

A new development for the Tikhonov Theorem in nonlinear singular perturbation systems[☆]

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ABSTRACT

This paper deals with the exponential stability of nonlinear perturbation systems under a new condition. A novel criterion of exponential stability of nonlinear systems is firstly given in a general form. In this criterion, a new kind of characteristic value is introduced, which makes the exponential stability measurable. Based on this criterion, a new development for the Tikhonov Theorem in nonlinear singular perturbation systems is presented.

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1. Introduction

Consider the nonlinear singular perturbation system [1]:

$$\dot{x} = F(x, z), \quad (1a)$$

$$\epsilon \dot{z} = G(x, z) + Q(x)u, \quad (1b)$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are slow and fast dynamic state variables respectively, $u \in \mathbb{R}^q$ the system input, F and G both vector functions, Q an $m \times q$ matrix, $\epsilon > 0$ a small parameter that is the ratio of the time scale of the fast dynamics to the time scale of the slow dynamics. Generally, (1a) and (1b) are called slow and fast dynamical subsystems respectively.

System (1) is a typical singular perturbation control system. Many physical systems are of this form, for instance, the modified IEEE type-2 voltage regulator [1,2], the armature-controlled DC motor [3], the flexible joint manipulator [4–8], and so on. The main property of these control systems is that the control input u has direct effect on the fast subsystem (1b) only, but the control objective is the slow subsystem (1a) usually. Therefore, if the Jacobian matrix $\frac{\partial F(x,z)}{\partial z}$ is singular, the linearized system of system (1) is uncontrollable. As a result, traditional linearization techniques are not suitable for these kinds of control systems. However, under certain conditions, system (1) can be approximated by two independent subsystems with different time scales, i.e., the boundary-layer system and the reduced system (see the Tikhonov Theorem, [3, Theorem 11.1]). So if we can divide the control input into two proper parts, they appear in the boundary-layer system and the reduced system individually. Therefore, it is able to make system (1) hold some desired capabilities by designing the boundary-layer system and the reduced system respectively. The composite control is based on this idea and at present, most control designs of singular perturbation system are composite control [1,7].

Since the singular perturbation systems are widely applied, their stability problem attracts lots of attention and the Tikhonov Theorem had ever been a main mean in these research [9–11]. The Tikhonov Theorem implies that whether the

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solution of system (1) can be approximated by the solutions of the reduced system and the boundary-layer system depends on the exponential stability of the boundary-layer system, and (or) the stability of the reduced system. So the major problem of the control design for system (1) is changed to the designing problems of the reduced system and the boundary-layer system for guaranteeing their stabilities. At present, there are many well-known techniques in analyzing the exponential stability of the reduced system and (or) the boundary-layer system, in which the Lyapunov direct method remains to be a main approach. However, the construction of Lyapunov functionals is well known as skillful and in addition, the techniques based on the Lyapunov direct method can neither be used to estimate the convergence rate nor be used to determine the attraction region of the state equilibrium points. In 2001, Qiao et al. [12] proposed a new approach, nonlinear measure, to the exponential stability analysis for nonlinear systems. In this criterion, one can qualify the exponential stability of the singular perturbation systems.

In this paper, we develop a new method to determine the exponential stability and boundedness of nonlinear systems, and present a proof of the Tikhonov Theorem in nonlinear singular perturbation systems by this new approach.

2. Stability of the nonlinear systems

In this section, we give a new criterion of exponential stability of nonlinear systems in a general form. In this criterion, we introduce a new kind of characteristic value, which makes the exponential stability measurable, such that the exponential decay of a solution to the equilibrium point can be described clearly. We also derive some new results relevant to the ultimate uniform boundedness of the solutions to the nonlinear systems.

We consider the following system:

$$\dot{v}(t) = H(t, v(t)), \quad v(t) \in \mathbb{R}^k, \quad t \geq t_0 \tag{2}$$

where H is absolutely continuous which guarantees the existence and uniqueness of the solution to this system.

