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# * Generalized Relative Dahlquist Constant with Applications in Stability Analysis of Nonlinear Systems

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**Dedicated to Professor CHEN Qingyi on the Occasion of his 80th Birthday**

**Abstract :** In this paper ,a novel concept named generalized relative Dahlquist constant is introduced for nonlinear operators ,and then a novel and effective approach to stability analysis of nonlinear systems is developed. With this new approach ,some sufficient conditions for the exponential stability of nonlinear systems are obtained ,and the exponential decay estimation of the solution is also proposed.

**Key words :** Generalized relative Dahlquist constant ; Generalized Dahlquist constant ; Exponential stability ;Nonlinear systems

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The stability of nonlinear systems is of great practical significance since most systems in the world are nonlinear ,and it has been an important research issue (see ,for example [1 ,2] and references therein) . In Peng[3] ,a concept named generalized Dahlquist constant was defined for nonlinear operators and successfully applied to the investigation of the asymptotic behaviors of nonlinear semigroup of Lipschitz operators. In this paper ,we further introduce a new concept named generalized relative Dahlquist constant ,and apply them to the stability analysis of nonlinear systems and obtain some sufficient conditions for the exponential stability of the systems.

Let $X$ be a Banach space endowed with the norm $\|\cdot\|$ and $\Omega$ be an open subset of $X$.

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Throughout this paper we consider the following system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)), t \geq 0, \tag{1}$$

where $F$ is a nonlinear operator defined on $\Omega$, and $x(t) \in \Omega$.

**Definition 1** Suppose $x^*$ is an equilibrium point of system (1). System (1) is called to be exponentially stable on a neighborhood $\Omega$ of $x^*$, if $\exists a > 0$, $\exists \varepsilon > 0$, such that $\| x(t) - x^* \| \leq \varepsilon e^{-at} \| x(0) - x^* \| (t \geq 0)$, where $x(t)$ is any solution of (1) initiated from $x(0)$.

**Definition 2** (Söderlind [7]) Let $X$ be a Banach space endowed with the norm $\| \cdot \|$, $\Omega$ be a subset of $X$, and $F: \Omega \to X$ be an operator.

If there exists a constant $M > 0$, such that $\| F(x) - F(y) \| \leq M \| x - y \|$, $\forall x, y \in \Omega$, then $F$ is said to be a Lipschitz operator. The constant $L(F) = \sup_{x, y \in \Omega, x \neq y} \frac{\| F(x) - F(y) \|}{\| x - y \|}$ is called to be the least upper bound Lipschitz constant of $F$.

Denote by $\mathrm{Lip}(\Omega, X)$ the set of all the Lipschitz operators from $\Omega$ to $X$.

**Definition 3** (Söderlind [7]) Suppose that $\Omega$ is a subset of a Banach space $X$, and $F$ is an operator in $\mathrm{Lip}(\Omega, \Omega)$. Denote by $I + sF$ the operator mapping every point $x \in \Omega$ onto $x + sF(x)$. The Dahlquist constant of $F$ is defined as $\mu(F) = \lim_{s \to 0^+} \frac{L(I + sF) - 1}{s}$.

The Dahlquist constant plays an important role in characterizing the stability of Lipschitz systems. Peng[4] proved that for the system (1), if the operator $F$ is of Lipschitz and satisfies $\mu(F) < 0$, then the system has a unique equilibrium point in $\Omega$ and the equilibrium point is exponentially stable. We expect there may be some quantity which can characterize the stability of general nonlinear systems.

**Definition 4** (Peng[3]) Suppose $\Omega$ is an open subset of Banach space $X$, $F: \Omega \to X$ is an operator, and $x^0$ is any fixed point in $\Omega$. The constant

$$\alpha(F) = \sup_{x, y \in \Omega, x \neq y} \frac{1}{\| x - y \|} \lim_{r \to +\infty} [\, \| (F + rI)x - (F + rI)y \| - r \| x - y \| \,] \tag{2}$$

is called to be the generalized Dahlquist constant of $F$ on $\Omega$.

