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Exponential stability of equilibria of differential equations with time-dependent delay and non-Lipschitz nonlinearity[☆]

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ABSTRACT

This paper studies stability of equilibria of differential equations with time-dependent delay and non-Lipschitz nonlinearity. For this class of problems, we develop a novel method of analysis, the relative nonlinear measure method. Using this method, we obtain a sufficient condition for exponential stability. Moreover, this condition is used to study the stability of the equilibrium of a neural network model. Finally, some examples illustrate that our results are improvement and extension of some existing ones.

1. Introduction

In this paper, we discuss exponential stability of differential equations with the form

$$\begin{aligned} &\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(u(t)) + G(u(t-\tau(t))), \quad t \geq t_0 \\ &u(t) = \phi(t), \quad t \in [t_0 - b, t_0] \end{aligned} \tag{1.1}$$

where $t_0 \geq 0$ and $b > 0$ are constants, F and G are nonlinear partially Lipschitz continuous operators from an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, $u(t) \in \Omega$ for $t \geq t_0$, the delay function $\tau(t)$ satisfies $0 \leq \tau(t) \leq b$ for $t \geq t_0$, and $\phi(\cdot) \in C([t_0 - b, t_0], \Omega)$ is an initial function with the norm $\|\phi\| = \sup_{t_0 - b \leq s \leq t_0} \|\phi(s)\|$, here $C([t_0 - b, t_0], \Omega)$ denotes the space of all continuous functions from $[t_0 - b, t_0]$ into Ω.

Stability analysis of the delay differential equations is important for many problems in applications. Many excellent results were obtained in this area [1–14]. Some such results are obtained using Lyapunov functions [1,4,10]. Constructing a Lyapunov function may be a difficult task. Besides, some existing results rely on restrictive conditions on the coefficients of the problem such as Lipschitz continuity [8], strict monotonicity [11], and boundedness [7,14]. This paper presents a new method of stability analysis for (1.1) with minimal assumptions on F and G. The latter are only required to be partially Lipschitz continuous and delay $\tau(t)$ is assumed to be bounded. Our method also does not require constructing a Lyapunov function. The method is used to obtain the sufficient condition for exponential stability of the equilibria of (1.1), which provides estimates for the exponential decay of solutions. We apply these results to study a class of neural networks with time-dependent delays.

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The paper is arranged as follows. Section 2 presents exponential stability results of equilibria of (1.1). In Section 3, we apply these results to a class of neural networks with time-varying delays. Conclusions are drawn in Section 4.

2. Main results

We start by introducing some notations, definitions and basic results that will be employed throughout the paper.

$\mathbb{R}^n$ is endowed with the l^1-norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Ω is an open subset in $\mathbb{R}^n$ and Ω_i denotes the projection of the subset Ω on the ith axis of $\mathbb{R}^n$.

Definition 1. The function $h : \Omega_i \to \mathbb{R}$ is said to be partially Lipschitz continuous on Ω_i, if for any $x \in \Omega_i$ there exists a constant $L_x > 0$ such that

$$|h(y) - h(x)| \le L_x |y - x|, \quad \forall y \in \Omega_i.$$

The constant

$$L_{\Omega_i}(h, x) = \sup_{y\in\Omega_i, y\neq x} \frac{|h(y) - h(x)|}{|y - x|}$$

is called minimal partial Lipschitz constant of h with respect to x on Ω_i. Particularly, if $\Omega_i = \mathbb{R}$, then the function h is called partially Lipschitz continuous.

The operator $f = (f_1, f_2, \ldots, f_n) : \Omega \to \mathbb{R}^n$ is partially Lipschitz continuous on Ω if each function f_i is partially Lipschitz continuous on Ω_i.

Remark 1. It is obvious that each Lipschitz continuous function is partially Lipschitz continuous. However, some partially Lipschitz continuous functions may not be Lipschitz continuous.

Definition 2 (*[15]*)**.** Let f be an operator form Ω to $\mathbb{R}^n$ and $x^0 \in \Omega$. Then

(1) the constant

$$m_\Omega(f) = \sup_{x,y\in\Omega, x\neq y} \frac{\langle f(x) - f(y), \operatorname{sgn}(x - y)\rangle}{\|x - y\|_1}$$

is called the nonlinear measure of f on Ω;

(2) the constant

$$m_\Omega(f, x^0) = \sup_{x\in\Omega, x\neq x^0} \frac{\langle f(x) - f(x^0), \operatorname{sgn}(x - x^0)\rangle}{\|x - x^0\|_1}$$

is called the relative nonlinear measure of f at x^0 on Ω. Here $\langle\cdot,\cdot\rangle$ denotes the inner product and $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \ldots, \operatorname{sgn}(x_n))^T$ denotes the sign vector of $x \in \mathbb{R}^n$, where $\operatorname{sgn}(r)$ is the usual sign function of any $r \in \mathbb{R}$.

Remark 2. The constants $m_\Omega(f)$ and $m_\Omega(f, x^0)$ are allowed to be infinite. However, if f is Lipschitz continuous on Ω, then $m_\Omega(f) < \infty$. If f is partially Lipschitz continuous with respect to x^0 on Ω, then $m_\Omega(f, x^0) < \infty$.

It is useful to notice that, for any point $x \in \mathbb{R}^n$,

$$\begin{cases} \|x\|_1 = \langle x, \operatorname{sgn}(x)\rangle & \text{and} \\ \|x\|_1 \ge \langle x, \operatorname{sgn}(y)\rangle & \text{for all } y \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Definition 3. Let x^* be an equilibrium point of the system (1.1) and Ω an open neighborhood of x^*. We say that x^* is exponentially stable on Ω if there exist two positive constants α and M such that

$$\|u(t) - x^*\|_1 \le M e^{-\alpha(t-t_0)} \sup_{t_0-b\le s\le t_0} \|\phi(s) - x^*\|_1, \quad t \ge t_0,$$

where $u(t)$ is the solution of the system (1.1) initiated from the function $\phi(\cdot) \in C([t_0 - b, t_0], \Omega)$.

Moreover, if $\Omega = \mathbb{R}^n$, i.e., x^* is exponentially stable on the whole space $\mathbb{R}^n$, the system (1.1) is said to be globally exponentially stable.