Proposition 1. *Suppose that for a given point $e \in \mathbb{R}^k$, there exists a vector norm $\| \cdot \|$ in \mathbb{R}^k and an open ball $\Omega = \{v \in \mathbb{R}^k : \|v - e\| < r\}$, such that*

$$\alpha(H, \Omega, e) =: \sup_{t \in [t_0, \infty)} \lim_{s \rightarrow 0^+} \frac{\|v - e + sH(t, v)\| - \|v - e\|}{s\|v - e\|} < 0, \tag{3}$$

then e is an exponentially stable equilibrium point of system (2), and any solution $v(t)$ starting in Ω (i.e., $v(t_0) \in \Omega$) satisfies

$$\|v(t) - e\| \leq e^{\alpha(H, \Omega, e)(t-t_0)} \|v(t_0) - e\|, \quad \forall t \geq t_0. \tag{4}$$

Proof. We assume that $v(t)$ is a solution to system (2) with $v(t_0) \in \Omega$. Since $v(t)$ is continuous, there exists a constant $\delta > t_0$ such that $\forall t \in [t_0, \delta)$, $v(t) \in \Omega$. Let $\delta_m = \sup\{\delta > t_0 : \forall t \in [t_0, \delta), v(t) \in \Omega\}$. Notice that $v(t)$ is differentiable for all $t \geq t_0$, then the function $\|v(t) - e\|$ is absolutely continuous on $[t_0, \delta_m)$, so that it is differentiable a.e. At a differentiable point t , we have

$$\begin{aligned} \frac{d\|v(t) - e\|}{dt} &= \lim_{s \rightarrow 0^+} \frac{\|v(t+s) - e\| - \|v(t) - e\|}{s} \\ &= \lim_{s \rightarrow 0^+} \frac{\|v(t) - e + \int_t^{t+s} H(\lambda, v(\lambda))d\lambda\| - \|v(t) - e\|}{s} \\ &\leq \lim_{s \rightarrow 0^+} \left(\frac{\|v(t+s) - e + sH(t, v(t))\| - \|v(t) - e\|}{s} + \frac{1}{s} \left\| \int_t^{t+s} H(\lambda, v(\lambda))d\lambda - sH(v(t), t) \right\| \right) \\ &= \lim_{s \rightarrow 0^+} \frac{\|v(t+s) - e + sH(t, v(t))\| - \|v(t) - e\|}{s}. \end{aligned} \tag{5}$$

Since $v(t) \in \Omega$ for all $t \in [t_0, \delta_m)$,

$$\frac{d\|v(t) - e\|}{dt} \leq \alpha(H, \Omega, e)\|v(t) - e\|, \quad t \in [t_0, \delta_m), \text{ a.e.}$$

As a result, we have

$$\|v(t) - e\| \leq e^{\alpha(H, \Omega, e)(t-t_0)} \|v(t_0) - e\|, \quad t \in [t_0, \delta_m). \tag{6}$$

Next We prove that $\delta_m = +\infty$, which leads to inequality (4). Assume that $\delta_m < +\infty$. Since $\alpha(H, \Omega, e) < 0$ and inequality (6), we have

$$\begin{aligned} \|v(\delta_m) - e\| &= \lim_{t \rightarrow \delta_m} \|v(t) - e\| \\ &\leq e^{\alpha(H, \Omega, e)(\delta_m-t_0)} \|v(t_0) - e\| < re^{\alpha(H, \Omega, e)(\delta_m-t_0)} < r \end{aligned}$$

i.e., $v(\delta_m) \in \Omega$. Since $v(t)$ is continuous, there exists a constant $\beta > 0$ that $v(t) \in \Omega$ for all $t \in (\delta_m - \beta, \delta_m + \beta)$. Noticing that $v(t) \in \Omega$ for all $t \in [t_0, \delta_m)$, we have $v(t) \in \Omega$ for all $t \in [t_0, \delta_m + \beta)$, which contradicts with the definition of δ_m . Therefore, $\delta_m = +\infty$.

