prove our result on the stability of a positive order-1 periodic solution $(\phi(t), \psi(t))$ of system (1.1).

Theorem 3. *For any $p, q \in (0, 1)$, if $h_2 < [(1-p)^{-dh/\lambda}(1-q)^{-ah/\lambda} - 1]/pbh$. Then $(\phi(t), \psi(t))$ is orbitally asymptotically stable and enjoys the property of asymptotic* phase.

Proof. Comparing with system (2.7), we have

$$P(x,y) = [a - by]x, \quad Q(x,y) = \left[\frac{\lambda bx(t)}{1+bhx(t)} - d\right]y,$$

and $\xi(x, y) = -px$, $\eta(x, y) = -qy$, $\varphi(x, y) = x - h_2$, $(\phi(T), \psi(T)) = (h_2, \eta_0)$ and $(\phi(T^+), \psi(T^+)) = ((1 - p)h_2, (1 - q)\eta_0)$. Thus

$$\frac{\partial P}{\partial x} = a - by, \quad \frac{\partial Q}{\partial y} = \frac{\lambda bx(t)}{1 + bhx(t)} - d \tag{3.5}$$

and

$$\frac{\partial \xi}{\partial x} = -p, \quad \frac{\partial \eta}{\partial y} = -q, \quad \frac{\partial \varphi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial y} = 0. \tag{3.6}$$

Further, from (3.5) and (3.6), it follows that

$$\begin{aligned}\kappa &= \frac{((\partial\eta/\partial y)(\partial\varphi/\partial x) - (\partial\eta/\partial x)(\partial\varphi/\partial y) + (\partial\varphi/\partial x))P_+ + ((\partial\xi/\partial x)(\partial\varphi/\partial y) - (\partial\xi/\partial y)(\partial\varphi/\partial x) + (\partial\varphi/\partial y))Q_+}{(\partial\varphi/\partial x)P + (\partial\varphi/\partial y)Q} \\ &= \frac{P^+(\phi(T^+), \psi(T^+))(1-q)}{P(\phi(T), \psi(T))} = \frac{(1-p)(1-q)[a - (1-q)b\eta_0]}{a - b\eta_0}\end{aligned}$$

and

$$\begin{aligned}\mu &= \kappa \exp \int_0^T \left[a - b\varphi(t) + \frac{\lambda b\phi(t)}{1 + bh\phi(t)} - d\right] \mathrm{d}t = \kappa \exp \int_0^T \left(\frac{\dot{\phi(t)}}{\phi(t)} + \frac{\dot{\varphi(t)}}{\varphi(t)}\right) \mathrm{d}t \\ &= \kappa \exp \int_0^T \mathrm{d}\ln(\phi(t)\varphi(t)) = \kappa \frac{\phi(T)\varphi(T)}{\phi(0)\varphi(0)} = \frac{a - (1-q)b\eta_0}{a - b\eta_0}.\end{aligned}$$

From the geometrical construction of the phase space of the system (2.1), we know that $\eta_0 < a/b$. Therefore, it follows that $0 < \mu < 1$. So, by Lemma 1, the order-1 periodic solution $(\phi(t), \psi(t))$ of system (1.1) is orbitally asymptotically stable and has the asymptotic phase property. This completes the proof of this theorem. □

Remark 1. We must note that the periodic solution $(\phi(t), \varphi(t))$ is locally stable rather than globally stable. That is, the solutions with different initial conditions may approach different states.

4. Existence and stability of positive periodic solutions

In this subsection, we give the sufficient conditions for the existence and stability of positive periodic solutions in the cases of $h_2 \leq d/(\lambda b - bdh)$ and $h_1 < d/(\lambda b - bdh) < h_2$.

4.1. The case of $h_2 \leq d/(\lambda b - bdh)$

On the existence of a positive periodic solution of system (1.1), we have the following theorem.

