Some Characterizations of Global Exponential Stability of a Generic Class of Continuous-Time Recurrent Neural Networks

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***Abstract*—This paper reveals two important characterizations of global exponential stability (GES) of a generic class of continuous-time recurrent neural networks. First, we show that GES of the neural networks can be fully characterized by global asymptotic stability (GAS) of the networks plus the condition that the maximum abscissa of spectral set of Jacobian matrix of the neural networks at the unique equilibrium point is less than zero. This result provides a very useful and direct way to distinguish GES from GAS for the neural networks. Second, we show that when the neural networks have small state feedback coefficients, the supremum of exponential convergence rates (ECRs) of trajectories of the neural networks is exactly equal to the absolute value of the maximum abscissa of spectral set of Jacobian matrix of the neural networks at the unique equilibrium point. Here, the supremum of ECRs indicates the potentially fastest speed of trajectory convergence. The obtained results are helpful in understanding the essence of GES and clarifying the difference between GES and GAS of the continuous-time recurrent neural networks.**

***Index Terms*—Continuous-time recurrent neural networks, exponential convergence rate (ECR), global asymptotic stability (GAS), global exponential stability (GES), global stability.**

I. Introduction

CONSIDER THE generic continuous-time recurrent neural networks modeled by the following nonlinear differential equation:

$$\dot{x}(t) = -Dx(t) + W \cdot f\left(Ax(t) + u_1\right) + u_2, x(0) \in R^n \quad (1)$$

where the diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ with $d_i > 0$ being the state feedback coefficients, W and A are two real matrices in which W is an $n \times n$ interconnection matrix and A is a diagonal matrix or, alternatively, A is an $n \times n$ interconnection matrix and W is a diagonal matrix, $x(t) = (x_1(t), x_2(t),\ldots,x_n(t))^{\mathrm{T}}$ is the neural network state, $u_1,u_2 \in R^n$ are two constant external input vectors, and $f(x)=(f_1(x_1), f_2(x_2),\ldots,f_n(x_n))^{\mathrm{T}}$ with $f_i : R \longrightarrow R$ being the activation function used in the ith neuron, which is normally assumed to be nonlinear, continuous, and bounded. The model (1) uniformly describes various continuous-time recurrent neural networks studied in literature. For examples, it describes the Hopfield-type neural networks and their various extensions (corresponding to $u_1 = \theta$ and $A = I$) [1]–[22], and it also casts the recurrent back-propagation networks (corresponding to $u_2 = \theta$ and $W = I$) [13], [23]. In this paper, we study global exponential stability (GES) or global asymptotic stability (GAS) of the generic continuous-time recurrent neural networks (1).

It is known that when applying neural networks to practical problems in optimization, control, and signal processing, one needs to design a neural network of the form (1) that has a unique equilibrium state with GAS or GES [4], [7], [9], [17]. Hence, exploring global stability of neural networks has become an important and attractive research topic. For instance, global stability of the continuous-time recurrent neural networks is intensively studied in [1]–[22]. In this paper, for studying global stability purpose, we will assume that the equilibrium point of system (1) is unique. Denote the unique equilibrium point henceforth by x^*. Moreover, we assume that the nonlinear activation mapping f is locally continuously differentiable at x^* (i.e., continuously differentiable in one neighborhood of x^*) and that there is a positive constant $K > 0$ such that for any $x \in R^n$

$$\left\| f(x) - f(x^*) \right\| \leq K \cdot \left\| x - x^* \right\|. \quad (2)$$

Note that since $f(x)$ is very often component-wisely defined in application, the condition (2) can be equivalently written as [34]

$$\left| f_i(x_i) - f_i\left(x_i^*\right) \right| \leq K \cdot \left| x_i - x_i^* \right|, \qquad i=1,2,\ldots,n. \quad (3)$$

Most popular activation functions, like sigmoid functions, saturation functions, and Lipschitz continuous activation functions, all satisfy the condition (2). In this paper, we will study global stability of system (1) by uncovering a fundamental, necessary, and sufficient condition for GES of the system.

Recall that system (1) is said to be globally exponentially stable in R^n whenever there exist two constants $\gamma > 0$ and

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$M > 0$ such that for any $x(0) \in R^n$ and any $t > 0$

$$\|x(t) - x^*\| \leq M e^{-\gamma t} \|x(0) - x^*\|. \quad (4)$$

The system is said to be globally asymptotically stable in R^n if it is locally stable in the sense of Lyapunov and globally attractive. GES and GAS are two different types of global stability. If system (1) is globally exponentially stable, then it is globally asymptotically stable. The inverse is, however, not true in general [1], [19] (see also the example provided in Section IV). Both GES and GAS of continuous-time neural networks have been widely studied in existing literature. For example, in [4]–[13], researchers have discussed GAS of the continuous-time recurrent neural networks and presented many different sufficient conditions. In [1]–[4], [14]–[22], researchers have studied GES of the neural networks and provided various sufficient conditions. In these researches, a general observation is that sufficient conditions proposed previously for GAS in a paper were shown to be able to guarantee GES of the same networks later in another paper. For instance, the sufficient conditions for GAS of Hopfield-type neural networks, developed in [9], [10], [21], [22], and [24], were shown later to be sufficient also for GES of the networks in [19]; the condition for GAS in [16] was shown also to be for GES of the neural networks later in [4]. Zhang *et al.* [1] showed that the similar conditions for GAS in [7] and [16] actually imply GES of the networks. Nevertheless, in very often cases, when researchers presented a sufficient condition for GAS of a neural network, they were unable to make sure whether the condition is also sufficient for GES of the network, due to the lack of direct criteria to distinguish GES from GAS. It is known that neural networks with GES are much more robust than those with only GAS [36], and their convergence speeds are more easy to estimate. Thus, it is an important and interesting problem to seek a simple criterion for distinguishing GES from GAS for the generic continuous-time neural networks (1). To the best of our knowledge, there is still no such criterion. In this paper, we will provide such a criterion.

In (4), the parameter γ is very often named as the exponential convergence rate (ECR) of trajectories of system (1), and M is named as the convergence coefficient. It is easily seen that whenever the equilibrium point x^* is globally exponentially stable, the speed of trajectory convergence of system (1) is mainly determined by the parameter γ [1], [2], [26]. Therefore, for quantitatively assessing convergence property of the neural networks, the estimation of ECR of the networks has been widely studied in [1], [17], [25]–[28]. In these studies, researchers derived various estimations of ECR for different continuous-time neural networks and revealed that the larger the ECR of a neural network, the faster the neural network converges. Therefore, in order to reveal the potentially fastest speed of the trajectory convergence of system (1), it is needed to know the supremum of the ECRs of trajectories of system (1). In this paper, we call the supremum of ECRs of system (1) as the essential ECR (e-ECR). Clearly, as compared with any specific ECR estimated in the existing literature, the e-ECR is an intrinsic feature to characterize the fastest exponential convergence speed of the system. If η is the e-ECR of system (1), then for any sufficiently small $\varepsilon > 0$, $\eta - \varepsilon$ is an ECR. Therefore, there is a specific convergence coefficient $M(\varepsilon) > 0$ such that the following sharp exponential convergence estimation holds for any $x(0) \in R^n$ and any $t > 0$:

$$\|x(t) - x^*\| \leq M(\varepsilon) e^{-(\eta-\varepsilon)\cdot t} \|x(0) - x^*\|. \quad (5)$$

It is not difficult to see that any known ECR of system (1) estimated in the existing literature is only a special case of $\eta - \varepsilon$, corresponding to a specific choice of ε. Therefore, it is important to deduce exactly the e-ECR of system (1). To our knowledge, the e-ECR of system (1) has not been precisely calculated in the existing literature. We will provide an exact evaluation of the e-ECR of system (1) in this paper.