Inspired by the above definition, we now introduce the following new concept.

**Definition 5** Suppose $\Omega$ is an open subset of Banach space $X$, $F: \Omega \to X$ is an operator, and $x^0$ is any fixed point in $\Omega$. The constant

$$\alpha(F, x^0) = \sup_{x \in \Omega, x \neq x^0} \frac{1}{\| x - x^0 \|} \lim_{r \to +\infty} [\, \| (F + rI)x - (F + rI)x^0 \| - r \| x - x^0 \| \,] \tag{3}$$

is called to be the generalized relative Dahlquist constant of $F$ at $x^0$.

It can be easily proved that the funciton defined as $f(r) = \| (F + rI)x - (F + rI)y \| - r \| x - y \|$, $r > 0$ is monotone decreasing, thus the limit $\lim_{r \to +\infty} f(r)$ exists, and therefore the generalized relative Dahlquist constant is well defined.

Unlike the Dahlquist constant, the definition of the generalized (relative) Dahlquist constant is not limited to Lipschitz operators, which can be seen from the following example.

**Example 1** In system (1), let $\Omega = X = \mathbf{R}$ (the set of real numbers) and $F(x(t)) = -\frac{x^3(t)}{2} - x(t)$, $x \in \mathbf{R}$. It is easy to show that $L(F) = +\infty$, i.e. $F \notin \mathrm{Lip}(\Omega, \Omega)$, thus the Dahlquist constant of $F$ cannot be defined, while $\alpha(F) = \alpha(F,0) = -1$, $\alpha(F,x^0) \le 1$ ($\forall x^0 \in \mathbf{R}$, $x^0 \ne 0$).

It is clear that $\forall x^0 \in \Omega$, $\alpha(F,x^0) \le \alpha(F)$ holds, and forthermore, the inequality may hold strictly, which can be demonstrated by the following example.

**Example 2** Consider the nonlinear system (1) where $X = \Omega = \mathbf{R}$ and the function $F$ is defined as $F(x) = -\frac{5\pi x}{\pi^2+9} + \sin^2 x$. Obviously, $x^* = 0$ is an equilibrium point of the system. We can deduce that

$$\alpha(F) = \sup_{x \ne y} \frac{1}{\|x-y\|} \lim_{r \to +\infty} [\|(F+rI)x - (F+rI)y\| - r\|x-y\|]$$
$$= -\frac{5\pi}{\pi^2+9} + \sup_{x \ne y} \frac{\sin^2 x - \sin^2 y}{x-y} = -\frac{5\pi}{\pi^2+9} + 1 > 0,$$

and

$$\alpha(F,0) = \sup_{x \ne 0} \frac{1}{\|x\|} \lim_{r \to +\infty} [\|(F+rI)x\| - r\|x\|] = -\frac{5\pi}{\pi^2+9} + \sup_{x \ne 0} \frac{\sin^2 x}{x} < 0.$$

Consequently, $\alpha(F,0) < \alpha(F)$ holds.

The generalized (relative) Dahlquist constant have the following useful properties which can be readily verified.

(i) $\alpha(G+H) \le \alpha(G) + \alpha(H)$, $\alpha(G+H,x^0) \le \alpha(G,x^0) + \alpha(H,x^0)$, $\forall G, H: \Omega \to X$;

(ii) $\alpha(kG) = k\alpha(G)$, $\alpha(kG,x^0) = k\alpha(G,x^0)$, $\forall G: \Omega \to X$, $\forall k \ge 0$;

(iii) $\alpha(G+lI) = \alpha(G) + l$, $\alpha(G+lI,x^0) = \alpha(G,x^0) + l$, $\forall G: \Omega \to X$ and $\forall l \in \mathbf{R}$.