Theorem 4. *For any $p, q \in (0, 1)$, if*

$$0 < \tau < \frac{a}{b} W_0 \left(-\exp\left(\frac{L_1 - a}{a}\right)\right) + \frac{a}{b} := \tau_0$$

holds. Then system (1.1) has a positive order-1 periodic solution.

Proof. Let the trajectory $O^+(E^*, t_0)$ of system (1.1) start from the initial point $E^*((1-p)h_2, a/b)$ intersects the section $\sum_2$ at the point $F^*(h_1, y^*)$. Then, from (2.8), we can determine y^* from the following relation

$$a \ln\left(\frac{y^* b}{a}\right) - b\left(y^* - \frac{a}{b}\right) = \frac{\lambda}{h} \ln\left(\frac{1 + bhh_1}{1 + (1-p)bhh_2}\right) - d\ln\left(\frac{h_1}{(1-p)h_2}\right) = L_1,$$

$$-\frac{by^*}{a} \exp(-\frac{by^*}{a}) = -\exp\left(\frac{L_1 - a}{a}\right).$$

Let

$$f(x) = \frac{\lambda}{h} \ln \left(\frac{1 + bhx}{1 + (1-p)bhh_2} \right) - d \ln \left(\frac{x}{(1-p)h_2} \right).$$

We note that $f((1-p)h_2) = 0$and

$$f'(x) = \frac{(\lambda b - bdh)x - d}{x(1 + bhx)}.$$

From $\lambda > dh$, it follows that $f'(x) < 0$ for any $x \in ((1-p)h_2,\ d/(\lambda b - bdh))$. Therefore, $L_1 < 0$. So, we only need to consider the branch $W_0(z)$ of the Lambert W function. That is, we obtain that

$$y^* = -\frac{a}{b} W_0 \left(-\exp\left(\frac{L_1 - a}{a} \right) \right).$$

Therefore, for any $0 < \tau < \tau_0 = a/b - y^*$, the trajectory $O^+(S, t_0)$ from the point $S((1-p)h_2, y)$ $(y \in (0,\ a/b))$ on the section $\sum_1$ which will intersect with $\sum_2$ and $\sum_3$ infinite times due to the impulsive effects.

For any $0 < \tau < \tau_0$, suppose the trajectory $O^+(S_1, t_0)$ of system (1.1) starts from the initial point $S_1(h_1, \tau)$ intersects $\sum_3$ at the point $S_2(h_2, \tau^*)$. Then, by (2.8), we can determine τ^* from the following relation

$$a \ln \left(\frac{\tau^*}{\tau} \right) - b\left(\tau^* - \tau\right) = \frac{\lambda}{h} \ln \left(\frac{1 + bhh_2}{1 + bhh_1} \right) - d \ln \left(\frac{h_2}{h_1} \right) = L_2,$$

$$-\frac{b\tau^*}{a} \exp(-\frac{b\tau^*}{a}) = -\frac{b\tau}{a} \exp\left(\frac{L_2 - b\tau}{a} \right).$$

Similarly, it is easy to prove that $L_2 < 0$. So, we only need to consider the branch $W_0(z)$ of the Lambert W function. Therefore, we have

$$\tau^* = -\frac{a}{b} W_0 \left(-\frac{b\tau}{a} \exp\left(\frac{L_2 - b\tau}{a} \right) \right).$$