In this paper, through developing a necessary and sufficient condition for GES of the continuous-time recurrent neural networks (1), we provide a simple criterion for distinguishing GES from GAS for system (1). We also reveal that for the generic neural networks (1) with small state feedback coefficients, the e-ECR of system (1) is exactly equal to the absolute value of the maximum abscissa of spectral set of Jacobian matrix of system (1) at the equilibrium point x^*. The obtained results are helpful in comprehending global stability of the generic continuous-time recurrent neural networks (1).

II. Basic Properties of $M_{\|\cdot\|}[T, x^*, V]$

In this section, we will introduce a functional $M_{\|\cdot\|}[T, x^*, V]$ for the nonlinear operator $T: V \subset R^n \to R^n$, which can actually be regarded as a nonlinear generalization of the logarithmic norm of matrices. For this purpose, we first review some important properties of the logarithmic norm of matrices.

Suppose that A is an $n \times n$ real matrix and $\|\cdot\|$ is a given vector norm in R^n. Then, the subordinated matrix norm and logarithmic norm of A are, respectively, defined as

$$\|A\| = \max_{\|x\|=1} \|Ax\| \quad \mu(A) = \lim_{h \to 0^+} \frac{\|I + h \cdot A\| - 1}{h}$$

where I is the $n \times n$ identity matrix. Let Ψ denote the set of all equivalent vector norms of $\|\cdot\|$ in R^n (note that in R^n, all vector norms are equivalent), let $\sigma(A)$ denote the spectral set of A (i.e., all eigenvalues of A), and let $\alpha(A) = \max\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ denote the maximum abscissa of $\sigma(A)$. It is known that $\alpha(A) = \inf_{\|\cdot\| \in \Psi} \mu(A)$ [32], [33]. This implies that for any sufficiently small $\varepsilon > 0$, there exists an equivalent norm $\|\cdot\|_\varepsilon \in \Psi$ such that the subordinated logarithmic norm $\mu(A)$ satisfies $|\mu(A) - \alpha(A)| \leq \varepsilon$. Here, a vector norm $\|\cdot\|_*$ is said to be equivalent with $\|\cdot\|$ whenever there exist two constants $K_2 \geq K_1 > 0$ such that $K_1\|x\|_* \leq \|x\| \leq K_2\|x\|_*$ for any $x \in R^n$. For two arbitrary real $n \times n$ matrices B and C, it is obvious that $\mu(C) - \mu(B) \leqslant \mu(C - B)$ and $\mu(C - B) \leqslant \|C - B\|$ hold [32]. This then implies that

$$|\mu(C) - \mu(B)| \leqq \|C - B\|. \quad (6)$$

As nonlinear generalizations of the norm and logarithmic norm of matrices, two useful nonlinear functionals can be introduced for nonlinear operators. Suppose that $T: V \subset R^n \to R^n$ is a nonlinear operator satisfying the following condition:

$$\|Tx - Tx^*\| \leq K \cdot \|x - x^*\| \qquad \forall x \in V \quad (7)$$

where $K > 0$ is a constant, V is a subset of R^n, and x^* is the unique equilibrium point of system (1). Denote

$$L_{\|\cdot\|}(T, x^*, V) = \sup_{x \neq x^*, x \in V} \frac{\|Tx - Tx^*\|}{\|x - x^*\|}. \quad (8)$$

Then, $L_{\|\cdot\|}(T, x^*, V)$ is a nonnegative functional determined by four parameters $T, x^*, \|\cdot\|$, and V [34]. For any $\beta > 0$, we have $L_{\|\cdot\|}(\beta T, x^*, V) = \beta L_{\|\cdot\|}(T, x^*, V)$. In addition, for any nonlinear operator $G : V \subset R^n \to R^n$ satisfying (7), i.e., $\|Gx - Gx^*\| \leq K \cdot \|x - x^*\|$ for all $x \in V$, we have

$$L_{\|\cdot\|}(T + G, x^*, V) \leq L_{\|\cdot\|}(T, x^*, V) + L_{\|\cdot\|}(G, x^*, V). \quad (9)$$

Let $f(t) = ((L_{\|\cdot\|}(I + tT, x^*, V) - 1)/t), t > 0$. It is then easy to see that $f(t)$ is a monotonically increasing function in $t > 0$ and the following estimations hold:

$$-L_{\|\cdot\|}(T, x^*, V) \leq f(t) \leq L_{\|\cdot\|}(T, x^*, V) \qquad \forall t > 0.$$

This shows the existence of the limit $\lim_{h \to 0^+}((L_{\|\cdot\|}(I + hT, x^*, V) - 1)/h)$. Let

$$M_{\|\cdot\|}[T, x^*, V] = \lim_{h \to 0^+} \frac{L_{\|\cdot\|}(I + hT, x^*, V) - 1}{h}. \quad (10)$$

$L_{\|\cdot\|}(T, x^*, V)$ and $M_{\|\cdot\|}[T, x^*, V]$ can then be regarded as nonlinear generalizations of the norm and logarithmic norm of matrices, respectively. With respect to $M_{\|\cdot\|}[T, x^*, V]$, there are the following properties.

1) For any given matrix A, if $V \subset R^n$ contains the origin θ as an interior point (i.e., V contains a neighborhood of θ), then $L_{\|\cdot\|}(A, \theta, V) = \|A\|$ and $M_{\|\cdot\|}[A, \theta, V] = \mu(A)$.
2) $M_{\|\cdot\|}[T, x^*, V] \leq L_{\|\cdot\|}(T, x^*, V)$.
3) $M_{\|\cdot\|}[T + G, x^*, V] \leq M_{\|\cdot\|}[T, x^*, V] + M_{\|\cdot\|}[G, x^*, V]$.

Hereinafter, we present simple proofs of these properties. When the origin θ is an interior point of V, then there is a constant $r > 0$ such that the spherical surface $B(r) = \{x \in R^n : \|x\| = r\} \subset V$. Since A is a matrix, $\|A\| = \sup_{x \in B(r)}(\|Ax\|/\|x\|) = \sup_{x \neq \theta, x \in R^n}(\|Ax\|/\|x\|)$. Thus, we have $L_{\|\cdot\|}(A, \theta, V) = \|A\|$. Moreover, for any $h > 0$, $L_{\|\cdot\|}(I + hA, \theta, V) = \|I + hA\|$. This implies that $M_{\|\cdot\|}[A, \theta, V] = \mu(A)$. Since for any $h > 0$, $L_{\|\cdot\|}(I + hT, x^*, V) \leq 1 + h \cdot L_{\|\cdot\|}(T, x^*, V)$, it is easy to imply that $M_{\|\cdot\|}[T, x^*, V] \leq L_{\|\cdot\|}(T, x^*, V)$. For any $h > 0$, the following inequalities hold:

$$\begin{aligned} &\frac{L_{\|\cdot\|}(I + h(T+G), x^*, V) - 1}{h} \\ &\leq \frac{L_{\|\cdot\|}\left(\frac{1}{2}I + hT, x^*, V\right) + L_{\|\cdot\|}\left(\frac{1}{2}I + hG, x^*, V\right) - 1}{h} \\ &= \frac{\frac{1}{2}\left(L_{\|\cdot\|}(I + 2hT, x^*, V) + L_{\|\cdot\|}(I + 2hG, x^*, V)\right) - 1}{h} \\ &\leq \frac{\left(L_{\|\cdot\|}(I + 2hT, x^*, V) - 1\right) + \left(L_{\|\cdot\|}(I + 2hG, x^*, V)\right) - 1\right)}{2h}. \end{aligned}$$

Thus, we can easily deduce that $M_{\|\cdot\|}[T + G, x^*, V] \leq M_{\|\cdot\|}[T, x^*, V] + M_{\|\cdot\|}[G, x^*, V]$.