**Lemma 1** If $\alpha(F) < 0$, then $F$ is a one-to-one mapping on $\Omega$.

**Proof** Suppose $x_1, x_2 \in \Omega$ satisfy $F(x_1) = F(x_2)$ while $x_1 \ne x_2$. Then, in view of (2), we have

$$\alpha(F) = \sup_{x,y \in \Omega, x \ne y} \frac{1}{\|x-y\|} \lim_{r \to +\infty} [\|(F+rI)x - (F+rI)y\| - r\|x-y\|]$$
$$\ge \frac{1}{\|x_1-x_2\|} \lim_{x \to +\infty} [\|(F+rI)x_1 - (F+rI)x_2\| - r\|x_1-x_2\|] = 0$$

in contradiction to $\alpha(F) < 0$. Thus, $F$ is a one-to-one mapping on $\Omega$.

**Lemma 2** If $x^* \in \Omega$ is an equilibrium point of the system (1) and $\alpha(F,x^*) < 0$, then there is no other equilibrium point in $\Omega$ than $x^*$ (that is, the equilibrium point of system (1) is unique in $\Omega$).

**Proof** Otherwise, let $\tilde{x} \in \Omega$ be any other equilibrium point of (1) different from $x^*$. Then $F(\tilde{x}) = F(x^*) = 0$, $\tilde{x} \ne x^*$. By (3), we infer that

$$\alpha(F,x^*) = \sup_{x \in \Omega, x \ne x^*} \frac{1}{\|x-x^*\|} \lim_{r \to +\infty} [\|(F+rI)x - (F+rI)x^*\| - r\|x-x^*\|]$$
$$\ge \frac{1}{\|\tilde{x}-x^*\|} \lim_{r \to +\infty} [\|(F+rI)\tilde{x} - (F+rI)x^*\| - r\|\tilde{x}-x^*\|] = 0$$

in contradiction to $\alpha(F,x^*) < 0$. Thus $\tilde{x} = x^*$, and therefore the equilibrium point of (1) is unique in $\Omega$.

Lemmas 1 and 2 indicate that, the generalized (relative) Dahlquist constant can exactly characterize the uniqueness of equilibrium point of nonlinear systems.

**Theorem 1** If the operator $F$ in the system (1) satisfies $\alpha(F) < 0$, then (1) has at most one equilibrium point in $\Omega$. Moreover, any two solutions $x(t)$ and $y(t)$ respectively initiated from $x(0) = x_0 \in \Omega$ and $y(0) = y_0 \in \Omega$ satisfy

$$\| x(t) - y(t) \| \leq e^{\alpha(F)t} \| x_0 - y_0 \|, \forall t \geq 0. \tag{4}$$

**Proof** Since $\alpha(F) < 0$, it follows from Lemma 1 that $F$ is one-to-one in $\Omega$, therefore there is at most one point $u \in \Omega$ such that $F(u) = 0$, i.e. there is at most one equilibrium point of (1) in $\Omega$. Assume $x(t)$ and $y(t)$ are the solutions of (1) respectively under the initial conditions $x(0) = x_0 \in \Omega$ and $y(0) = y_0 \in \Omega$. We have $(e^{rt}x(t))'_t = re^{rt}x(t) + e^{rt}Fx(t) = e^{rt}(F + rI)x(t)$ for all $t \geq 0$ and $r > 0$. $\forall x_0, y_0 \in \Omega, r > 0, t > s \geq 0$,

$$e^{rt}[x(t) - y(t)] = e^{rs}[x(s) - y(s)] + \int_s^t e^{ru}[(F + rI)x(u) - (F + rI)y(u)]du,$$

then

$$e^{rt} \| x(t) - y(t) \| - e^{rs} \| x(s) - y(s) \| \leq \int_s^t e^{ru} \| (F + rI)x(u) - (F + rI)y(u) \| du.$$

