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Stability Analysis of Age-Structured Disabled Population Dynamics*

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Abstract. In this note, we present a characteristic of an ordered Banach space with base and then give a necessary and sufficient condition for the disabled population system to decay exponentially.

Key words: ordered Banach space, positive operator semigroup, disabled population system, asymptotic stability.

Let X be a real Banach space with norm $\|\cdot\|$. Given a $v \in X^*$, we can show that the binary relation $\leqslant_v$ defined on X by the formula

$$x \leqslant_v y \iff \|x-y\| \leqslant v(y)-v(x) \tag{1}$$

is a partial order, and thus X is an ordered Banach space with positive cone

$$X_v^+ := \{z \in X : \|z\| \leqslant v(z)\}.$$

The following proposition shows that any ordered Banach space with base** is subordinate to a partial order of this kind.

Proposition 1. *An ordered Banach space X with positive cone X^+ has a base if and only if there exists a $v \in X^*$ such that $X^+ \subseteq \{x \in X : \|x\| \leqslant v(x)\}$.*

By Proposition 1, the following assertion is immediate.

Proposition 2. *Suppose that $\{T(t)\}_{t\geqslant 0}$ is a positive C_0-semigroup on an ordered Banach space X with base K. Let $v \in X^*$ be determined as in Proposition 1. If $v \in D(A^*)$ and $A^*v \leqslant \lambda v$ for some real number λ, then*

$$\|T(t)x\| \leqslant \|v\| \cdot e^{\lambda t} \cdot \|x\| \qquad \forall x \in X^+,\ t \geqslant 0. \tag{2}$$

Therefore, if X^+ is a generating cone, then there exists a constant $\alpha \geqslant 1$ such that $\|T(t)\| \leqslant \alpha\|v\| \cdot e^{\lambda t}$ for all $t \geqslant 0$.

In this note, we adopt the terminology in [1] and refer the reader to [1] and [2] for details about positive C_0-semigroups.

Now we use Propositions 1 and 2 to study the asymptotic behavior of the age-structured disabled population system, which is described by the equations (see [3])

$$\begin{cases} \dfrac{\partial \phi_1(r,t)}{\partial t} + \dfrac{\partial \phi_1(r,t)}{\partial r} = \sigma(r)\phi_2(r,t) - (\mu(r)+\delta(r))\phi_1(r,t), \\ \dfrac{\partial \phi_2(r,t)}{\partial t} + \dfrac{\partial \phi_2(r,t)}{\partial r} = \delta(r)\phi_1(r,t) - (\widetilde{\mu}(r)+\sigma(r))\phi_2(r,t), \\ \phi_1(r,0) = \phi_{10}(r), \quad \phi_2(r,0) = \phi_{20}(r), \\ \phi_1(0,t)+\phi_2(0,t) = \beta \displaystyle\int_{r_1}^{r_2} h(r)k(r)\phi_1(r,t)\,dr + \widetilde{\beta}\int_{r_1}^{r_2} \widetilde{h}(r)\widetilde{k}(r)\phi_2(r,t)\,dr. \end{cases} \tag{3}$$

Here ϕ_1 and ϕ_2 are the normal and disabled population density distributions, respectively; μ and $\widetilde{\mu}$ are the age-specific mortality moduli of normal and disabled people, respectively; $\sigma(r), \delta(r) \in [0,1]$

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**Recall that a subset K of the positive cone X^+ of an ordered Banach space X is called a base if it is bounded, closed, and convex and if for each $x \in X^+$ there exists a unique number $\lambda(x) \geqslant 0$ such that $x \in \lambda(x)K$ (e.g., see [1]).

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are the recovery rate and the disabled rate, respectively, at age r; $k(r), \widetilde{k}(r) \in (0,1)$ are the proportions of normal and disabled female populations, respectively, at age r; and the functions h and $\widetilde{h}$ have unit L^1-norm and represent the birth modes of normal females and disabled females, respectively. Next, the constant $r_m > 0$ is the lifespan of humans, $[r_1, r_2] \subset (0, r_m)$ is the birth interval of females, and the constants β and $\widetilde{\beta}$ are the birth rates of normal and disabled people, respectively, which depend on the government population policy. According to their physical meaning ([3], [4]), the functions $\mu, \widetilde{\mu}, h, \widetilde{h}, k$, and $\widetilde{k}$ are nonnegative and satisfy the following conditions:

(i) $\mu_* \leqslant \mu(r), \widetilde{\mu}(r) < \infty$ for all $r \in [0, r_m)$.