Inequality (6) shows that $v(t) \equiv e$ if $v(t_0) = e$. Substituting it into (2), we have $\dot{e} = H(e) = 0$. So e is an equilibrium point of (2). Inequality (4) shows that the equilibrium point e is exponentially stable. \square

Remark 1. (i) The limitation in (3) does exist. In fact, it is easy to prove that for all $x, y \in \mathbb{R}^k (x \neq 0)$, function $f(s) = \frac{\|x-sy\|-\|x\|}{s\|x\|}$ is monotonically increasing in $(0, +\infty)$, and $\forall s \in (0, +\infty), |f(s)| \leq \|x\|^{-1}\|y\|$. So $\lim_{s \rightarrow 0^+} f(s)$ exists.
 (ii) If system (2) is linear, and that the norm $\|\cdot\|$ used in (3) is an l^2 norm, then $\alpha(H, \Omega, e)$ defined in (3) is a kind of matrix measure of H [13]. So the characteristic value $\alpha(H, \Omega, e)$ is a kind of nonlinear extension of matrix measures. A similar term with l^1 norm has been introduced in [12] that is named as nonlinear measure. The meaning of introducing index $\alpha(H, \Omega, e)$ is obvious: in a nonlinear system, the exponential stability and its corresponding process of exponential decay can be described by $\alpha(H, \Omega, e)$ quantitatively as we do in the linear system.

Notice that $\alpha(H, \Omega, e)$ depends on not only the function H , the domain Ω and the point e , but also the selection of the norm $\|\cdot\|$. In fact, the freedom in choosing the norm makes $\alpha(H, \Omega, e)$ valuable in applications. When $\|\cdot\|$ is l^2 -norm, we have

$$\begin{aligned} \alpha(H, \Omega, e) &= \sup_{\substack{v \in \Omega/e \\ t \in [t_0, \infty)}} \lim_{s \rightarrow 0^+} \frac{\|v - e + sH(t, v)\| - \|v - e\|}{s\|v - e\|} \\ &= \sup_{\substack{v \in \Omega/e \\ t \in [t_0, \infty)}} \lim_{s \rightarrow 0^+} \frac{\|v - e + sH(t, v)\|^2 - \|v - e\|^2}{s\|v - e\|(\|v - e + sH(t, v)\| + \|v - e\|)} \\ &= \sup_{\substack{v \in \Omega/e \\ t \in [t_0, \infty)}} \lim_{s \rightarrow 0^+} \frac{\|v - e\|^2 + 2\langle v - e, sH(t, v) \rangle + s^2\|H(t, v)\|^2 - \|v - e\|^2}{s\|v - e\|(\|v - e + sH(t, v)\| + \|v - e\|)} \\ &= \sup_{\substack{v \in \Omega/e \\ t \in [t_0, \infty)}} \lim_{s \rightarrow 0^+} \frac{2\langle v - e, sH(t, v) \rangle + s^2\|H(t, v)\|^2}{s\|v - e\|(\|v - e + sH(t, v)\| + \|v - e\|)} \\ &= \sup_{\substack{v \in \Omega/e \\ t \in [t_0, \infty)}} \frac{\langle v - e, H(t, v) \rangle}{\|v - e\|^2}. \end{aligned} \tag{7}$$

In the following proposition, we give a result about the boundedness of the solution $v(t)$ to some nonlinear systems. Without losing generality, here we just consider the situation that $e = 0$ for convenience.

Proposition 2. Consider the following system:

$$\dot{v}(t) = H(t, v(t)) + B(t, v(t)), \quad v(t) \in \mathbb{R}^k, \quad t \geq t_0. \tag{8}$$

Suppose that H is absolutely continuous and $\Omega = \{v \in \mathbb{R}^k : \|v\| < r, r > 0\}$. $B(t, v(t))$ is bounded in $\mathbb{R}^+ \times \Omega$ such that

$$b = \sup_{(t, v) \in \mathbb{R}^+ \times \Omega} \|B(t, v(t))\|,$$

where $\|\cdot\|$ is a vector norm. If $\alpha(H, \Omega, 0) < 0$ and $-\alpha(H, \Omega, 0)^{-1}b < r$, then the solution $v(t)$ to (8) exponentially converges to

$$\Gamma = \{v \in \mathbb{R}^k : \|v\| \leq -\alpha(H, \Omega, 0)^{-1}b\}$$

if $v(t_0) \in \Omega$.