$M_{\|\cdot\|}[T, x^*, V]$ is an important functional that can be used to characterize the exponential stability property of nonlinear continuous dynamical systems. More precisely, with respect to the trajectories starting from V, we have the following lemma.

Lemma 1: Consider the nonlinear continuous dynamical system: $\dot{x}(t) = T(x(t)), x(0) \in R^n$. Suppose that $x^* \in R^n$ is the unique equilibrium point of the system (i.e., $Tx^* = \theta$) and V is an arbitrary open subset of R^n. Then, no matter whether $x^* \in V$, whenever a trajectory $x(t)$ satisfies $x(t) \in V$ for all $0 \leq t \leq b$ (here, b is a given constant), the following estimation holds for any $0 \leq t \leq b$:

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| \cdot e^{M_{\|\cdot\|}[T, x^*, V] \cdot t}. \quad (11)$$

Proof: Denote $v(t) = x(t) - x^*$. For any $t \leq b$, we have [33]

$$\begin{aligned} \frac{d\|v(t)\|}{dt} &= \lim_{h \to 0^+} \frac{\|v(t+h)\| - \|v(t)\|}{h} \\ &= \lim_{h \to 0^+} \frac{\|v(t) + hv'(t)\| - \|v(t)\|}{h} \\ &= \lim_{h \to 0^+} \frac{\|x(t) - x^* + hx'(t)\| - \|x(t) - x^*\|}{h} \\ &= \lim_{h \to 0^+} \frac{\|x(t) - x^* + h(Tx(t) - Tx^*)\| - \|x(t) - x^*\|}{h} \\ &\leq \lim_{h \to 0^+} \frac{L_{\|\cdot\|}(I + hT, x^*, V) - 1}{h} \cdot \|x(t) - x^*\| \\ &= M_{\|\cdot\|}[T, x^*, V] \cdot \|x(t) - x^*\| \\ &= M_{\|\cdot\|}[T, x^*, V] \cdot \|v(t)\|. \end{aligned}$$

This shows that $\|v(t)\| \leq \|v(0)\| \cdot e^{M_{\|\cdot\|}[T, x^*, V] \cdot t}$ for any $t \leq b$. Consequently, for any $t \leq b$, we have

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| \cdot e^{M_{\|\cdot\|}[T, x^*, V] \cdot t}. \qquad \blacksquare$$

III. Main Results

Recall that K-function and L-function are also useful tools for characterizing stability properties of differential equations. Here, the nonnegative function $\phi : R^+ \to R^+$ is said to be a K-function if it is continuous, strictly increasing, and $\phi(0) = 0$; the nonnegative function $\varphi : R^+ \to R^+$ is said to be an L-function if it is continuous, strictly decreasing, $\varphi(0) < \infty$, and $\varphi(r) \to 0$ as $r \to \infty$. For the autonomous system $\dot{x}(t) = T(x(t)), x(0) \in R^n$, a well-known result is that if the equilibrium point x^* of the system is globally asymptotically stable in R^n and $\alpha(T'(x^*)) < 0$, then there exist a K-function ϕ and an L-function φ such that $\|x(t) - x^*\| \leq \phi(\|x(0)\|)\varphi(t)$ for any $t > 0$ and any $x(0) \in R^n$ [31]. This indicates that x^* is globally uniformly asymptotically stable in R^n (see also [30, pp. 144–146]). Here, x^* is said to be globally uniformly asymptotically stable if, for each pair of positive number M, ε with M being arbitrarily large and ε being arbitrarily small, there exists a finite positive number $Q(M, \varepsilon)$ such that $\|x(t) - x^*\| \leq \varepsilon$ for any $t \geq Q(M, \varepsilon)$ and any $\|x(0)\| \leq M$. In the proof of the following Theorem 1, we will use this assertion.

Whenever a nonlinear continuous dynamical system is exponentially stable, we have the following lemma.

Lemma 2: Consider the nonlinear continuous dynamical system: $\dot{x}(t) = T(x(t)), x(0) \in R^n$. Suppose that x^* is the unique GES equilibrium point of the system, $V \subset R^n$ is a bounded set and takes x^* as an interior point, and T is locally continuously differentiable at x^*. Then, the e-ECR η of the trajectories of the system starting from V satisfies $\eta \le -\alpha(T'(x^*))$. Here, $T'(x^*)$ is the Jacobian matrix of T at x^*.

Proof: Let $\beta > 0$ represent an ECR of the trajectories of the system starting from V. Denote $z(t) = x(t) - x^*$, $G(z(t)) = T(z(t) + x^*)$, and $V_1 = \{x - x^* : x \in V\}$. Then, we have $G(\theta) = \theta$ and θ is the unique equilibrium point of the following nonlinear continuous dynamical system:

$$\dot{z}(t) = G(z(t)) = G'(\theta)(z(t)) + (G - G'(\theta))(z(t)), \; z(0) \in V_1. \tag{12}$$

Here, θ represents the origin. We can easily see that, in (12), θ is exponentially stable in V_1 and β is an ECR of the trajectories of (12) starting from V_1. Since $G(z)$ is Frechet differentiable at $z = \theta$ and $G(\theta) = \theta$, by the definition of Frechet differential, we obtain

$$\lim_{z \to \theta} \frac{\|Gz - G'(\theta)z\|}{\|z\|} = \lim_{z \to \theta} \frac{\|Gz - G\theta - G'(\theta)z\|}{\|z\|} = 0. \tag{13}$$

Since (12) is exponentially stable with the ECR β and $(G - G^{'}(\theta))(z) = o(\|z\|)$ as $z \to \theta$, based on [35], all eigenvalues of $G^{'}(\theta)$ have real parts less than $-\beta$. This implies that $\beta \le -\alpha(G^{'}(\theta)) = -\alpha(T'(x^*))$. Therefore, $\eta \le -\alpha(T'(x^*))$ holds. ■

For the convenience of discussion, we will set $Tx = -Dx + Wf(Ax + u_1) + u_2$ for any $x \in R^n$ in system (1) henceforth. Based on special structures of the neural networks (1) and Lemmas 1 and 2, we can now establish our main result of this paper as follows.