Proposition 1. *The solutions of the time-delayed system (1.1) exist in the global time interval* $[t_0, \infty)$.

Proof. The solutions of the time-delayed system (1.1) locally exist [16]. Then the system (1.1) enjoys a solution $x(t, \phi)$ satisfying $x(t_0, \phi) = \phi$ for $t \in [t_0, t^*(\phi))$ where $t^*(\phi) \in (t_0, +\infty)$ or $t^*(\phi) = +\infty$ such that $[t_0, t^*(\phi))$ is the maximal right existence interval of the solution $x(t, \phi)$. Let $T_0 \in (t_0, \infty)$ be any finite time such that $x(t, \phi)$ is a solution of the system (1.1) for $t \in [t_0, T_0)$. Since F, G is partially Lipschitz continuous, there exist constants $L_\Omega(F, 0), L_\Omega(G, 0) > 0$ such that

$$\|F(u(t)) - F(0)\| \le L_\Omega(F, 0)\|u(t)\| \quad \text{and} \quad \|G(u(t)) - G(0)\| \le L_\Omega(G, 0)\|u(t)\|$$

for any $u(t) \in \Omega$. Let $\mathcal{H}(u(t)) = F(u(t)) - G(u(t-\tau(t)))$. Hence,

$$\begin{aligned}\|\mathcal{H}(u(t)) - \mathcal{H}(0)\| &= \|F(u(t)) + G(u(t-\tau(t))) - F(0) - G(0)\| \\ &\leq \|F(u(t)) - F(0)\| + \|G(u(t-\tau(t))) - G(0)\| \\ &\leq L_\Omega(F,0)\|u(t)\| + L_\Omega(G,0)\|u(t-\tau(t))\|.\end{aligned}$$

Moreover we have

$$\|\mathcal{H}(u(t))\| \leq \|\mathcal{H}(0)\| + L_\Omega(F,0)\|u(t)\| + L_\Omega(G,0)\|u(t-\tau(t))\| \quad \text{for } u(t) \in \Omega.$$

By integrating two sides of (1.1) from t_0 to $t < T_0$, we have

$$u(t) = u(t_0) + \int_{t_0}^{t} \mathcal{H}(u(s))\mathrm{d}s, \quad \forall t \in [t_0, T_0) \tag{2.2}$$

and $u(t) = \phi(t)$ for $t \in [t_0 - b, t_0]$. Consequently, we derive

$$\begin{aligned}\|u(t)\| &\leq \|\phi\| + \int_{t_0}^{t} (\|\mathcal{H}(0)\| + L_\Omega(F,0)\|u(s)\| + L_\Omega(G,0)\|u(s-\tau(s))\|)\mathrm{d}s \\ &\leq [\|\phi\| + (T_0 - t_0)\|\mathcal{H}(0)\|] + L_\Omega(F,0)\int_{t_0}^{t} \|u(s)\|\mathrm{d}s + L_\Omega(G,0)\int_{t_0}^{t} \|u(s-\tau(s))\|\mathrm{d}s \\ &\leq [\|\phi\| + (T_0 - t_0)\|\mathcal{H}(0)\|] + L_\Omega(F,0)\int_{t_0}^{t} \|u(s)\|\mathrm{d}s + L_\Omega(G,0)\int_{t_0-b}^{t} \|u(s)\|\mathrm{d}s \\ &= \alpha + \beta\int_{t_0}^{t} \|u(s)\|\mathrm{d}s\end{aligned}$$

where $\alpha = \|\phi\| + (T_0 - t_0)\|\mathcal{H}(0)\| + bL_\Omega(G,0)\|\phi\|$, $\beta = L_\Omega(F,0) + L_\Omega(G,0)$. By the Gronwall inequality, we obtain

$$\|u(t)\| \leq \alpha \mathrm{e}^{\beta t} \leq \alpha \mathrm{e}^{\beta T_0} < \infty, \quad \forall t \in [t_0, T_0). \tag{2.3}$$

Then, the solution $x(t, \phi)$ is bounded for $t \in [t_0, t^*(\phi))$ if $T_0 = t^*(\phi)$ is finite. From the continuation theorem we can conclude that the solution $x(t, \phi)$ exits in the global time interval $[t_0, \infty)$. □

We can easily derive the following proposition for the time-delayed system (1.1) by the concept of relative nonlinear measure.

Proposition 2. *Suppose that $x^* \in \Omega$ is an equilibrium point of the system (1.1). If $m_\Omega(F+G, x^*) < 0$, then there is no other equilibrium point in Ω than x^*, namely, the equilibrium point of the system (1.1) is unique in Ω.*

Proof. Assume that $u \in \Omega$ is any other equilibrium point of the system (1.1) different from x^*, i.e., $(F+G)(x^*) = (F+G)(u) = 0$, $u \neq x^*$. Then, we have

$$\begin{aligned}m_\Omega(F+G, x^*) &= \sup_{x \in \Omega, x \neq x^*} \frac{\langle (F+G)(x) - (F+G)(x^*), \operatorname{sgn}(x - x^*)\rangle}{\|x - x^*\|_1} \\ &\geq \frac{\langle (F+G)(u) - (F+G)(x^*), \operatorname{sgn}(u - x^*)\rangle}{\|u - x^*\|_1} = 0,\end{aligned}$$

which contradicts $m_\Omega(F+G, x^*) < 0$. □

Remark 3. It is worth noting that the system (1.1) may have no equilibrium since coefficient operators F and G are only assumed to be partially Lipschitz continuous. However, there exist many known methods to the existence of equilibria of nonlinear systems, such as, contraction mapping principle and Brouwer's fixed point theorem [17].

Lemma 1 *([18, Lemma D, p. 389–390]). Let γ and p be constants with $0 < p < \gamma$. Let $v(\cdot)$ be a continuous nonnegative function satisfying*

$$v'(t) \leq -\gamma v(t) + p\|v_t\| \quad \text{for } t \geq t_0.$$

Then we have

$$v(t) \leq \|v_{t_0}\|\mathrm{e}^{-\lambda(t-t_0)} \quad \text{for all } t \geq t_0$$

where $\|v_{t_0}\| = \sup_{t_0 - b \leq s \leq t_0} \|v(s)\|$ and λ is the unique positive solution of

$$\lambda = \gamma - p\mathrm{e}^{\lambda b}.$$

Lemma 2 ([8]). *If $a > c \geq 0$, for every nonnegative real number b, the equation*

$$0 = \lambda - a + c\mathrm{e}^{\lambda b}$$

has a unique positive solution.