Let the point $E_1((1-p)h_2, \alpha_1)$ on $\sum_1$ and $\alpha_1 < (1-q)\tau^*$. Suppose that the trajectory $O^+(E_1, t_0)$ of system (1.1) intersects $\sum_2$ at the point $F_1(h_1, \beta_1)$, and then jumps to the point $F_1^+(h_1, \beta_1 + \tau)$ due to the impulsive $\triangle y(t) = \tau$. Further, the trajectory $O^+(E_1, t_0)$ intersects $\sum_3$ at the point $G_1(h_2, \gamma)$ and jumps to the point $E_2((1-p)h_2, \alpha_2)$ on $\sum_1$, where $\alpha_2 = (1-q)\gamma_1$. At the state E_2, the trajectory intersects $\sum_2$ and $\sum_3$ at the points $F_2(h_1, \beta_2)$, $F_2^+(h_1, \beta_2 + \tau)$ and $G_2(h_2, \gamma_2)$, respectively. By $\alpha_1 < (1-q)\tau^*$, $\tau < \beta_1 + \tau$ and the geometrical construction of the phase space of system (2.1), we obtain that the point E_2 is above the point E_1. Further, the point G_2 is above the point G_1. Hence, from (2.5) we have $\gamma_2 = P_2(q, \tau, \gamma_1)$ and

$$\gamma_1 - P_2(q, \tau, \gamma_1) = \gamma_1 - \gamma_2 < 0. \tag{4.1}$$

On the other hand, for any $0 < \tau < \tau_0$, suppose that the trajectory $O^+(A_1, t_0)$ from the initial point $A_1((1-p)h_2, a/b)$ intersects $\sum_2$ and $\sum_3$ at the points $B_1(h_1, \mu_1)$, $B_1^+(h_1, \mu_1 + \tau)$ and $C_1(h_2, \nu_1)$, respectively. At the state C_1, the trajectory $O^+(A_1, t_0)$ jumps to the point $A_2((1-p)h_2, (1-q)\nu_1)$ on the section $\sum_1$ and then intersects $\sum_2$ and $\sum_3$ at the points $B_2(h_1, \mu_2)$, $B_2^+(h_1, \mu_2 + \alpha)$ and $C_2(h_2, \nu_2)$ again. In view of the geometrical construction of the phase space of system (1.1), we obtain that the point C_2 is under the point C_1 for any $p, q \in (0,\ 1)$ and $\tau \in (0,\ \tau_0)$. Thus, from (2.5) we have

$$\nu_1 - P_2(q, \tau, \nu_1) = \nu_1 - \nu_2 > 0. \tag{4.2}$$

By (3.5) and (3.6), it follows that the Poincaré map (2.5) has a fixed point, that is the system (1.1) has a positive order-1 periodic solution. This completes the proof of this theorem. □

Remark 2. Considering the geometrical construction of the phase space of the system (2.1), if $\tau > \tau_0$, the trajectory which starting from the point $S((1-p)h_2, y)(y < a/b)$ may intersect $\sum_3$ finite times. So, $0 < \tau < \tau_0$ is a sufficient condition for system (1.1) has a positive order-1 periodic solution.

Let $(\phi(t), \varphi(t))$ be a positive order-1 periodic solution of system (1.1) which starts from the point (h_2, λ_0). Next, we will state and prove our result on the stability of solution $(\phi(t), \varphi(t))$.

Theorem 5. *For any $p, q \in (0, 1)$ and $0 < \tau < \tau_0$, if*

$$\mu = \left| \frac{[a + aW_1(\lambda_0) - b\tau][1 - (1-q)b\lambda_0]W_1(\lambda_0)}{a(a - b\lambda_0)[aW_1(\lambda_0) - b\tau][1 + W_1(\lambda_0)]} \right| < 1 \tag{4.3}$$

holds, where τ_0 and $W_1(\lambda_0)$ are given in Theorem 4 and (4.7), respectively. Then $(\phi(t), \varphi(t))$ is a locally orbitally asymptotically stable positive order-1 periodic solution of system (1.1) and which has asymptotic phase property.