Theorem 1: Assume that x^* is the unique equilibrium point of system (1) and f is locally continuously differentiable at x^*. Then, system (1) is globally exponentially stable in R^n if and only if it is globally asymptotically stable and satisfies $\alpha(T'(x^*)) < 0$. Furthermore, denote $\lambda = \min\{d_1, d_2, \ldots, d_n, -\alpha(T'(x^*))\}$. Then, whenever x^* is globally exponentially stable, the e-ECR η of the trajectories of system (1) starting from R^n satisfies $\lambda \le \eta \le -\alpha(T'(x^*))$. In this case, for any sufficiently small $\varepsilon > 0$, there is a constant $L(\varepsilon) > 0$ such that for any $x(0) \in R^n$

$$\|x(t) - x^*\| \le L(\varepsilon) \cdot e^{-(\lambda - \varepsilon)\cdot t} \|x(0) - x^*\|. \tag{14}$$

Proof: By Lemma 2, we can easily imply that system (1) is globally asymptotically stable and satisfies $\alpha(T'(x^*)) < 0$ whenever (1) is globally exponentially stable in R^n. Thus, we only need to prove that system (1) is globally exponentially stable if it is globally asymptotically stable and satisfies $\alpha(T'(x^*)) < 0$. Since the proof for this assertion is very complicated, we introduce its sketch first before it proceeds. In the proof, the full space R^n will be divided into two parts: a bounded closed set $B(2b)$ and an unbounded set $E(2b)$, which satisfy $B(2b) \cup E(2b) = R^n$. We will prove that the trajectories starting from each set converge exponentially to x^*. In doing so for the bounded set $B(2b)$, we need one time of norm changing: from $\|\cdot\|$ to a specific equivalent norm $\|\cdot\|_\varepsilon$. Through such a norm changing, we deduce the explicit exponential convergence estimation as shown in (19). Finally, by using some mathematical skills, we prove the exponential convergence of the trajectories starting from any point in $E(2b)$. The details of the proof are as follows.

Denote $c = -\mu(-D) = \min_i\{d_i\}$. Then, $\lambda = \min\{c, -\alpha(T'(x^*))\}$. Since $f(x)$ is a bounded nonlinear map, we have $\sup\{\|f(x)\| : x \in R^n\} < \infty$. This implies that

$$\begin{aligned}
&\limsup_{\|x\| \longrightarrow \infty} \frac{\|Wf(Ax + u_1) + u_2 - x^*\|}{\|x - x^*\|} \\
&\quad = \limsup_{\|x\| \longrightarrow \infty} \frac{\|Wf(Ax + u_1) + u_2 - x^*\|}{\|x\|} \cdot \frac{\|x\|}{\|x - x^*\|} \\
&\quad \le \limsup_{\|x\| \longrightarrow \infty} \frac{\|Wf(Ax + u_1)\|}{\|x\|} \\
&\quad \le \limsup_{\|x\| \longrightarrow \infty} \frac{\|W\| \, \|f(Ax + u_1)\|}{\|x\|} = 0.
\end{aligned}$$

Therefore, for any $\varepsilon \in (0, \lambda)$, there exists a positive number $b > 0$ such that for any $x \in R^n$ with $\|x\| \geqslant b$, we have $((\|Wf(Ax + u_1) + u_2 - x^*\|)/(\|x - x^*\|)) \le \varepsilon$. Denote $E(b) = \{x \in R^n : \|x\| > b\}$, $B(2b) = \{x \in R^n : \|x\| \le 2b\}$, $E(2b) = \{x \in R^n : \|x\| > 2b\}$, and $Hx = Wf(Ax + u_1) + u_2$ for all $x \in R^n$. We then have $L_{\|\cdot\|}(H, x^*, E(b)) \le \varepsilon$. This implies that

$$\begin{aligned}
M_{\|\cdot\|}[T, x^*, E(b)] &\le \mu(-D) + M_{\|\cdot\|}[H, x^*, E(b)] \\
&\le -c + L_{\|\cdot\|}(H, x^*, E(b)) \\
&< -\lambda + \varepsilon < 0.
\end{aligned}$$

According to Lemma 1, for any $x(0) \in E(2b) \subset E(b)$ and any positive number $h > 0$, whenever $x(t) \in E(2b)$ for any $t \in [0, h)$, we have

$$\begin{aligned}
\|x(h) - x^*\| &\le e^{M_{\|\cdot\|}[T, x^*, E(b)]\cdot h} \cdot \|x(0) - x^*\| \\
&\le e^{(-\lambda + \varepsilon)\cdot h} \|x(0) - x^*\|.
\end{aligned} \tag{15}$$

We proceed to prove that the trajectories starting from $B(2b)$ and $E(2b)$ converge exponentially to x^*, respectively.

Let us fix ε as above. Then, there is a specific equivalent vector norm $\|\cdot\|_\varepsilon$ such that the subordinated logarithmic norm $\mu(T'(x^*))$ satisfies $|\mu(T'(x^*)) - \alpha(T'(x^*))| \le (\varepsilon/2)$. By (6), the logarithmic norm $\mu(T'(x))$ is a continuous function of Jacobian matrix $T'(x)$, $x \in R^n$. Since T is continuously differentiable in one local neighborhood of x^*, $\mu(T'(x))$ is continuous in the local neighborhood of x^* as well. This then shows that there is a spherical neighborhood $B(x^*, r) = \{x \in R^n : \|x - x^*\|_\varepsilon \le r\} \subset B(2b)$ such that T is continuously differentiable in $B(x^*, r)$, and for any $x \in B(x^*, r)$, we have $|\mu(T'(x)) - \mu(T'(x^*))| \le (\varepsilon/2)$. Thus, for any $x \in B(x^*, r)$, we have

$$\begin{aligned}
|\mu(T'(x)) - \alpha(T'(x^*))| &\le |\mu(T'(x)) - \mu(T'(x^*))| \\
&\quad + |\mu(T'(x^*)) - \alpha(T'(x^*))| \le \varepsilon.
\end{aligned}$$

This deduces that for any $x \in B(x^*, r)$, $\mu(T'(x)) \le \alpha(T'(x^*)) + \varepsilon$ holds. Since T is continuously differentiable in

$B(x^*, r)$, T is Lipschitz continuous on $B(x^*, r)$. Based on the study in [33], if we denote

$$L(T) = \sup_{x \neq y, x, y \in B(x^*, r)} \frac{\|Tx - Ty\|_\varepsilon}{\|x - y\|_\varepsilon}$$

then the limit $M[T] = \lim_{h \to 0^+}((L(I + hT) - 1)/h)$ exists and $M[T] = \sup_{x \in B(x^*, r)} \mu(T'(x))$ [33, p. 672]. As $L_{\|\cdot\|_\varepsilon}(I + hT, x^*, B(x^*, r)) \leq L(I + hT)$ holds for any $h > 0$, this implies that $M_{\|\cdot\|_\varepsilon}[T, x^*, B(x^*, r)] \leq M[T]$. Therefore, we obtain

$$\begin{aligned} M_{\|\cdot\|_\varepsilon}[T, x^*, B(x^*, r)] &\leq \sup_{x \in B(x^*, r)} \mu(T'(x)) \\ &\leq \alpha(T'(x^*)) + \varepsilon < 0. \end{aligned}$$

For the given positive number $\widetilde{r} < r$, denote $S(x^*, \widetilde{r}) = \{x \in R^n : \|x - x^*\|_\varepsilon < \widetilde{r}\}$. Then, $S(x^*, \widetilde{r}) \subset B(x^*, r)$. Since $M_{\|\cdot\|_\varepsilon}[T, x^*, B(x^*, r)] < 0$, it is easy to see that for any $x(0) \in S(x^*, \widetilde{r})$ and any $t > 0$, $x(t) \in S(x^*, \widetilde{r})$. By Lemma 1, for any $x(0) \in S(x^*, \widetilde{r})$ and any $t > 0$, we thus have

$$\begin{aligned} \|x(t) - x^*\|_\varepsilon &\leq e^{M_{\|\cdot\|_\varepsilon}[T, x^*, B(x^*, r)] \cdot t} \|x(0) - x^*\|_\varepsilon \\ &\leq e^{(\alpha(T'(x^*)) + \varepsilon)t} \|x(0) - x^*\|_\varepsilon. \end{aligned} \quad (16)$$

Hereinafter, we will further show that the same conclusion is also true for all trajectories starting from $B(2b)$.