Proposition 3. *Let Ω be a neighborhood of the equilibrium x^* of the system (1.1). If, for some diagonal matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0$,*

$$m_{A^{-1}(\Omega)}(FA, A^{-1}x^*) + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*) < 0, \tag{2.4}$$

then x^* is exponentially stable on Ω. Moreover, the exponential decay estimation of the solution $x(t)$ initiated from the function $\phi(\cdot) \in C([t_0 - b, t_0], \Omega)$ is determined by

$$\|x(t) - x^*\|_1 \leq \mathrm{e}^{-\lambda(t-t_0)} \sup_{t_0-b\leq s\leq t_0} \|\phi(s) - x^*\|_1 \quad \text{for all } t \geq t_0, \tag{2.5}$$

where λ is the unique positive solution of the equation

$$0 = \lambda \cdot \min_{1\leq i\leq n} a_i + m_{A^{-1}(\Omega)}(FA, A^{-1}x^*) + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*) \cdot \mathrm{e}^{b\lambda}.$$

Proof. Let $u(t) = x(t) - x^*$ for all $t \geq t_0$. From (2.1) we can easily conclude that, for any $s > 0$,

$$\frac{\|u(t)\|_1 - \|u(t-s)\|_1}{s} \leq \frac{1}{s}\langle u(t) - u(t-s), \mathrm{sgn}(u(t))\rangle,$$

which implies that the derivatives of $\|u(t)\|_1$ exist almost everywhere in $(t_0, +\infty)$ because the function $t \mapsto \|u(t)\|_1$ is absolutely continuous in $(t_0, +\infty)$ [19, Theorem 6.9, p. 129]. From (1.1), the derivatives of $\|u(t)\|_1$ satisfy almost everywhere in $(t_0, +\infty)$ that

$$\begin{aligned}
\frac{\mathrm{d}\|u(t)\|_1}{\mathrm{d}t} &\leq \left\langle \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \mathrm{sgn}(u(t))\right\rangle \\
&= \langle F(x(t)) + G(x(t-\tau(t))), \mathrm{sgn}(u(t))\rangle \\
&= \langle F(x(t)) + G(x(t-\tau(t))) - (F(x^*) + G(x^*)), \mathrm{sgn}(u(t))\rangle \\
&= \langle F(x(t)) - F(x^*), \mathrm{sgn}(u(t))\rangle + \langle G(x(t-\tau(t))) - G(x^*), \mathrm{sgn}(u(t))\rangle \\
&= \langle F(x(t)) - F(x^*), \mathrm{sgn}(A^{-1}u(t))\rangle + \langle G(x(t-\tau(t))) - G(x^*), \mathrm{sgn}(u(t))\rangle \\
&\leq m_{A^{-1}(\Omega)}(FA, A^{-1}x^*)\|A^{-1}u(t)\|_1 + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*)\|A^{-1}x(t-\tau(t)) - A^{-1}x^*\|_1 \\
&\leq \left(m_{A^{-1}(\Omega)}(FA, A^{-1}x^*)\|u(t)\|_1 + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*)\|u_t\|\right)\left(\min_{1\leq i\leq n} a_i\right)^{-1},
\end{aligned}$$

where $\|u_t\| = \sup_{t-b\leq s\leq t} \|u(s)\|_1$. Owing to $m_{A^{-1}(\Omega)}(FA, A^{-1}x^*) + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*) < 0$ and $L_{A^{-1}(\Omega)}(GA, A^{-1}x^*) \geq 0$, by Lemmas 1 and 2, we have

$$\|u(t)\|_1 \leq \mathrm{e}^{-\lambda(t-t_0)} \sup_{t_0-b\leq s\leq t_0} \|u(s)\|_1,$$

where λ is the unique positive solution of the equation

$$\lambda = -m_{A^{-1}(\Omega)}(FA, A^{-1}x^*) \cdot \left(\min_{1\leq i\leq n} a_i\right)^{-1} - L_{A^{-1}(\Omega)}(GA, A^{-1}x^*) \cdot \left(\min_{1\leq i\leq n} a_i\right)^{-1} \mathrm{e}^{b\lambda}.$$

Therefore, the inequality (2.5) is derived. □

Remark 4. The existing results assume the coefficient operators to be Lipschitz continuous [8,15], strictly increasing [11] and bounded [7,14]. Our results remove these restrictions of coefficient operators. Consequently, Proposition 3 is the improved version of [7,8,15,11,14]. Moreover, even if F and G are Lipschitz continuous, the condition (2.4) is more efficient than $m_{A^{-1}(\Omega)}(FA) + L_{A^{-1}(\Omega)}(GA) < 0$ of Theorem 1 of [8] because $m_{A^{-1}(\Omega)}(FA, A^{-1}x^*) + L_{A^{-1}(\Omega)}(GA, A^{-1}x^*)$ may be strictly less than $m_{A^{-1}(\Omega)}(FA) + L_{A^{-1}(\Omega)}(GA)$. For this, we present a simple illustrative example.

Example 1. Consider the following time-varying delay system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \sin^2(u(t)) - \frac{4\pi u(t)}{\pi^2 + 4} + ku(t - \tau(t)), \tag{2.6}$$

where $0 < k < \frac{4\pi}{\pi^2+4} - \sup_{r\neq 0} \frac{\sin^2 r}{r}$ is a constant.