Proof. Suppose that $v(t)$ is a solution to system (8). It is easy to see that $\|v(t)\|$ is absolutely continuous in $[t_0, +\infty)$, so that it is differentiable in $[t_0, +\infty)$ a.e. Similar to the proof of inequality (5), we have

$$\begin{aligned} \frac{d\|v(t)\|}{dt} &\leq \lim_{s \rightarrow 0^+} \frac{\|v(t) + s[H(t, v(t)) + B(t, v)]\| - \|v(t)\|}{s} \\ &\leq \lim_{s \rightarrow 0^+} \frac{\|v(t) + sH(t, v(t))\| - \|v(t)\|}{s} + \|B(t, v)\| \\ &\leq \alpha(H, \Omega, 0)\|v(t)\| + b, \end{aligned}$$

which holds at every differentiable point t . By simple integration we can derive that for all $t \geq t_0$,

$$\|v(t)\| \leq e^{\alpha(H, \Omega, 0)(t-t_0)}\|v(t_0)\| - b\alpha(H, \Omega, 0)^{-1}(1 - e^{\alpha(H, \Omega, 0)(t-t_0)}).$$

Since $\|v(t_0)\| < r$, $\alpha(H, \Omega, 0)^{-1} < 0$ and $-\alpha(H, \Omega, 0)^{-1}b < r$,

$$\|v(t)\| < r$$

which implies that $v(t) \in \Omega$, $\forall t \geq t_0$. Therefore, as

$$\sup_{v \in \Omega} \lim_{t \rightarrow +\infty} \|v(t)\| \leq -\alpha(H, \Omega, 0)^{-1}b,$$

we derive that $v(t)$ exponentially converges to Γ . \square

3. A new development for the Tikhonov Theorem

In this section, we present a new development for the Tikhonov Theorem based on the criterion and the boundedness proposition we proposed in the last section. This new development mainly concerned with the boundedness of the solution to the standard singular perturbation system [3]

$$\dot{x} = f(t, x, z, \epsilon), \quad x(t_0) = \xi(\epsilon), \tag{9a}$$

$$\epsilon \dot{z} = g(t, x, z, \epsilon), \quad z(t_0) = \eta(\epsilon). \tag{9b}$$

Functions f and g are continuously differentiable for $(t, x, z, \epsilon) \in [0, t_1] \times D_x \times D_z \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^m$ are open connected sets. $\xi(\epsilon)$ and $\eta(\epsilon)$ is smooth for ϵ and $t > 0$. Denote the solution to system (9) by $x(t, \epsilon)$ and $z(t, \epsilon)$.

Let $\epsilon = 0$ in Eq. (9b) then

$$0 = g(t, x, z, 0). \tag{10}$$

Suppose that Eq. (10) has a unique isolated real root

$$z = h(t, x)$$

for any $(t, x) \in [0, t_1] \times D_x$, where $h(t, x)$ is called “quasi-steady-state”. We obtain the reduced system

$$\dot{x} = f(t, x, h(t, x), 0), \quad x(t_0) = \xi(0). \tag{11}$$

Denote the solution to (11) by $\bar{x}(t)$. Define

$$\bar{z}(t) = h(t, \bar{x}(t))$$

as the quasi-steady-state for z when $x = \bar{x}$.

Let

$$y = z - h(t, x).$$

Then the full problem for in the new variables (x, y) is

$$\dot{x} = f(t, x, y + h(t, x), \epsilon), \quad x(t_0) = \xi(\epsilon) \tag{12a}$$

$$\begin{aligned} \epsilon \dot{y} &= g(t, x, y + h(t, x), \epsilon) - \epsilon \frac{\partial h}{\partial t} - \epsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \epsilon), \\ y(t_0) &= \eta(\epsilon) - h(t_0, \xi(\epsilon)). \end{aligned} \tag{12b}$$

Let $\tau = t/\epsilon$, then $\epsilon \frac{dy}{dt} = \frac{dy}{d\tau}$ and $\frac{dx}{dt} = \frac{1}{\epsilon}$. Therefore, (12b) can be presented as

$$\begin{aligned} \frac{dy}{d\tau} &= g(t, x, y + h(t, x), \epsilon) - \epsilon \frac{\partial h}{\partial t} - \epsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \epsilon) \\ y(t_0) &= \eta(\epsilon) - h(t_0, \xi(\epsilon)). \end{aligned} \tag{13}$$