In fact, since system (1) is an autonomous system with $\alpha(T'(x^*)) < 0$ and its equilibrium point x^* is globally asymptotically stable in R^n, the equilibrium point is globally uniformly asymptotically stable as well [30, pp. 144–146]. This implies that for all convergent trajectories starting from the bounded set $B(2b)$, they will converge uniformly to x^*. In other words, there is a positive constant $t_0 > 0$ such that for any $t \geqslant t_0$ and all $x(0) \in B(2b)$, we have $x(t) \in S(x^*, \widetilde{r})$.

From (2), we know that there exists a positive number $\tau > 0$ such that $L_{\|\cdot\|_\varepsilon}(T, x^*, R^n) \leq \tau$. Denote $M = e^{(\tau - (\alpha(T'(x^*)) + \varepsilon)) \cdot t_0}$. Then, M is a positive constant greater than one and is determined by ε. Since $\alpha(T'(x^*)) + \varepsilon < 0$, for any positive number $t \leq t_0$, we have $e^{-(\alpha(T'(x^*)) + \varepsilon) \cdot t} \leq e^{-(\alpha(T'(x^*)) + \varepsilon) \cdot t_0}$. By Lemma 1, for any $t \leq t_0$ and any initial value $x(0) \in B(2b)$ and $x(0) \notin S(x^*, \widetilde{r})$, we thus have

$$\begin{aligned} &\|x(t) - x^*\|_\varepsilon \\ &\quad \leq e^{M_{\|\cdot\|_\varepsilon}[T, x^*, R^n] \cdot t} \cdot \|x(0) - x^*\|_\varepsilon \\ &\quad \leq e^{L_{\|\cdot\|_\varepsilon}(T, x^*, R^n) \cdot t_0} \cdot \|x(0) - x^*\|_\varepsilon \\ &\quad \leq e^{\tau \cdot t_0} \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \cdot e^{-(\alpha(T'(x^*)) + \varepsilon) \cdot t} \|x(0) - x^*\|_\varepsilon \\ &\quad \leq e^{\tau \cdot t_0} \cdot e^{-(\alpha(T'(x^*)) + \varepsilon) \cdot t_0} \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \|x(0) - x^*\|_\varepsilon \\ &\quad \leq M \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \cdot \|x(0) - x^*\|_\varepsilon. \end{aligned} \quad (17)$$

From (16) and (17), it then follows that for any $t > t_0$ and any $x(0) \in B(2b)$ and $x(0) \notin S(x^*, \widetilde{r})$

$$\begin{aligned} &\|x(t) - x^*\|_\varepsilon \\ &\quad \leq e^{(\alpha(T'(x^*)) + \varepsilon) \cdot (t - t_0)} \|x(t_0) - x^*\|_\varepsilon \\ &\quad \leq e^{(\alpha(T'(x^*)) + \varepsilon) \cdot (t - t_0)} M e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t_0} \|x(0) - x^*\|_\varepsilon \\ &\quad \leq M \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \|x(0) - x^*\|_\varepsilon. \end{aligned} \quad (18)$$

Combining (16)–(18) shows that for any positive number $t > 0$ and any $x(0) \in B(2b)$

$$\|x(t) - x^*\|_\varepsilon \leq M \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \|x(0) - x^*\|_\varepsilon. \quad (19)$$

Since $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ are equivalent, there exists a constant $\widetilde{K} > 1$ such that

$$\|x(t) - x^*\| \leq M \cdot \widetilde{K} \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot t} \|x(0) - x^*\|. \quad (20)$$

Thus, the trajectories starting from $B(2b)$ converge exponentially to x^* with the ECR $-\alpha(T'(x^*)) - \varepsilon$.

Note that for any $x(0) \in E(2b)$, $\lim_{t \longrightarrow \infty} x(t) = x^*$. Therefore, for each given $x(0) \in E(2b)$, we can assume that there exists a positive number m such that $x(m) \in B(2b)$ and $x(t) \in E(2b)$ for any $t < m$. For each given positive number k, there are two possible cases: $k > m$ and $k \leqslant m$. For the given $x(0) \in E(2b)$ and the given positive number k, we consider the following two cases.

1) If $k > m$, then, based on (15) and (20), we have

$$\begin{aligned} \|x(k) - x^*\| &\leq \widetilde{K} \cdot M \cdot e^{(\alpha(T'(x^*)) + \varepsilon) \cdot (k - m)} \|x(m) - x^*\| \\ &\leq \widetilde{K} \cdot M \cdot e^{(-\lambda + \varepsilon) \cdot (k - m)} e^{(-\lambda + \varepsilon) \cdot m} \|x(0) - x^*\| \\ &\leq \widetilde{K} \cdot M \cdot e^{(-\lambda + \varepsilon) \cdot k} \|x(0) - x^*\|. \end{aligned} \quad (21)$$

2) If $k \leq m$, then, based on (15), we have

$$\|x(k) - x^*\| \leq e^{(-\lambda + \varepsilon) \cdot k} \|x(0) - x^*\|. \quad (22)$$

Since $\widetilde{K} > 1$ and $M > 1$, from (21) and (22), we deduce that for any $x(0) \in E(2b)$ and any positive number k

$$\|x(k) - x^*\| \leq \widetilde{K} \cdot M \cdot e^{(-\lambda + \varepsilon) \cdot k} \|x(0) - x^*\|. \quad (23)$$

Thus, the trajectories starting from $E(2b)$ converge exponentially to x^*. By (20) and (23), we can further imply that for any $x(0) \in R^n$ and any positive number t

$$\|x(t) - x^*\| \leq \widetilde{K} \cdot M \cdot e^{-(\lambda - \varepsilon) \cdot t} \|x(0) - x^*\|. \quad (24)$$

Since $-(\lambda - \varepsilon) < 0$, x^* is globally exponentially stable in R^n with the ECR $\lambda - \varepsilon$. Since $\varepsilon > 0$ can be arbitrary, we conclude that the e-ECR of trajectories starting from R^n is not less than λ. Based on lemma 2, we imply that the e-ECR η of the trajectories starting from R^n satisfies $\lambda \leq \eta \leq -\alpha(T'(x^*))$. With this, the proof of Theorem 1 is completed. ■

Remark 1:

1) Theorem 1 reveals that GES of the continuous-time neural networks modeled in (1) can be fully characterized by GAS of the networks plus $\alpha(T'(x^*)) < 0$. This provides a very direct and simple means to distinguish GES from GAS for system (1). For example, whenever we want to distinguish GES from GAS for the networks (1), it is only needed to compute $\alpha(-D + Wf'(Ax^* + u_1)A)$ and to see if it is negative.
2) It is pointed out in [4] and [29] that the ECR of the trajectories of some simplified model of system (1) is at least $(\min\{d_1, d_2, \ldots, d_n\}/2)$. Theorem 1, however,

shows that the e-ECR η of the trajectories of system (1) is either exactly equal to $-\alpha(T'(x^*))$ or satisfies $\min\{d_1, d_2, \ldots, d_n\} \leq \eta < -\alpha(T'(x^*))$. Thus, in this case, Theorem 1 directly improves on the estimations of [4] and [29].