Let $F(r) = \sin^2 r - \frac{4\pi r}{\pi^2+4}$ and $G(r) = kr$ for $r \in \mathbb{R}$. Obviously, $u^* = 0$ is an equilibrium point of the system (2.6). Moreover, F and G are Lipschitz continuous on $\mathbb{R}$ because of

$$L_{\mathbb{R}}(F) = \sup_{r\in\mathbb{R}} |F'(r)| = 1 + \frac{4\pi}{\pi^2+4} < \infty \quad \text{and} \quad L_{\mathbb{R}}(G) = \sup_{r\in\mathbb{R}} |G'(r)| = k < \infty.$$

By Definition 2, we have

$$m_{\mathbb{R}}(F) = 1 - \frac{4\pi}{\pi^2+4} > 0 \quad \text{and} \quad m_{\mathbb{R}}(F, 0) = \sup_{r\neq 0} \frac{\sin^2 r}{r} - \frac{4\pi}{\pi^2+4} < 0.$$

Taking $A = 1$, we have $m_{A^{-1}(\mathbb{R})}(FA) + L_{A^{-1}(\mathbb{R})}(GA) > 0$, but $m_{A^{-1}(\mathbb{R})}(FA, 0) + L_{A^{-1}(\mathbb{R})}(GA, 0) = \sup_{r\neq 0} \frac{\sin^2 r}{r} - \frac{4\pi}{\pi^2+4} + k < 0$.

3. Applications to neural networks with time-varying delays

In this section, we apply the results obtained in the last section to the neural network model described by the following differential equations with time-varying delays:

$$\begin{aligned} &\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i(u_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^{\tau} f_j^{\tau}(u_j(t - \tau_{ij}(t))) + I_i, \quad t \geq 0, \\ &u_i(t) = \phi_i(t) \in C([-b, 0], \Omega_i), \end{aligned} \tag{3.1}$$

where $i = 1, 2, \ldots, n$ and $n \geq 2$ is the number of neurons in the networks; $u_i(t)$ denote the neuron state vectors; $a_i(\cdot)$ are the appropriately behaved functions; $W = (w_{ij})_{n\times n}$ and $W^{\tau} = (w_{ij}^{\tau})_{n\times n}$ denote the normal and delayed connection weight matrices, f_i and f_i^{τ} the normal and delayed activation functions, respectively; $\tau_{ij}(t) \geq 0$ are the time-varying delays caused during the switching and transmission processes and $b = \sup\{\tau_{ij}(t) : 1 \leq i, j \leq n, t \in \mathbb{R}\} < \infty$; I_i are the constant external inputs.

The model (3.1) is a kind of generalization of Cohen–Grossberg neural networks (CGNNs) firstly proposed by Cohen and Grossberg in [20]. Many famous neural network models, for example, Hopfield-type neural networks with time-varying delays [8], cellular neural networks with time-varying delays [21–24] and bi-directional associative memory neural networks with discrete delays [25], are its special cases. The stability analysis of the neural network model is fundamental for networks designs and applications [26,27,25,28–33,4,34–38,8,15,39–44,11,45–48,14,49,50,21–23,51,52,24]. For the neural network model (3.1), we only adopt the following general assumptions:

(H_1) $a_i(\cdot)$ are partially Lipschitz continuous on Ω_i and there exist constants $\lambda_i > 0, L_{1i}, M_{1i} \geq 0$ such that

$$(r_1 - r_2)[a_i(r_1) - a_i(r_2)] \geq \lambda_i (r_1 - r_2)^2 \quad \text{and} \quad |a_i^{-1}(r)| \leq L_{1i}|r| + M_{1i}$$

for all $r, r_1, r_2 \in \Omega_i$ and $i = 1, 2, \ldots, n$;

(H_2) f_i and f_i^{τ} are partially Lipschitz continuous on Ω_i and satisfy respectively

$$|f_i(r)| \leq L_{2i}|r| + M_{2i} \quad \text{and} \quad |f_i^{\tau}(r)| \leq L_{2i}^{\tau}|r| + M_{2i}^{\tau}, \quad r \in \mathbb{R}, \ i = 1, 2, \ldots, n,$$

where $L_{2i}, M_{2i}, L_{2i}^{\tau}$ and M_{2i}^{τ} are nonnegative constants;

(H_3) $\rho(K) < 1$, where $\rho(K)$ denote the spectral radius of matrix $K = (K_{ij})_{n\times n}, K_{ij} = L_{1i}(|w_{ij}|L_{2j} + |w_{ij}^{\tau}|L_{2j}^{\tau})$.

Under the assumptions (H_1)–(H_3), the model (3.1) has at least one equilibrium by virtue of Theorem 3.1 in [53]. Let $F : \Omega \to \mathbb{R}^n$ by $F_i(u) = -a_i(u_i) + \sum_{j=1}^{n} w_{ij} f_j(u_j)$ and $G : \Omega \to \mathbb{R}^n$ by $G_i(u) = \sum_{j=1}^{n} w_{ij}^{\tau} f_j^{\tau}(u_j) + I_i$. Then, the model (3.1) is equivalently rewritten as the system (1.1).

Proposition 4. *Assume that Ω is an open neighborhood of the equilibrium u^* of the model (3.1). If there exist a set of real numbers $d_i > 0$ $(i = 1, 2, \ldots, n)$ such that*

$$\max_{1\leq i\leq n} \frac{1}{\lambda_i} \left\{ L_i \sum_{j=1}^{n} \frac{d_j}{d_i} |w_{ji}| + L_i^{\tau} \sum_{j=1}^{n} \frac{d_j}{d_i} |w_{ji}^{\tau}| \right\} < 1, \tag{3.2}$$

where $L_i = L_{\Omega_i}(f_i, u_i^*)$ and $L_i^{\tau} = L_{\Omega_i}(f_i^{\tau}, u_i^*)$, then u^* is the unique equilibrium point of the model (3.1) in Ω.