Proof. From the geometrical construction of the phase space of the system (2.1), we know that $\lambda_0 < a/b$. Therefore, for any point $C_k(h_2, y_k)$ on $\sum_3$ and $0 < y_k < a/b$, the trajectory of system (1.1) starts from the point C_k intersects $\sum_1$, $\sum_2$ and $\sum_3$ at the points $A_{k+1}((1-p)h_2, (1-q)y_k)$, $B_{k+1}(h_1, y^*_{k+1})$, $B^+_{k+1}(h_1, y^*_{k+1} + \tau)$ and $C_{k+1}(h_2, y_{k+1})$ respectively, then $0 < y_{k+1} = P_2(q, \tau, y_k) < a/b$. From (2.8), the relation between A_{k+1} and B_{k+1} is

$$\begin{aligned} & a \ln\left(\frac{y^*_{k+1}}{(1-q)y_k}\right) - b(y^*_{k+1} - (1-q)y_k) \\ &= \frac{\lambda}{h} \ln\left(\frac{1 + bhh_1}{1 + (1-p)bhh_2}\right) - d \ln\left(\frac{h_1}{(1-p)h_2}\right) \\ &= L_1, -\frac{by^*_{k+1}}{a} \exp(-\frac{by^*_{k+1}}{a}) = -\frac{(1-q)by_k}{a} \exp\left(\frac{L_1 - (1-q)by_k}{a}\right). \end{aligned}$$

From the proof of Theorem 4, we note that $L_1 < 0$ for any $p \in (0, 1)$. Since $0 < y_k, y^*_{k+1} < a/b$, here we only need to consider the branch $W_0(z)$ of the Lambert W function. Thus, we obtain that

$$y^*_{k+1} = -\frac{a}{b} W_0\left(-\frac{(1-q)by_k}{a} \exp\left(\frac{L_1 - (1-q)by_k}{a}\right)\right) := \omega(y_k). \tag{4.4}$$

Next, we calculate the relation of the points B^+_{k+1} and C_{k+1}. From (2.8), it follows that

$$a \ln\left(\frac{y_{k+1}}{y^*_{k+1} + \tau}\right) - b(y_{k+1} - (y^*_{k+1} + \tau)) = \frac{\lambda}{h} \ln\left(\frac{1 + bhh_2}{1 + bhh_1}\right) - d \ln\left(\frac{h_2}{h_1}\right) = L_2,$$

$$\frac{by_{k+1}}{a} \exp(-\frac{by_{k+1}}{a}) = -\frac{b(\omega(y_k) + \tau)}{a} \exp\left(\frac{L_2 - b(\omega(y_k) + \tau)}{a}\right).$$

Similarly to the proof of Theorem 4, it is easy to prove that $L_2 < 0$. Therefore, by $0 < y_k, y_{k+1} < a/b$, we have

$$y_{k+1} = -\frac{a}{b} W_0\left(-\frac{b(\omega(y_k) + \tau)}{a} \exp\left(\frac{L_2 - b(\omega(y_k) + \tau)}{a}\right)\right) := P_2(q, \tau, y_k). \tag{4.5}$$

From (2.6), the derivative of the Poincaré map (4.5) with respect to y_k, it follows that

$$\frac{\partial P_2(q, \tau, y_k)}{\partial y_k} = -\frac{aW_0(Z)}{bZ(1 + W_0(Z))} \frac{\partial Z}{\partial y_k},$$

where

$$Z = -\frac{b(\omega(y_k) + \tau)}{a} \exp\left(\frac{L_2 - b(\omega(y_k) + \tau)}{a}\right).$$

Further, by (2.6) and (4.4) the derivative of the Z with respect to y_k, we have

$$\frac{\partial Z}{\partial y_k} = \left(1 + W_1(y_k) - \frac{b\tau}{a}\right) \frac{(1 - \frac{(1-q)by_k}{a})W_1(y_k)}{(1 + W_1(y_k))y_k} \exp\left(\frac{L_2 - b(\omega(y_k) + \tau)}{a}\right), \tag{4.6}$$

where

$$W_1(y_k) = W_0\left(-\frac{(1-q)by_k}{a}\exp\left(\frac{L_1-(1-q)by_k}{a}\right)\right). \tag{4.7}$$