3) In real implementations of neural networks, it is normally required that the conditions for global stability of system (1) should be robust with respect to variations of the system parameters or any small perturbation [36]. Theorem 1 shows that for system (1), whenever the equilibrium point x^* is globally asymptotically stable rather than globally exponentially stable, we have $\alpha(T'(x^*)) = 0$. Such neural networks are clearly not robust to perturbations. Thus, they are mathematically interesting but are rarely used in engineering applications [36]. Particularly, whenever the sufficient condition for global stability of the neural network (1) is requested to be robust, we should seek such conditions that can ensure system (1) to be globally exponentially stable.
4) Consider the following general differential equation:

$$\dot{x}(t) = G\left(x(t)\right), \qquad x(0) \in R^n \tag{25}$$

where x^* is an equilibrium point and $G : R^n \to R^n$ is continuously differentiable at x^*. It is well known that, in (25), local exponential stability of x^* (namely, x^* is exponentially stable in one local neighborhood U of x^*) is equivalent to the exponential stability of the corresponding linearized system and equivalent to the condition $\alpha(G'(x^*)) < 0$. In general, however, the condition $\alpha(G'(x^*)) < 0$ cannot imply GES of (25). Moreover, to the best of our knowledge, even if we additionally assume that x^* is globally asymptotically stable in system (25), generally, we cannot conclude whether system (25) is globally exponentially stable or not. Thus, Theorem 1 actually reveals a special characterization possessed by the neural network system (1).
5) To some extent, the main results obtained in Theorem 1 can be regarded as the continuous-time counterpart of an earlier work of discrete-time neural networks by the same authors in [34].

In Theorem 1, we have discussed the evaluation of the e-ECR of the trajectories of system (1) starting from any state in R^n. If the trajectories are confined to be initiated from a bounded subset, the estimation of the e-ECR of system (1) can be further improved. We have the following result.

Corollary 1: Supposing that x^* is the unique GES equilibrium point of system (1) in R^n, $V \subset R^n$ is a given bounded subset that takes x^* as an interior point. Then, the e-ECR of the trajectories starting from V is exactly equal to $-\alpha(T'(x^*))$. Moreover, for any sufficiently small $\varepsilon > 0$, there is a constant $L(\varepsilon) > 0$ such that for any $x(0) \in V$

$$\|x(t) - x^*\| \leq L(\varepsilon) \cdot e^{(\alpha(T'(x^*))+\varepsilon)\cdot t} \|x(0) - x^*\| .$$

Here, $L(\varepsilon)$ might be varied when different V is selected.

Proof: Based on the proof of Theorem 1 and (20), for any sufficiently small $\varepsilon > 0$ and the given bounded convergent region (i.e., the bounded region from which the starting trajectories all converge to the equilibrium point) $V \subset R^n$, we can construct $B(2b)$ such that $V \subset B(2b)$ and the trajectories starting from $B(2b)$ converge exponentially to x^* with the ECR $-\alpha(T'(x^*)) - \varepsilon$. This implies that the e-ECR of the convergent trajectories starting from V is not less than $-\alpha(T'(x^*))$. By Lemma 2, however, the e-ECR of the trajectories starting from V is not larger than $-\alpha(T'(x^*))$. Thus, the e-ECR of the trajectories starting from V is exactly equal to $-\alpha(T'(x^*))$. This implies Corollary 1. ∎

It is not clear whether the result in Corollary 1 is still valid when the bounded convergent region V is replaced by R^n or by any unbounded convergent region. Nevertheless, whenever system (1) has small state feedback coefficients in the sense of $-d_i \leq \alpha(T'(x^*))$, $1 \leqslant i \leqslant n$, we can draw the following conclusion.

Corollary 2: In system (1), if x^* is globally exponentially stable in R^n, then the e-ECR of the trajectories starting from R^n is either exactly equal to $-\alpha(T'(x^*))$ or satisfies $\min\{d_1, d_2, \ldots, d_n\} \leq \eta < -\alpha(T'(x^*))$. More specifically, if system (1) has small state feedback coefficients in the sense of $-d_i \leq \alpha(T'(x^*))$, $1 \leqslant i \leqslant n$, then the e-ECR of trajectories starting from R^n is exactly equal to $-\alpha(T'(x^*))$. In this case, for any sufficiently small $\varepsilon > 0$, there is a constant $M(\varepsilon) > 0$ such that for any $x(0) \in R^n$

$$\|x(t) - x^*\| \leq M(\varepsilon) \cdot e^{(\alpha(T'(x^*))+\varepsilon)\cdot t} \|x(0) - x^*\| . \tag{26}$$

Proof: According to Theorem 1, the e-ECR of the trajectories starting from R^n is at least $\min\{d_1, d_2, \ldots, d_n, -\alpha(T'(x^*))\}$. By Lemma 2, we know, however, that the e-ECR of trajectories is not larger than $-\alpha(T'(x^*))$. Thus, if $-d_i \leq \alpha(T'(x^*))$ for all $1 \leqslant i \leqslant n$, then it is easy to imply that the e-ECR of trajectories starting from R^n is exactly equal to $-\alpha(T'(x^*))$. ∎

Remark 2:

1) Corollary 2 shows that for the continuous-time recurrent neural networks (1) with small state feedback coefficients, the e-ECR of the trajectories starting from R^n is determined uniquely by the maximum abscissa of spectral set of Jacobian matrix of the neural networks at the unique equilibrium point [i.e., the maximum real part of eigenvalues of the Jacobian matrix $T'(x^*)$]. Thus, for the neural networks with small state feedback coefficients, we can compute exactly the e-ECR of the trajectories of system (1) starting from R^n through computing $\alpha(T'(x^*))$.
2) Suppose that in Corollary 2, the equilibrium point of system (1) is changed to $\overline{x}_*$ when the external inputs u_1 and u_2 are changed to $\overline{u}_1$ and $\overline{u}_2$. Then, correspondingly, the e-ECR of trajectories will be changed to $\alpha(-D + Wf'(A\overline{x}_* + \overline{u}_1)A)$. This reveals that the exponential convergence properties of system (1) can be affected seriously by the external inputs. In addition, the result actually provides also the formula for quantitatively characterizing the change of exponential convergence properties of system (1) when the external inputs are

varied. A simple example showing this fact is in the following:

$$\dot{x}(t)=-Dx(t)+W\cdot f(x(t))+u,\qquad t>0,\qquad x(0)\in R^3$$

where $D=\mathrm{diag}(1,2,3)$, $W=\begin{pmatrix}0.5 & 0.3 & 0.2\\ 0 & 0.5 & 0.4\\ 0 & 0 & 0.5\end{pmatrix}$, $f_i(x_i)=\sin(x_i)$ for $i=1,2,3$, and $u=(u_1,u_2,u_3)\in R^3$, where u represents an external input. Denote $Tx=-Dx+W\cdot f(x)+u$ for all $x\in R^3$. Since $\|W\|_2\le 0.85083$, according to [20], for any external input u, the system is globally exponentially stable. We can see that, when $u=(0,0,0)$, the equilibrium point is $x^*=(0,0,0)$ and $\alpha(T'(x^*))=-0.5$. By Corollary 2, the corresponding e-ECR of trajectories is equal to 0.5. However, when $u=(-1+0.5\pi,-0.9+\pi,-0.5+1.5\pi)$, the equilibrium point is $x^*=(0.5\pi,0.5\pi,0.5\pi)$ and $\alpha(T'(x^*))=-1$. Thus, the corresponding e-ECR of trajectories is equal to 1.