Proof. Let $H(u) = F(u) + G(u)$, $u \in \Omega$ and $P = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. To prove the uniqueness of equilibrium of the model (3.1) in Ω, it suffices to prove that $m_{\Omega}(PH, u^*) < 0$. In fact, for all $y \in \Omega$ and $y \neq u^*$, we have

$$\begin{aligned} &\langle PH(y) - PH(u^*), \mathrm{sgn}(y - u^*)\rangle \\ &\quad = \sum_{i=1}^{n} \left\{ \mathrm{sgn}(y_i - u_i^*) d_i \left[-(a_i(y_i) - a_i(u_i^*)) + \sum_{j=1}^{n} w_{ij}(f_j(y_j) - f_j(u_j^*)) + \sum_{j=1}^{n} w_{ij}^{\tau}(f_j^{\tau}(y_j) - f_j^{\tau}(u_j^*)) \right] \right\} \end{aligned}$$

$$\leq \sum_{i=1}^{n} d_i \left\{ -|a_i(y_i) - a_i(u_i^*)| + \sum_{j=1}^{n} (|w_{ij}|\,|f_j(y_j) - f_j(u_j^*)| + |w_{ij}^\tau|\,|f_j^\tau(y_j) - f_j^\tau(u_j^*)|) \right\}$$
$$\leq -\sum_{i=1}^{n} d_i\lambda_i|y_i - u_i^*| + \sum_{j=1}^{n}\sum_{i=1}^{n} d_i(|w_{ij}| \cdot L_j|y_j - u_j^*| + |w_{ij}^\tau| \cdot L_j^\tau|y_j - u_j^*|)$$
$$= -\sum_{i=1}^{n} d_i\lambda_i|y_i - u_i^*| + \sum_{j=1}^{n} \left\{ L_j|y_j - u_j^*| \sum_{i=1}^{n} d_i|w_{ij}| + L_j^\tau|y_j - u_j^*| \sum_{i=1}^{n} d_i|w_{ij}^\tau| \right\}$$
$$= -\sum_{i=1}^{n} \left\{ d_i \left(\lambda_i - L_i \sum_{j=1}^{n} \frac{d_j}{d_i}|w_{ji}| - L_i^\tau \sum_{j=1}^{n} \frac{d_j}{d_i}|w_{ji}^\tau| \right) \right\} |y_i - u_i^*|.$$

It follows directly from the inequality (3.2) that $m_\Omega(PH, u^*) < 0$. Consequently, by Proposition 2, we conclude that the equilibrium point of the model (3.1) is unique in Ω. □

Proposition 5. *Suppose that Ω is a neighborhood of the equilibrium point u^* of the model (3.1). If there exist a set of real numbers $d_i > 0\,(i = 1, 2, \dots, n)$ such that the inequality (3.2) holds, then u^* is exponentially stable on Ω, and the exponential decay estimate of any solution initiated from $\phi(\cdot) \in C([-b, 0], \Omega)$ is governed by*

$$\|u(t) - u^*\|_1 \leq \frac{\max_{1\leq i\leq n} d_i}{\min_{1\leq i\leq n} d_i} e^{-\sigma t} \sup_{-b\leq s\leq 0} \|\phi(s) - u^*\|_1, \quad t \geq 0. \tag{3.3}$$

Here σ is the unique positive solution of $\sigma \cdot \min_{1\leq i\leq n} \frac{1}{c_i} - 1 + ke^{b\sigma} = 0$ with

$$c_i = \lambda_i - L_i \sum_{j=1}^{n} \frac{d_j}{d_i}|w_{ji}| \quad \textit{and} \quad k = \max_{1\leq i\leq n} \left\{ c_i^{-1} L_i^\tau \sum_{j=1}^{n} \frac{d_j}{d_i}|w_{ji}^\tau| \right\},$$

where $L_i = L_{\Omega_i}(f_i, u_i^*)$ and $L_i^\tau = L_{\Omega_i}(f_i^\tau, u_i^*)$. Particularly, if the inequality (3.2) holds on the whole space $\mathbb{R}^n$, then the model (3.1) is globally exponentially stable with the exponential decay estimate (3.3).

Proof. Let $P = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \dots, d_n^{-1})$ and $A = \operatorname{diag}(c_1^{-1}, c_2^{-1}, \dots, c_n^{-1})$. It follows from (3.2) that $c_i = \lambda_i - L_i v_i \sum_{j=1}^{n} \frac{d_j}{d_i}|w_{ji}| > 0,\ i = 1, 2, \dots, n$. Similar to the proof of Proposition 4, we can infer that, for all $y \in A^{-1}P^{-1}(\Omega)$,

$$\langle P^{-1}F(PAy) - P^{-1}F(u^*), \operatorname{sgn}(y - A^{-1}P^{-1}u^*)\rangle$$
$$\leq \sum_{i=1}^{n} d_i \left\{ -|a_i(d_i^{-1}c_i^{-1}y_i) - a_i(u_i^*)| + \sum_{j=1}^{n} |w_{ij}|\,|f_j(d_j^{-1}c_j^{-1}y_j) - f_j(u_j^*)| \right\}$$
$$\leq \sum_{i=1}^{n} d_i \left\{ -d_i^{-1}c_i^{-1}\lambda_i|y_i - d_ic_iu_i^*| + \sum_{j=1}^{n} |w_{ij}| \cdot L_jd_j^{-1}c_j^{-1}|y_j - d_jc_ju_j^*| \right\}$$
$$= -\sum_{j=1}^{n} c_j^{-1} \left(\lambda_j - L_j \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}| \right) |y_j - d_jc_ju_j^*|$$
$$= -\|y - A^{-1}P^{-1}u^*\|_1,$$

which implies that $m_{P^{-1}A^{-1}(\Omega)}(P^{-1}FPA, A^{-1}P^{-1}u^*) \leq -1$.

For all $y \in A^{-1}P^{-1}(\Omega)$, we deduce that

$$\|P^{-1}G(PA)y - P^{-1}Gu^*\|_1 = \sum_{i=1}^{n} \left| d_i \sum_{j=1}^{n} w_{ij}^\tau (f_j^\tau(d_j^{-1}c_j^{-1}y_j) - f_j^\tau(u_j^*)) \right|$$
$$\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{n} d_i|w_{ij}^\tau| \cdot L_j^\tau d_j^{-1}c_j^{-1} \right) |y_j - d_jc_ju_j^*|,$$

and thus,

$$L_{A^{-1}P^{-1}(\Omega)}(P^{-1}GPA, A^{-1}P^{-1}u^*) \leq \max_{1\leq j\leq n} \left\{ c_j^{-1}L_j^\tau \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}^\tau| \right\} = k.$$

Hence, by (3.2) we have

$$\begin{aligned}
& m_{P^{-1}A^{-1}(\Omega)}(P^{-1}FPA, A^{-1}P^{-1}u^*) + L_{A^{-1}P^{-1}(\Omega)}(P^{-1}GPA, A^{-1}P^{-1}u^*) \\
&\quad \le -1 + \max_{1\le j\le n} \frac{L_j^\tau \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}^\tau|}{c_j} \\
&\quad = \max_{1\le j\le n} \left\{ \frac{-\lambda_j + L_j \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}| + L_j^\tau \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}^\tau|}{\lambda_j - L_j \sum_{i=1}^{n} \frac{d_i}{d_j}|w_{ij}|} \right\} < 0.
\end{aligned}$$