Note that $(\phi(t), \varphi(t))$ be a positive order-1 periodic solution of system (1.1) which starts from the point (h_2, η_0), so η_0 is a fixed point of $P_2(q, \tau, y_k)$. Therefore from (4.4)–(4.6), we have

$$\frac{\partial P_2(q,\tau,\eta_0)}{\partial y_k} = \frac{[a+aW_1(\lambda_0)-b\tau][1-(1-q)b\lambda_0]W_1(\lambda_0)}{a(a-b\lambda_0)[aW_1(\lambda_0)-b\tau][1+W_1(\lambda_0)]}.$$

If (4.3) holds, then $|\partial P_2(q, \tau, \lambda_0)/\partial y_k| < 1$. Thus, the solution $(\phi(t), \varphi(t))$ of system (1.1) is orbitally asymptotically stable and has the asymptotic phase property. This completes the proof of this theorem. □

Remark 3. From the proof of Theorem 5, we note that the solution $(\phi(t), \varphi(t))$ of system (1.1) is unstable if $\mu > 1$ and is the critical case when $\mu = 1$.

4.2. The case of $h_1 \le d/(\lambda b - bdh) < h_2$

In order to discuss this case, let

$$g(x) = \frac{\lambda}{h}\ln\left(\frac{1+bhx}{1+bhh_1}\right) - d\ln\left(\frac{x}{h_1}\right).$$

It is easy to see that $g(h_1) = 0$ and

$$g'(x) = \frac{(\lambda b - bdh)x - d}{x(1+bhx)}.$$

So, we get that $g'(x) < 0$ for any $x \in (h_1,\ d/(\lambda b - bdh))$ and $g'(x) > 0$ for any $x \in (d/(\lambda b - bdh),\ \infty)$.

On the other hand, by $\lambda > dh$, we have

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\frac{1}{h}\left[\ln\left(\frac{1+bhx}{1+bhh_1}\right)^{\lambda} - \ln\left(\frac{x}{h_1}\right)^{dh}\right]$$

$$= \lim_{x\to\infty}\frac{1}{h}\ln\left[\left(\frac{(1+bhx)h}{(1+bhh1)x}\right)^{dh}\left(\frac{1+bhx}{1+bhh1}\right)^{\lambda-dh}\right] = \infty.$$

Therefore, there is a constant $H^* \in (d/(\lambda b - bdh),\ \infty)$ such that $g(x) < 0$ for any $x \in (h_1,\ H^*)$, where H^* is a solution of the following equation

$$\left(\frac{1+bhx}{1+bhh_1}\right)^{\lambda} = \left(\frac{x}{h_1}\right)^{dh}.$$

On the existence of a positive periodic solution of system (1.1), we have the following result.

Theorem 6. For any $p, q \in (0,\ 1)$, if $h_2 < H^*$ and $0 < \tau < \tau^*$, where $\tau^* = \rho^* - \rho_0$ if the trajectory $O^+((h_2, a/b), 0)$ of system (2.1) intersects with $\sum^{h_1}$ and $\rho^* > \rho_0$; otherwise, $\tau^* = a/b - \rho_0$. ρ^* and ρ_0 are given in (4.8) and (4.9), respectively. Then system (1.1) has a positive order-1 periodic solution.

Proof. From the geometrical construction of the phase space of the system (2.1), if the trajectory Γ of system (2.1) which starts from $Q_2(h_2, a/b)$ on $\sum_3$ intersects with $\sum_2$ at the point $Q_1(h_1, \rho^*)$, where $\rho^* \le a/b$. Let $\rho = a/b$, by (2.8), we have

$$a\ln\left(\frac{\rho}{\rho^*}\right) - b\left(\rho-\rho^*\right) = \frac{\lambda}{h}\ln\left(\frac{1+bhh_2}{1+bhh_1}\right) - d\ln\left(\frac{h_2}{h_1}\right) = L_2,$$

$$-\frac{b\rho}{a}\exp\left(-\frac{b\rho}{a}\right) = -\frac{b\rho^*}{a}\exp\left(\frac{L_2-b\rho^*}{a}\right).$$