(ii) $h(r) = \widetilde{h}(r) = 0$ for all $r \notin [r_1, r_2]$, and $\int_0^{r_m} h(r)\,dr = \int_0^{r_m} \widetilde{h}(r)\,dr = 1$.

(iii) $k_* \leqslant k(r), \widetilde{k}(r) < 1$ for all $r \in [0, r_m)$.

Here μ_* and k_* are positive constants.

For $p \in [1, \infty]$, let X_p be the product space $L^p[0, r_m) \times L^p[0, r_m)$ equipped with the norm $\|\phi\|_p = \|\phi_1\|_p + \|\phi_2\|_p$. Then X_p is clearly an ordered Banach space with positive cone $X_{p+} = L^p_+[0, r_m) \times L^p_+[0, r_m)$, where $L^p_+[0, r_m)$ is the set of nonnegative functions in $L^p[0, r_m)$. Define an operator $A_p\colon D(A_p) \subset X_p \to X_p$ by the formula

$$A_p\phi = \begin{pmatrix} -\phi_1' - (\mu + \delta)\phi_1 + \sigma\phi_2 \\ -\phi_2' - (\widetilde{\mu} + \sigma)\phi_2 + \delta\phi_1 \end{pmatrix}, \tag{4}$$

where the domain $D(A_p)$ consists of all vectors $\phi = (\phi_1, \phi_2)^T \in X_p$ such that ϕ_1 and ϕ_2 are absolutely continuous, have derivatives $\phi_1', \phi_2' \in L^p[0, r_m)$, and satisfy

$$\phi_1(0) + \phi_2(0) = \beta \int_{r_1}^{r_2} h(r)k(r)\phi_1(r)dr + \widetilde{\beta} \int_{r_1}^{r_2} \widetilde{h}(r)\widetilde{k}(r)\phi_2(r)dr.$$

Then we can rewrite system (3) in the equivalent shorthand form

$$u'(t) = A_1 u(t), \quad t > 0, \qquad u(0) = \phi_0, \quad u(t) \in X_1. \tag{5}$$

By [4], the operator A_p $(1 \leqslant p < \infty)$ generates a positive C_0-semigroup on X_p, which is denoted by $\{T_p(t)\}_{t \geqslant 0}$ in what follows. To derive the asymptotic stability of system (3), we define a functional $B\colon X_p \to \mathbb{R}$ and an operator $K\colon X_p \to X_p$ by setting

$$B\phi = \beta \int_{r_1}^{r_2} h(r)k(r)\phi_1(r)\,dr + \widetilde{\beta} \int_{r_1}^{r_2} \widetilde{h}(r)\widetilde{k}(r)\phi_2(r)\,dr, \tag{6}$$

$$(K\phi)(r) = \exp\left(\int_0^r \begin{pmatrix} -\mu(s) - \delta(s) & \sigma(s) \\ \delta(s) & -\widetilde{\mu}(s) - \sigma(s) \end{pmatrix} ds \right) \cdot \phi(r). \tag{7}$$

One can readily verify that both B and K are bounded.

Lemma 3. *The equation*

$$2 = \beta \int_{r_1}^{r_2} e^{-\lambda r} h(r)k(r)\phi_1(r)\,dr + \widetilde{\beta} \int_{r_1}^{r_2} e^{-\lambda r} \widetilde{h}(r)\widetilde{k}(r)\phi_2(r)\,dr, \tag{8}$$

where $(\phi_1, \phi_2)^T = Ke$ and $e(r) = (1,1)^T$ for all $r \in [0, r_m)$, has a unique real solution λ.