Therefore, by Proposition 3 we deduce that the solution of time-varying delay system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = P^{-1}FP(x(t)) + P^{-1}GP(x(t-\tau(t))), \quad t \ge 0 \tag{3.4}$$

satisfies

$$\|x(t) - P^{-1}u^*\|_1 \le \mathrm{e}^{-\sigma t} \sup_{-b\le s\le 0} \|x(s) - P^{-1}u^*\|_1, \quad t \ge 0.$$

Noticing that $x(t) = P^{-1}u(t)$ is the solution of the system (3.4) whenever $u(t)$ is a solution of the model (3.1), we derive that the model (3.1) is exponentially stable on Ω and the exponential decay estimate is governed by (3.3). □

Remark 5. It is significant to incorporate the adjustable parameters d_i $(i = 1, 2, \ldots, n)$ into (3.2) in Propositions 4 and 5. Particularly, compared with the condition

$$\max_{1\le i\le n} \frac{1}{\lambda_i} \left\{ L_i \sum_{j=1}^{n} |w_{ji}| + L_i^\tau \sum_{j=1}^{n} |w_{ji}^\tau| \right\} < 1, \tag{3.5}$$

the stability criterion (3.2) can be applied to wider neural network models, that is to say, the stability criterion (3.2) is more effective than the condition (3.5). For this, we provide an illustrative example.

Example 2. Consider the neural network model:

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = -3u_1(t) + \dfrac{3}{4}f_1(u_1(t)) + \dfrac{2}{15}f_2(u_2(t-\tau_{12}(t))), \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = -2u_2(t) + \dfrac{2}{7}f_2(u_2(t)) + \dfrac{3}{4}f_1(u_1(t-\tau_{21}(t))), \end{cases} \tag{3.6}$$

where $f_1(r) = f_2(r) = r(1+\cos r)$, $r \in \mathbb{R}$.

$u^* = (0, 0)^T$ is an equilibrium point of (3.6). For any $y \in \mathbb{R}$, the inequality

$$\begin{aligned}
|f_i(x) - f_i(y)| &\le |x(1+\cos x) - y(1+\cos y)| \le |x+y| + |x\cos x - y\cos y| \\
&\le |x-y| + |x\cos x - y\cos x| + |y\cos x - y\cos y| \\
&\le (2|x-y| + |y|\,|\cos x - \cos y|) \le (2+|y|)|x-y|
\end{aligned} \tag{3.7}$$

implies that f_i are partially Lipschitz continuous. $L_{\mathbb{R}}(f_i, 0) = \sup_{r\ne 0} \frac{|f_i(r)-f_i(0)|}{|r-0|} = \sup_{r\ne 0}|1+\cos r| = 2$, $w_{12} = w_{21} = w_{11}^\tau = w_{22}^\tau = 0$, $w_{11} = \frac{3}{4}$, $w_{22} = \frac{2}{7}$, $w_{12}^\tau = \frac{2}{15}$ and $w_{21}^\tau = \frac{3}{4}$, $L_{11} = 1/3$, $L_{12} = 1/2$, $L_{21} = L_{22} = L_{21}^\tau = L_{22}^\tau = 2$. It is easily computed that $\rho(K) = 1/2 < 1$. Take $\lambda_1 = \lambda_2 = 2$ and $d_1 = 5$, $d_2 = 1$. We have

$$\max_{1\le i\le 2} \frac{1}{\lambda_i} \left\{ L_{\mathbb{R}}(f_i, 0) \sum_{j=1}^{2} \frac{d_j}{d_i}|w_{ji}| + L_{\mathbb{R}}(f_i^\tau, 0) \sum_{j=1}^{2} \frac{d_j}{d_i}|w_{ji}^\tau| \right\} = \max\left\{ \frac{9}{10}, \frac{20}{21} \right\} = \frac{3}{4} < 1.$$

According to Proposition 5, $u^* = (0, 0)^T$ is globally exponentially stable. However,

$$\max_{1\le i\le 2} \frac{1}{\lambda_i} \left\{ L_{\mathbb{R}}(f_i) \sum_{j=1}^{2} |w_{ji}| + L_{\mathbb{R}}(f_i^\tau) \sum_{j=1}^{2} |w_{ji}^\tau| \right\} = \max\left\{ \frac{3}{2}, \frac{44}{105} \right\} > 1.$$

Remark 6. It is necessary and important to discuss how to verify the condition (3.2). In what follows, we suggest an optimization method.

Noticing that

$$(3.2) = \max_{1\le i\le n} \frac{1}{\lambda_i}\left\{L_i\sum_{j=1}^{n}\frac{d_j}{d_i}|w_{ji}| + L_i^\tau\sum_{j=1}^{n}\frac{d_j}{d_i}|w_{ji}^\tau|\right\} < 1$$

$$\Longleftrightarrow d_i\lambda_i - \sum_{j=1}^{n}(L_i|w_{ji}| + L_i^\tau|w_{ji}^\tau|)d_j > 0, \quad i = 1, 2, \ldots, n.$$

We define $f_i(d_1, d_2, \ldots, d_n) = d_i\lambda_i - \sum_{j=1}^{n}(L_i|w_{ji}| + L_i^\tau|w_{ji}^\tau|)d_j,\ i = 1, 2, \ldots, n$. Then, whether the condition (3.2) holds for some d_i is equivalent to whether the following optimization problem

$$\begin{cases}\min\ -d_{n+1}\\ \text{s.t.} f_i - d_{n+1} \ge 0, \quad i = 1, 2, \ldots, n\\ d_i > 0, \quad i = 1, 2, \ldots, n, n+1\end{cases} \tag{3.8}$$

enjoys a solution $(d_1^*, d_2^*, \ldots, d_n^*, d_{n+1}^*)^T$. For methods to the optimization problem (3.8), we refer to literature [54].