Then for almost all $t \geq 0$, we infer that

$$(e^{rt} \| x(t) - y(t) \|)'_t \leq e^{rt} \| (F + rI)x(t) - (F + rI)y(t) \|.$$

Therefore, we have

$$\| x(t) - y(t) \|'_t \leq \| (F + rI)x(t) - (F + rI)y(t) \| - r \| x(t) - y(t) \|.$$

Let $r \to +\infty$, then

$$\| x(t) - y(t) \|'_t \leq \alpha(F) \| x(t) - y(t) \|. \tag{5}$$

Integrating the above inequality over $[0, t]$, we obtain $\| x(t) - y(t) \| \leq e^{\alpha(F)t} \| x_0 - y_0 \|$, i.e. the two solutions $x(t)$ and $y(t)$ satisfy the inequality (4).

**Theorem 2** Suppose $x^* \in \Omega$ is an equilibrium point of the system (1). If the operator $F$ satisfies $\alpha(F, x^*) < 0$, then $x^*$ is the unique equilibrium point of system (1) in $\Omega$, $x^*$ is exponentially stable, and furthermore, the exponential decay estimation of any solution $x(t)$ initiated from $x(0) = x_0 \in \Omega$ satisfies

$$\| x(t) - x^* \| \leq e^{\alpha(F, x^*)t} \| x_0 - x^* \|, \forall t \geq 0. \tag{6}$$

**Proof** By Lemma 2, we infer that $x^*$ is the unique equlibrium point of system (1) in $\Omega$. We have $F(x^*) = 0$. Suppose $x(t)$ is the solution of system (1) initiated from $x(0) = x_0 \in \Omega$. Analogous to the proof in Theorem 1, we can readily justify that

$$\| x(t) - x^* \|'_t \leq \alpha(F, x^*) \| x(t) - x^* \|. \tag{7}$$

Integration of the above inequality from 0 to $t$ yields

$$\| x(t) - x^* \| \leq e^{\alpha(F, x^*)t} \| x_0 - x^* \|.$$

Since $\alpha(F, x^*) < 0$, we infer that the equilibrium point $x^*$ is exponentially stable in $\Omega$, and the exponential decay estimation of the solution $x(t)$ is governed by the inequality (6).

**Remark 1** It is obvious that in Example 1, zero is an equilibrium point of the system. Since $\alpha(F) = \alpha(F, 0) = -1 < 0$, by Theorem 1 or 2, we can readily infer that 0 is the unique equilibrium point, and the zero equilibrium point is globally exponentially stable in the real number space.

**Remark 2** Since $\alpha(F, x^*) \leq \alpha(F)$, the condition $\alpha(F) < 0$ can sufficiently guarantee the exponential stability of the system (1).

**Remark 3** It should be specially emphasized that the generalized realtive Dahlquist constant is sometimes more precise and efficient than the generalized Dahlquist constant in quantifying the exponential stability of equilibrium point of nonlinear systems, in that the generalized relative Dahlquist constant $\alpha(F, x^*)$ may be strictly less than the generalized Dahlquist constant $\alpha(F)$. For example, 0 is obviously an equilibrium point in Example 2; since the generalized Dahlquist constant $\alpha(F) > 0$, Theorem 1 goes invalid; while the generalized relative Dahlquist constant $\alpha(F, 0) < 0$, then by Theorem 2, we can infer that 0 is the unique equilibrium point and the zero equilibrium point is globally exponentially stable in real number space.

Theorems 1 and 2 show that the generalized (relative) Dahlquist constant do have characterized the exponential stability of nonlinear systems in Banach space, and thus provide a new approach to stability analysis of nonlinear systems.

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# 广义相对 Dalquist 数及其在非线性系统稳定性分析中的应用

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**摘要:对非线性算子引入了一个新概念——广义相对** Dalquist **数,建立了一般的非线性系统稳定性分析的一种新方法. 借助这一新方法,得到了非线性系统指数稳定的充分条件,并给出了解的指数衰减估计**.

**关键词:广义相对** Dalquist **数;广义** Dalquist **数;指数稳定性;非线性系统**