From $g(x) < 0$ for any $x \in (h_1,\ H^*)$, it follows that $L_2 < 0$. Hence, we only need to consider the branch $W_0(z)$ of the Lambert W function. Therefore, we have

$$-\frac{b}{a}\rho = -1 = W_0\left(-\frac{b\rho^*}{a}\exp\left(\frac{L_2 - b\rho^*}{a}\right)\right).$$

By Definition 3, we obtain that ρ^* satisfies the following equation

$$\rho^* \exp\left(\frac{L_2 - b\rho^*}{a}\right) = \frac{a}{b}\exp(-1). \tag{4.8}$$

Otherwise, the trajectory Γ does not intersect with $\sum_2$ and $\rho^* = a/b$.

On the other hand, suppose that the trajectory $O^+(C_1, t_0)$ starts from the point $C_1(h_2, a/b)$ which intersects $\sum_1$ and $\sum_2$ at the points $A_1((1-p)h_2, (1-q)a/b)$ and $B_1(h_1, \rho_0)$ due to impulsive effects. Similarly, we can obtain ρ_0 from the following relation

$$\rho_0 = -\frac{a}{b}W_0\left(-(1-q)\exp(\frac{L_1 - (1-q)a}{a})\right). \tag{4.9}$$

By the conditions of Theorem 6, we have $0 < \tau^* = \rho^* - \rho_0$. Thus, for any $\tau \in (0,\ \tau^*)$, the trajectory of system (1.1) which starts from the point $G_1(h_2, y_1)$ $(y_1 < a/b)$ will intersects with $\sum_2$ and $\sum_3$ infinite times due to the impulsive effects.

Similar to the proof of Theorem 4, we obtain that system (1.1) has a positive order-1 periodic solution and this completes the proof. □

Remark 4. From the geometrical construction of the phase space of the system (2.1), we note that the trajectory of system (1.1) may intersect with $\sum_3$ finite times if $\tau > \tau^*$. So, $0 < \tau < \tau^*$ is a sufficient condition for system (1.1) to have a positive order-1 periodic solution.

Finally, on the stability of a positive order-1 periodic solution of system (1.1), we have the following result.

Theorem 7. *Let $(\varphi(t), \psi(t))$ be a positive order-1 periodic solution of system* (1.1) *which starts from the point (h_2, λ_0). Suppose that conditions of* Theorem 6 *hold. If*

$$\nu = \left|\frac{[a + aW_1(\lambda_0) - b\tau][1 - (1-q)b\lambda_0]W_1(\lambda_0)}{a(a - b\lambda_0)[aW_1(\lambda_0) - b\tau][1 + W_1(\lambda_0)]}\right| < 1$$

holds, where $w_1(\eta_0)$ is given in (4.7). *T*hen *$(\varphi(t), \psi(t))$is locally orbitally asymptotically stable and has the asymptotic phase property.*

The proof of Theorem 7 is similar to that of Theorem 5, we therefore omit it here.

5. Example, numerical simulation and discussion

In this paper, we investigate a class of predator–prey models with state dependent impulsive effects. By using Poincaré map and Lambert W function, we give the criteria for the existence and stability of semi-trivial solution and positive periodic solution of system (1.1).

In order to testify the validity of our results, in system (1.1), let $a = 0.8, b = 0.6, \lambda = 0.5, d = 0.2, h = 0.02, \tau \geq 0$, $p, q \in (0,\ \ 1)$, $h_1 > 0$, $h_2 > 0$ and $(1-p)h_2 < h_1$. Obviously, we obtain that system (1.1) without impulsive effects has a stable focus (0.6720, 1.3333). Now, we consider the impulsive effects influence the dynamics of system (1.1).