Example 3. Consider the following neural network model:

$$\begin{cases}\dfrac{du_1(t)}{dt} = -4u_1(t) + \dfrac{1}{2}f_1(u_1(t)) + \dfrac{1}{4}f_2(u_2(t-\tau_{12}(t))),\\ \dfrac{du_2(t)}{dt} = -4u_2(t) + \dfrac{2}{7}f_2(u_2(t)) + \dfrac{1}{4}f_1(u_1(t-\tau_{21}(t))),\end{cases} \tag{3.9}$$

where $f_1(r) = f_2(r) = r(1+\sin r)$, $r \in \mathbb{R}$ and $\tau_{ij}(t) = 4|\cos t|,\ i, j = 1, 2, t \in \mathbb{R}$.

Obviously, $u^* = (0, 0)^T$ is an equilibrium point of the model (3.9) and $b = \max\{\tau_{ij}(t), i, j = 1, 2\} = 4$. Since $f_i'(r) = 1 + \sin r + r\cos r$ are not bounded, f_i are not globally Lipschitz continuous on $\mathbb{R}$. Moreover, f_i are neither bounded nor monotone. Hence, none of the existing stability criteria [55,8,40,21–24] can be applied. However, it can easily be verified that f_i are partially Lipschitz continuous on $\mathbb{R}$, $L_{11} = L_{12} = 1/4$, $L_{21} = L_{22} = L_{21}^\tau = L_{22}^\tau = 2$ and $\rho(K) = 1/4 < 1$, i.e, the model (3.9) satisfies the assumptions (H_1)–(H_3). Since $f_i = f_i^\tau$ and $L_{\mathbb{R}}(f_i, 0) = \sup_{r\neq 0}\frac{|f_i(r)-f_i(0)|}{|r-0|} = \sup_{r\neq 0}|1+\sin r| = 2$, then, with $\lambda_1 = \lambda_2 = 2$ and $\Omega = \mathbb{R}^2$, we have

$$\max_{1\le i\le 2}\frac{1}{\lambda_i}\left\{L_{\mathbb{R}}(f_i, 0)\sum_{j=1}^{2}|w_{ji}| + L_{\mathbb{R}}(f_i^\tau, 0)\sum_{j=1}^{2}|w_{ji}^\tau|\right\} = \max\left\{\frac{3}{4}, \frac{15}{28}\right\} = \frac{3}{4} < 1,$$

i.e., the condition (3.2) holds with $d_1 = d_2 = 1$. Hence, by virtue of Proposition 5, we can conclude that the equilibrium point $u^* = (0, 0)^T$ is globally exponentially stable and the exponential decay estimate is governed by

$$|u_1(t)| + |u_2(t)| \le e^{-\sigma t}\sup_{-4\le s\le 0}\|\phi(s)\|_1,$$

where $u(t) = (u_1(t), u_2(t))^T$ is the solution of (3.9) initiated from $\phi(\cdot) \in C([-4, 0], \mathbb{R}^2)$, σ is the unique positive solution of the equation $\frac{7}{10}\sigma - 1 + \frac{1}{2}e^{4\sigma} = 0$ (Fig. 1).

The following example shows that the present stability condition (3.2) is really sharper than that of [8,40] even if the activation functions are Lipschitz continuous.

Example 4. Consider the following neural network model:

$$\frac{du_i(t)}{dt} = -2\left(\frac{\pi+2}{\pi-2}\right)^4 u_i(t) + \sum_{j=1}^{n}w_{ij}f_j(u_j(t)) + \sum_{j=1}^{n}w_{ij}^\tau f_j^\tau(u_j(t-\tau_{ij}(t))), \quad i = 1, 2. \tag{3.10}$$

Here $b = \max_{1\le i,j\le 2}\{\tau_{ij}(t) \ge 0\} < \infty$, $f_i = f_i^\tau = f$ are defined by $f(x) = \sin^2 x + \frac{4\pi x}{\pi^2+4}$, $x \in [0, \infty)$ and the connection weight matrices W and W^τ are determined by

$$W = \begin{pmatrix}\dfrac{1}{4}\dfrac{L(f,0)}{L(f)-L(f,0)} & \dfrac{1}{4}\\ \dfrac{1}{2} & \dfrac{3}{2}\dfrac{L(f,0)}{L(f)-L(f,0)}\end{pmatrix} \qquad W^\tau = \begin{pmatrix}\dfrac{7}{4}\dfrac{L(f,0)}{L(f)-L(f,0)} & \dfrac{3}{4}\\ \dfrac{1}{2} & \dfrac{1}{2}\dfrac{L(f,0)}{L(f)-L(f,0)}\end{pmatrix},$$

where

$$L(f) = \sup_{x,y\in[0,+\infty), x\neq y}\frac{|f(x)-f(y)|}{|x-y|} = 1 + \frac{4\pi}{\pi^2+4} \tag{3.11}$$

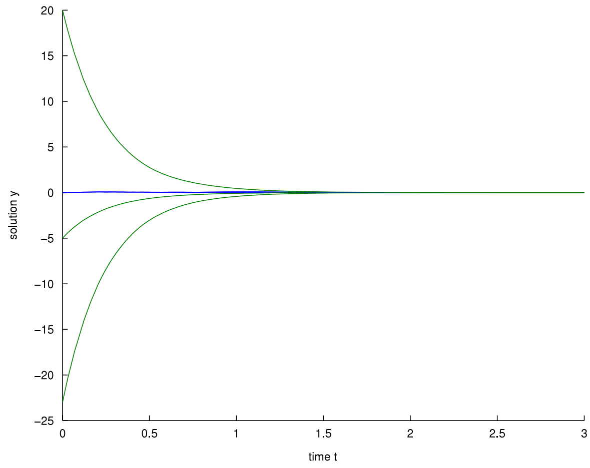

Fig. 1. The simulation for the solutions to the neural network model defined in Example 3.

and

$$L(f,0)=\sup_{x\in[0,+\infty),x\neq 0}\frac{|f(x)-f(0)|}{|x-0|}=\sup_{x\in(0,+\infty)}\frac{\sin^2 x}{x}+\frac{4\pi}{\pi^2+4}. \tag{3.12}$$