Theorem 1 can be generalized to the case of f not locally continuously differentiable at x^* in system (1). In fact, it is well known that when f is locally continuously differentiable at x^* in system (1), system (1) is locally exponentially stable at x^* if and only if $\alpha(T'(x^*))<0$. Therefore, in the case of f not locally continuously differentiable at x^*, instead of $\alpha(T'(x^*))<0$, we can use the condition that "x^* is locally exponential stable" to deduce the same conclusion of Theorem 1 through applying the same argument. We state this generalization as the following Theorem 2.

Theorem 2: Assume that x^* is the unique equilibrium point of system (1) and f is not locally continuously differentiable at x^*. Then, system (1) is globally exponentially stable in R^n if and only if it is globally asymptotically stable in R^n and is locally exponentially stable at x^*. Furthermore, if system (1) is locally exponentially stable in one neighborhood U of x^* with the ECR $\beta>0$, then the e-ECR of the trajectories of system (1) starting from R^n is at least $\lambda=\min\{d_1,d_2,\ldots,d_n,\beta\}$. In other words, for any sufficiently small $\varepsilon>0$, there is a constant $L(\varepsilon)>0$ such that for any $x(0)\in R^n$

$$\|x(t)-x^*\|\le L(\varepsilon)\cdot e^{-(\lambda-\varepsilon)\cdot t}\|x(0)-x^*\|. \tag{27}$$

IV. Specifications and Examples

In this section, we apply the main results established in the last section to distinguish GES from GAS for the generic neural networks (1) and to estimate ECR of convergent trajectories of networks (1).

First, we provide an example that is globally asymptotically stable rather than globally exponentially stable. Consider the following special case of networks (1):

$$\dot{x}(t)=-Dx(t)+f(x(t)),\qquad t>0,\qquad x(0)\in R^4 \tag{28}$$

where $D=\mathrm{diag}(1,1,1,1)$ and $f_i(x_i)=\sin(x_i)$ for $i=1,2,3,4$. It is easy to see that system (28) has the unique equilibrium point $x^*=(0,0,0,0)$. Denote $Tx=-Dx+f(x)$ for all $x\in R^4$. Then, we have $\alpha(T'(x^*))=0$. This implies that system (28) is not globally exponentially stable. However, according to the Appendix, system (28) is globally asymptotically stable.

Subsequently, we provide some examples to show that many sufficient conditions for GAS of the continuous-time neural networks presented in existing literature can actually imply GES of the networks, also. Let us consider the following simplified model of system (1):

$$\dot{x}(t)=-Dx(t)+Wf(x(t))+u,\qquad x(0)\in R^n \tag{29}$$

where W is assumed to be a symmetric matrix and $f(x)$ belongs to the family S of sigmoidal functions, i.e., for any $i=1,2,\ldots,n$, $f_i(x_i):R\to R$ is a C^1-function with $(df_i(x_i)/dx_i)>0$ for all $x_i\in R$ and $f_i(R)\subset(a_i,b_i)$ for some $a_i,b_i\in R$, $a_i<b_i$. System (29) is said to be absolutely stable whenever there is a GAS equilibrium point for every neuron activation function belonging to the class S, every constant input vector u, and any diagonal matrix D [8]. In [8], it is proved that system (29) is absolutely stable if and only if W is a negatively semidefinite matrix. By applying Theorem 1, we can now sharpen this result as follows.

Corollary 3: For system (29), if W is negatively semidefinite, then for any given neuron activation function belonging to the class S, the constant input vector u, and the diagonal matrix D, the unique equilibrium point x^* is globally exponentially stable in R^n. Moreover, the e-ECR of the trajectories of (29) starting from R^n is at least $\lambda=\min\{d_1,d_2,\ldots,d_n,-\alpha(-D+Wf'(x^*))\}$. Furthermore, for any sufficiently small $\varepsilon>0$, there is a constant $L(\varepsilon)>0$ such that for any $x(0)\in R^n$

$$\|x(t)-x^*\|\le L(\varepsilon)\cdot e^{-(\lambda-\varepsilon)\cdot t}\|x(0)-x^*\|. \tag{30}$$

We present a simple proof of this assertion. When W is negatively semidefinite, the study in [8] has proved that for any given neuron activation function belonging to the class S, the constant input vector u, and the diagonal matrix D, there is a unique equilibrium point x^* of (29) that is globally asymptotically stable in R^n. In the proof of Proposition 1 in [11], it was pointed out that when W is negatively semidefinite, not only is system (29) absolutely stable but also the unique equilibrium point x^* is locally exponentially stable (or equivalently $\alpha(-D+Wf'(x^*))<0$). Thus, by applying Theorem 1, we conclude that x^* is also globally exponentially stable in R^n, and the exponential convergence estimation satisfies (30).

Furthermore, let us consider a special case of (29), where $n=2$. For such a Hopfield-type neural network with two neurons, the study in [18] showed that for any fixed W, the neural network has a unique equilibrium point that is locally exponentially stable for any $f(x)$, u, and D if and only if the network is globally asymptotically stable for any $f(x)$ and for any u and D. In addition, it was shown that in [18], when the derivatives of $f_i(x_i)$ are bounded, then the neural network has a unique equilibrium point that is locally exponentially stable for any $f(x)$, u, and D if and only if the neural network is globally exponentially stable for any $f(x)$ and for any u and D. In fact, by applying Theorem 1 in this paper, we can further imply that for the given specific W, if the neural network has a unique equilibrium point that is locally exponentially stable for any $f(x)$ and for any u and D, then for the same W, the neural network is, in fact, globally exponentially stable for any $f(x)$ and for any u and D.

Finally, we discuss the evaluation of ECR of trajectories of neural networks. In Theorem 1, we have claimed that when system (1) is globally exponentially stable, the e-ECR η of the trajectories of system (1) starting from R^n is either equal to $-\alpha(T'(x^*))$ or is a value between $\min\{d_1, d_2, \ldots, d_n\}$ and $-\alpha(T'(x^*))$. Hereinafter, we provide several specification examples to show how the existing ECR estimations can be sharpened.