Example 1. Existence and stability of positive periodic solution with $\tau = 0$.

In system (1.1), let $p = 0.4$, $q = 0.3$, $\tau = 0$ and $h_2 = 1.1$. It is easy to compute that system (1.1) has the following semi-trivial periodic solution for $(k-1)T < t \leq kT (k = 1, 2, \ldots)$,

$$\begin{cases} \phi(t) = 0.66\exp[0.8(t - (k-1)T)], \\ \varphi(t) = 0, \end{cases} \tag{5.1}$$

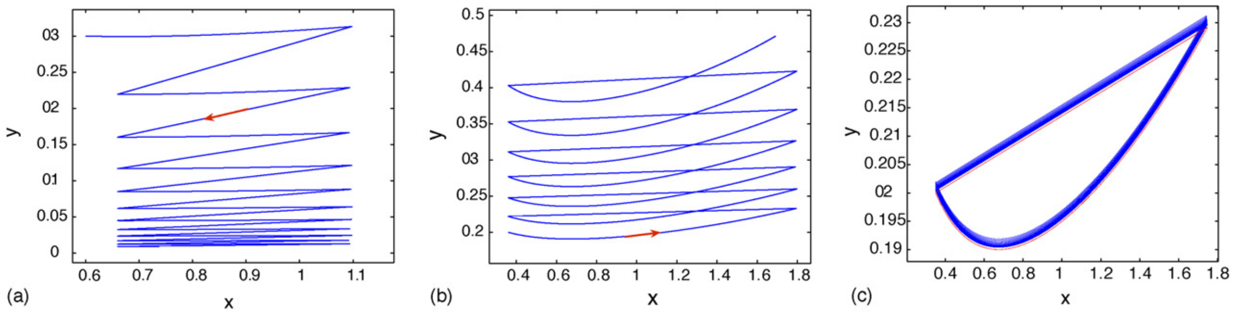

Fig. 1. The trajectory of system (1.1) with $\tau = 0$: (a) $p = 0.4$, $q = 0.3$ and $h_2 = 1.1$; (b) $p = 0.8$, $q = 0.05$ and $h_2 = 1.8$; (c) $p = 0.8$, $q = 0.13$ and $h_2 = 1.75$.

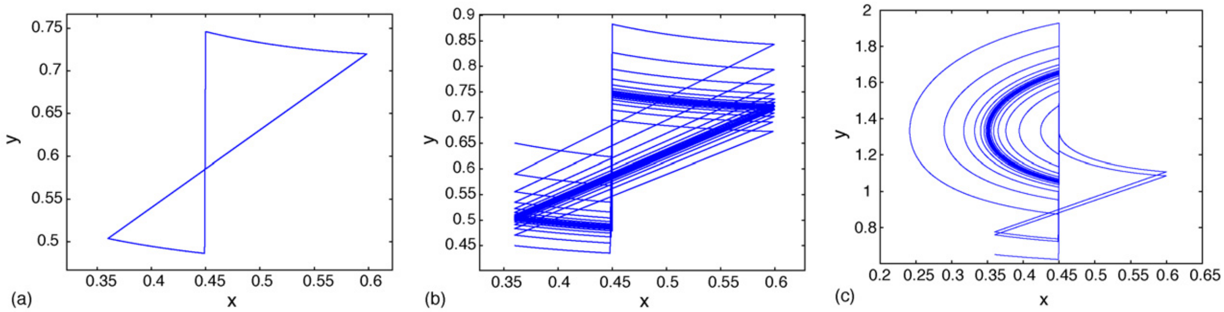

Fig. 2. The trajectory of system (1.1) with $p = 0.4$, $q = 0.3$, $h_1 = 0.45$, $h_2 = 0.6$: (a) $\tau = 0.26$; (b) $\tau = 0.26$; (c) $\tau = 0.6$.