Clearly, $u^*=(0,0)^T$ is an equilibrium point of the model (3.10). (3.11) and (3.12) imply that each activation function of the model (3.10) is Lipschitz continuous and $L(f)>L(f,0)$. It is a routine matter to check that the assumptions (H_1)–(H_3) hold with $\lambda_1=\lambda_2=\frac{2L(f)\cdot L(f,0)}{L(f)-L(f,0)}$ and $\rho(K)=0.0109<1$. Then, we have

$$\max_{1\leq i\leq 2}\frac{1}{\lambda_i}\left\{L(f_i,0)\sum_{j=1}^{2}\frac{d_j}{d_i}|w_{ji}|+L(f_i^\tau,0)\sum_{j=1}^{2}\frac{d_j}{d_i}|w_{ji}^\tau|\right\}=\frac{L(f)+L(f,0)}{2L(f)}<1,$$

where $d_1=d_2=\frac{3}{2}L(f,0)$. That is, the condition (3.2) holds. But, for arbitrary $d_1',d_2'>0$, there always holds that

$$\min_{1\leq i\leq 2}\frac{1}{\lambda_i}\left\{L(f_i)\sum_{j=1}^{2}\frac{d_j'}{d_i'}|w_{ji}|+L(f_i^\tau)\sum_{j=1}^{2}\frac{d_j'}{d_i'}|w_{ji}^\tau|\right\}=\min\left\{1+\frac{L(f)-L(f,0)}{2L(f,0)}\frac{d_2'}{d_1'},1+\frac{L(f)-L(f,0)}{2L(f,0)}\frac{d_1'}{d_2'}\right\}$$
$$>1. \tag{3.13}$$

Therefore, by Proposition 5, we can conclude that the model (3.10) is globally exponentially stable. But Theorem 3 in [8] and Theorem 2 in [40] are not effective for this example due to the inequality (3.13).

Remark 7. Non-Lipschitz continuous functions which is not differentiable in dome point have been extensively studied in [56,57]. Motivated by this, we give an example whose activation functions are not differentiable in dome point.

Example 5. Consider the following neural network model:

$$\begin{cases}u_1'(t)=-u_1(t)+0.5f_2^\tau(u_2(t-\tau_{12}(t)))+0.5,\\ u_2'(t)=-u_2(t)+0.5f_1^\tau(u_1(t-\tau_{21}(t)))+0.5,\end{cases} \tag{3.14}$$

where $f_1^\tau(r)=f_2^\tau(r)=f(r)=\sqrt{r}$, $r\geq 0$ and $\tau_{ij}(t)=|\sin t|,\ i,j=1,2,\ t\in\mathbb{R}$.

It is easily verified that the function $f(r)=\sqrt{r}$ is not globally Lipschitz continuous but partially Lipschitz continuous on $\mathbb{R}^+=\{x:x\geq 0\}$. Obviously, $f(r)$ is not differentiable at $r=0$. By computation, we know that the model (3.14) has four equilibrium points: $(1,1)^T$, $(0.6545,0.0955)^T$, $(0.25,0.25)^T$ and $(0.0955,0.6545)^T$.

In what follows, we only discuss exponential stability of the equilibrium $(1,1)^T$ on $\Omega=\{(u_1,u_2)^T:0.7<u_1,u_2<1.3\}$. For this, let $x_i(t)=u_i(t)-1,\ i=1,2$ and then the model (3.14) is transformed into

$$\begin{cases}x_1'(t)=-(x_1(t)+1)+0.5g_2^\tau(x_2(t-\tau_{12}(t)))+0.5,\\ x_2'(t)=-(x_2(t)+1)+0.5g_1^\tau(x_1(t-\tau_{21}(t)))+0.5,\end{cases} \tag{3.15}$$

where $g_1^\tau(x) = g_2^\tau(x) = f(x+1)$. Then $(0, 0)^T$ is the equilibrium point of the model (3.15). Accordingly, we discuss stability of the equilibrium point $(0, 0)^T$ on $\Omega' = \{(x_1, x_2) : -0.3 < x_1, x_2 < 0.3\}$. The inequality

$$|g_i^\tau(x)| = |f(x+1)| = \sqrt{x+1} = \frac{|x+1|}{\sqrt{x+1}} \leq 1.1952(|x|+1), \quad x \in (-0.3, 0.3)$$

implies $L_{2i}^\tau = 1.1952,\ i = 1, 2$. Take $L_{1i} = 1.5$ and then $\rho(K) = 0.8964 < 1$. This implies that the assumptions (H_1)–(H_3) hold for the model (3.15) with $\lambda_i = 1,\ i = 1, 2$. Moreover,

$$L_{\Omega_i'}(g_i^\tau, 0) = \sup_{x\in\Omega_i', x\neq 0} \frac{|f(x+1) - f(0)|}{|x|} = \sup_{x\in\Omega_i', x\neq 0} \frac{|\sqrt{x+1} - 1|}{|x|} = 0.5445 \quad (3.16)$$

where $\Omega_i' = \{x : -0.3 < x < 0.3\}$. Then

$$\max_{1\leq i\leq 2}\left\{\frac{1}{\lambda_i} L_{\Omega'}(g_i^\tau, 0) \sum_{j=1}^{2} \frac{d_j}{d_i}|w_{ji}^\tau|\right\} = \max\{0.1361, 0.5445\} = 0.5445 < 1,$$

holds for $d_1 = 2, d_2 = 1$. By Proposition 5, $(0, 0)^T$ is the exponentially stable equilibrium point of the model (3.15) on Ω'. Consequently, $(1, 1)^T$ is the exponentially stable equilibrium point of the model (3.14) on Ω.

4. Conclusions

In this paper, we have discussed stability of equilibria of differential equations with time-dependent delay and non-Lipschitz nonlinearity. Particularly, we have derived a new sufficient condition for exponential stability of the delay differential equations. Our stability condition is independent of Lipschitz continuity, boundedness and monotonicity of nonlinear coefficient operators and differentiability of delay function. Moreover, our results derived are applied to a neural network model with time-varying delays and a novel sufficient condition is obtained for exponential stability of equilibrium of the neural network model. Our stability condition is less conservative because activation functions are not required to be bounded, Lipschitz continuous, differentiable and monotonic. Examples are presented to illustrate the effectiveness of our results.

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