Proposition 4. *A necessary and sufficient condition that all solutions $\phi = (\phi_1, \phi_2)^T$ of Eq. (3) with $\phi_{10}, \phi_{20} \in L^\infty[0, r_m)$ exponentially decay as $t \to \infty$ is that the unique real solution λ^* of Eq. (8) be negative.*

Proof. Let $\chi_{[r_1, r_2]}$ be the characteristic function of the interval $[r_1, r_2]$. Consider the operator $A^\#$ on X_1 defined by the formula

$$A^\#\varphi = \begin{pmatrix} \varphi_1' - (\mu + \delta)\varphi_1 + \delta\varphi_2 + \beta\varphi_1(0)\chi_{[r_1, r_2]} h\, k \\ \varphi_2' - (\mu + \sigma)\varphi_2 + \sigma\varphi_1 + \widetilde{\beta}\varphi_2(0)\chi_{[r_1, r_2]} \widetilde{h}\, \widetilde{k} \end{pmatrix}, \tag{9}$$

where the domain $D(A^\#)$ consists of all $\varphi = (\varphi_1, \varphi_2)^T \in X_1$ such that φ_1 and φ_2 are absolutely continuous, have derivatives $\varphi_1', \varphi_2' \in X_1$, and satisfy $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(r_m^-) = \varphi_2(r_m^-) = 0$.

It is plain to show that $A_\infty \subseteq (A^\#)^*$, and hence $A^\#$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X_1. One can readily verify that the function $c=(c_1,c_2)^T$ defined by $c(r)=e^{-\lambda^* r}(Ke)(r)(r\in[0,r_m))$ is absolutely continuous and satisfies $c_1(0)=c_2(0)=1$. Moreover, it follows from Lemma 3 that $c\in D(A_\infty)$. Note that $A_\infty\subseteq (A^\#)^*$ and $c\in X_{1+}$, and so $((A^\#)^*c)(r)=(A_\infty c)(r)=\lambda^* c$. Thus, Proposition 2 gives $\|S(t)\|\leqslant c_{\min}^{-1}\cdot\|c\|_\infty\cdot e^{\lambda^* t}$, where $c_{\min}=\inf\limits_{r\in[0,r_m)} c(r)$.

Let $\{T_\infty(t)\}_{t\geqslant 0}$ be the adjoint semigroup of $\{S(t)\}_{t\geqslant 0}$. Clearly, A_∞ is contained in the w^*-generator of $T_\infty(t)$. Obviously, the analytic representation of $\{T_p(t)\}_{t\geqslant 0}$ for all $p\in[0,\infty]$ is the same; i.e., $T_p(t)\phi=T_\infty(t)\phi$ for all $t\geqslant 0$ and $p\in[1,\infty)$ whenever $\phi\in X_\infty$. Hence if the initial value satisfies $\phi_0=(\phi_{10},\phi_{20})^T\in X_\infty$, then the inequality

$$\int_0^{r_m}|\phi_1(r,t)|\,dr+\int_0^{r_m}|\phi_2(r,t)|\,dr=\|T_1(t)\phi_0\|_1=\|T_\infty(t)\phi_0\|_1$$
$$\leqslant r_m\|T_\infty(t)\phi_0\|_\infty\leqslant\frac{r_m\|c\|_\infty}{c_{\min}}\cdot e^{\lambda^* t}\|\phi_0\|_\infty$$

holds for the corresponding solution $\phi(r,t)=(\phi_1(r,t),\phi_2(r,t))^T$ of Eq. (3). Therefore, the solution $\phi(r,t)$ with initial value $\phi_0=(\phi_{10},\phi_{20})^T\in X_\infty$ decays exponentially as $t\to+\infty$, provided that $\lambda^*<0$.

Conversely, suppose that $\lambda^*\geqslant 0$. One can readily show that the above-defined c belongs to $X_\infty\cap D(A_p)$ and $A_pc=\lambda^*c$ for all $p\in[1,\infty)$. Hence it follows that

$$T_p(t)c=\int_0^t T_p(r)A_pc\,\mathrm{d}r+c=\lambda^*\int_0^t T_p(r)c\,\mathrm{d}r+c\geqslant c,\qquad t\geqslant 0.$$

Consequently, the solution $\phi(r,t)=(T_p(t)c)(r)$ with $\phi_0=c$ cannot decay. The proof of the proposition is complete. □

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