where $T = 1.25 \ln(5/3)$. From Theorem 1, we can easily compute $\mu \approx 0.7253 < 1$. So, condition (3.2) holds. Therefore, from Theorem 1, system (1.1) has a typical stable of semi-trivial periodic solution (5.1), which shown in Fig. 1(a). However, if we choose $p = 0.8$, $q = 0.05$, $\tau = 0$ and $h_2 = 1.8$ in system (1.1), we can compute $\mu \approx 1.0827 > 1$. In this case, system (1.1) has an unstable semi-trivial periodic solution. Numerical simulation of the result can be seen in Fig. 1(b). Further, we choose $p = 0.8$, $q = 0.13$ and $h_2 = 1.75$, we can compute $[(1-p)^{-dh/\lambda}(1-q)^{-ah/\lambda} - 1]/pbh \approx 1.8211 > 1.75 = h_2$. So, by Theorems 2 and 3, system (1.1) has a locally stable positive periodic solution, which can be seen in Fig. 1(c).

Example 2. Existence and stability of positive periodic solutions with $h_2 \leq \lambda/(\lambda b - bdh)$.

In system (1.1), let $p = 0.4$, $q = 0.3$, $h_1 = 0.75$ and $h_2 = 1.1$. We easily compute $\tau^* = -W_0(-0.3666) + W_0(-0.3265) \approx 0.3178$. So, system (1.1) has a positive order-1 or order-2 periodic solution when $\tau < \tau^*$, and may have no positive order-1 periodic solution when $\tau > \tau^*$. Which are shown in Fig. 2.

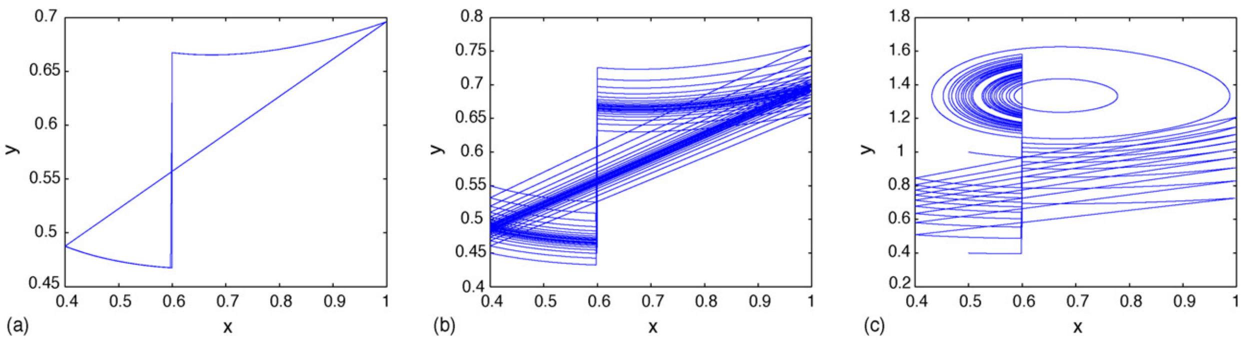

Fig. 3. The trajectory of system (1.1) with $p = 0.4$, $q = 0.3$, $h_1 = 0.6$ and $h_2 = 1$: (a) $\tau = 0.2$; (b) $\tau = 0.2$; (c) $\tau = 0.3$.

Example 3. Existence and stability of positive periodic solutions with $h_1 \leq (b_2/a_{21}) < h_2$.

In system (1.1), let $p = 0.6$, $q = 0.3$, $\tau = 0.2$, $h_1 = 0.6$ and $h_2 = 1$. We note that system (1.1) has a positive order-1 periodic solution which is orbitally asymptotically stable and has the asymptotic phase property when $\tau = 0.2$, and system (1.1) may not have a positive order-1 periodic solution when $\tau = 0.3$. These are shown in Fig. 3.

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