Consider the following simplified model of system (1):

$$\dot{x}(t) = Wf(x(t)) + u, \qquad x(0) \in R^n. \tag{31}$$

We assume that (31) has a unique GES equilibrium point. Denote $\widetilde{T}x = Wf(x) + u$ for all $x \in R^n$. It is then easy to see that $\widetilde{T}$ is a bounded mapping, i.e., $\widetilde{T}(R^n)$ is a bounded set. Thus, we can find a bounded set $V \subset R^n$ such that $\widetilde{T}(R^n) \subset V$ and V contains the unique equilibrium point x^* as an interior point. By applying Corollary 1, we can thus conclude that the e-ECR of trajectories of system (31) starting from R^n is exactly equal to $-\alpha(Wf'(x^*))$. That is, for any sufficiently small $\varepsilon > 0$, there is a constant $L(\varepsilon) > 0$ such that for any $x(0) \in R^n$

$$\|x(t)-x^*\| \leq L(\varepsilon)\cdot e^{-(-\alpha(Wf'(x^*))-\varepsilon)\cdot t}\|x(0)-x^*\|. \tag{32}$$

We further consider the neural networks discussed in [9]

$$\dot{x}(t) = -Dx(t) + Wf(x(t)) + u \tag{33}$$

where W is assumed to be a triangular matrix with $W_{ii} = 0, i = 1, 2, \ldots n$, and $f : R^n \rightarrow R^n$ is a continuous function such that $f_1(x)$ is a constant and $f_i(x) = f_i(x_1, x_2, \ldots, x_{i-1}), i = 2, \ldots, n$. Reference [9] showed that (33) is globally asymptotically stable. Furthermore, the study in [19] later showed that it is, in fact, globally exponentially stable also. However, ECR of (33) has not been discussed yet. If we assume that $f(x)$ in (33) is a bounded function and it is locally continuously differentiable at x^*, then we can see that for the Jacobian matrix of (33) at the unique equilibrium x^*, we have $\alpha(-D + Wf'(x^*)) = \alpha(-D) = -\min\{d_1, d_2, \ldots, d_n\}$. Thus, applying Corollary 2, we conclude that the e-ECR of (33) is exactly equal to $d^* = \min\{d_1, d_2, \ldots, d_n\}$, and, furthermore, there is a constant $L(\varepsilon) > 0$ such that for any $x(0) \in R^n$

$$\|x(t) - x^*\| \leq L(\varepsilon) \cdot e^{-(d^*-\varepsilon)\cdot t}\|x(0) - x^*\|.$$

The following neural networks were studied in [15]:

$$\dot{x}(t) = -x(t) + Wf(x(t)) + u, \qquad x(0) \in R^n \tag{34}$$

where $f_i(x_i) : R \rightarrow R$ is assumed to be continuously differentiable and $\sup_{x_i \in R} f_i'(x_i) = 1,\ i = 1, 2, \ldots n$. In [15, Lemma 4], it is proved that whenever $\langle Wx, x\rangle > 0$ (or < 0) holds for every $x \in R^n$, then the real part of eigenvalues of $Wf'(x^*)$ is positive (or negative). Thus, whenever $\langle Wx, x\rangle > 0$ (or < 0) holds for every $x \in R^n$, then $\alpha(-I + Wf'(x^*)) = -1 + \alpha(Wf'(x^*)) > -1$ (or < -1). By applying Corollary 2 and Theorem 1, we can now get the following result.

Corollary 4: If system (34) has a unique equilibrium point x^* that is globally exponentially stable, then the e-ECR η either satisfies $\eta = 1 - \alpha(Wf'(x^*)) < 1$ or satisfies $1 \leq \eta \leq 1 - \alpha(Wf'(x^*))$. Specially, if $\langle Wx, x\rangle > 0$ holds for any $x \in R^n$, then $\eta = 1 - \alpha(Wf'(x^*)) < 1$. If $\langle Wx, x\rangle < 0$ holds for any $x \in R^n$, then $1 \leq \eta \leq 1 - \alpha(Wf'(x^*))$.

Based on the results in [20], we have known that if $\mu_1(W) < 1$ (or $\mu_\infty(W) < 1$), then (34) is globally exponentially stable with the ECR $1 - \mu_1(W)^+$(or $1 - \mu_\infty(W)^+$). Here, $\mu_1(W)^+ = \max\{0, \mu_1(W)\}$ and $\mu_\infty(W)^+ = \max\{0, \mu_\infty(W)\}$. It is easy to see that the aforementioned result has refined these known ECR estimations.

V. Conclusion

In this paper, we have revealed some important characterizations of GES of the generic continuous-time recurrent neural networks (1). Our main finding is that the GES property of system (1) can be fully characterized by its GAS property plus $\alpha(T'(x^*)) < 0$. Here, $\alpha(T'(x^*))$ is the maximum abscissa of the spectral set of Jacobian matrix of the system at the unique equilibrium point x^*. The established result provides a very useful and simple method to distinguish GES from GAS for system (1). Our other finding is that whenever system (1) is globally exponentially stable with small state feedbacks, the supremum of ECRs of the system, η, which has been defined in this paper as the e-ECR, is exactly equal to $-\alpha(T'(x^*))$. This finding provides us with a precise measurement of the optimal exponential bound of convergence of system (1). The results obtained are helpful in understanding the essence of GES of the generic continuous-time recurrent neural network (1).

Appendix

In the appendix, we will prove that system (28) is globally asymptotically stable. It is easy to see that we only need to prove that the following system is globally asymptotically stable:

$$\dot{x}(t) = -x(t) + \sin(x(t)), \qquad t > 0, \qquad x(0) \in R. \tag{35}$$

Let $x(t)$ denote any solution of system (35). Then, we have

$$x(t) = e^{-t}x(0) + \int_0^t e^{-s}\sin(x(t-s))\,ds, \qquad t \geq 0.$$

Furthermore, for any $t \geq 0$, we have

$$\begin{aligned}|x(t)| &\leq e^{-t}|x(0)| + \int_0^t e^{-s}|\sin(x(t-s))|\,ds \\ &\leq |x(0)| + 1.\end{aligned}$$

This concludes that the solution $x(t)$ is bounded for any $t \geq 0$. Let us define a function $V(x) : R \rightarrow R$ as follows:

$$V(x) = 0.5x^2 + \cos(x) - 1, \qquad x \in R. \tag{36}$$

Denote $r(t) = V(x(t)), t > 0$. It is easy to see that $r(t)$ is continuously differentiable and its derivative has the following property for any $t \geq 0$:

$$\begin{aligned} r'(t) &= x(t)x'(t) - \sin(x(t))\, x'(t) \\ &= -\left(x(t) - \sin(x(t))\right)^2 \leq 0. \end{aligned} \tag{37}$$

This implies that $r(t)$ is monotonously decreased for $t \geq 0$. Thus, the limit $\lim_{t\to\infty} r(t)$ exists, and it is a bounded number. Furthermore, for any $t \geq 0$, we have

$$\begin{aligned} r''(t) &= -2\left(x(t) - \sin(x(t))\right)\left(x'(t) - \cos(x(t))\, x'(t)\right) \\ &= 2\left(x(t) - \sin(x(t))\right)^2 \left(1 - \cos(x(t))\right). \end{aligned} \tag{38}$$

Since $x(t)$ is bounded for any $t \geq 0$, $r''(t)$ is bounded for any $t \geq 0$ as well. This implies that $r(t)$ is uniformly continuous for all $t \geq 0$. According to the Barbalat lemma [38], we have $\lim_{t\to\infty} r'(t) = 0$. Suppose that α is one of accumulation points of $x(t)$ as $t \to \infty$. According to (37), we have $\alpha = \sin(\alpha)$. It implies that $\alpha = 0$. Therefore, $\lim_{t\to\infty} x(t) = 0$ holds, i.e., the equilibrium point $x^* = 0$ is globally attractive. In addition, it is easy to see that the function $V(x(t))$ is positive definite and $V'(x(t))$ is negatively semidefinite. This implies that the equilibrium point $x^* = 0$ is stable. Thus, the equilibrium point x^* is globally asymptotically stable.

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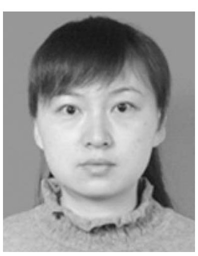

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