Since t and x vary slowly in the time scale τ , they can be given as

$$\begin{aligned} t &= t_0 + \epsilon \tau \\ x &= x(t_0 + \epsilon \tau, \epsilon). \end{aligned}$$

Let $\epsilon = 0$ and freeze t and x at $t = t_0$ and $x = \xi(0)$, so that (13) can be reduced as

$$\frac{dy}{d\tau} = g(t_0, \xi(0), y + h(t_0, \xi(0)), 0), \quad y(t_0) = \eta(0) - h(t_0, \xi(0)). \tag{14}$$

Then we obtain the boundary-layer system

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0) \tag{15}$$

where $(t, x) \in [0, t_1] \times D_x$ are treated as fixed parameters.

Theorem 1 (Tikhonov Theorem for Finite Time Interval [3]). Consider the singular perturbation problem of (9) and let $z = h(x, t)$ be an isolated root of (10). Assume that the following conditions are satisfied for all

$$[t, x, z - h(t, x), \epsilon] \in [0, t_1] \times D_x \times D_y \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$, in which D_x is convex and D_y contains the origin:

- (1) Functions f, g , their first partial derivatives with respect to (x, y, ϵ) , and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ ;
- (2) The reduced problem (11) has a unique solution $\bar{x}(t) \in S$, for $t \in [t_0, t_1]$, where S is a compact subset of D_x ;
- (3) The origin is an exponentially stable equilibrium point of the boundary-layer model (15), uniformly in (t, x) ; let $\mathcal{R}_y \subset D_y$ be the region of attraction of (14) and Ω_y be a compact subset of \mathcal{R}_y .

Then there exists a positive constant ϵ^* such that for all $\eta(0) - h(t_0, \xi(0)) \in \Omega_y$ and $0 < \epsilon < \epsilon^*$, the singular perturbation problem of (9) has a unique solution $x(t, \epsilon), z(t, \epsilon)$ on $[t_0, t_1]$, and

$$\begin{aligned} x(t, \epsilon) - \bar{x}(t) &= O(\epsilon) \\ z(t, \epsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\epsilon) &= O(\epsilon) \end{aligned}$$

hold uniformly for $t \in [t_0, t_1]$, where $\hat{y}(\tau)$ is the solution of the boundary-layer model (14). Moreover, given any $t_b > t_0$, there is $\epsilon^{**} \leq \epsilon^*$ such that

$$z(t, \epsilon) - h(t, \bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, t_1]$ whenever $\epsilon < \epsilon^{**}$.

Theorem 2 (Tikhonov Theorem for Infinite Time Interval [3]). Consider the singular perturbation problem of (9) and let $z = h(x, t)$ be an isolated root of (10). Assume that the following conditions are satisfied for all

$$[t, x, z - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$, in which D_x is convex and D_y contains the origin:

- (1) On any compact subset of $D_x \times D_y$, functions f, g , their first partial derivatives with respect to (x, y, ϵ) , and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ .
- (2) The origin is an exponentially stable equilibrium point of the reduced system (11). There is a Lyapunov function $V(t, x)$ that satisfies the condition Theorem 4.9 in [3] for (11) for $(t, x) \in [0, \infty) \times D_x$ and $\{W_1(x) \leq c\}$ is a compact set of D_x ;
- (3) The origin is an exponentially stable equilibrium point of the boundary-layer model (15), uniformly in (t, x) ; let $\mathcal{R}_y \subset D_y$ be the region of attraction of (14) and Ω_y be a compact subset of \mathcal{R}_y .

Then, for each compact set $\Omega_x \subset \{W_2(x) \leq \rho c, 0 < \rho < 1\}$ there is a positive constant ϵ^* such that for all $t_0 \geq 0, \xi(0) \in \Omega_x, \eta(0) - h(t_0, \xi(0)) \in \Omega_y$, and $0 < \epsilon < \epsilon^*$, the singular perturbation problem of (9) has a unique solution $x(t, \epsilon), z(t, \epsilon)$ on $[t_0, \infty)$, and

$$\begin{aligned} x(t, \epsilon) - \bar{x}(t) &= O(\epsilon) \\ z(t, \epsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\epsilon) &= O(\epsilon) \end{aligned}$$

hold uniformly for $t \in [t_0, \infty)$, where $\hat{y}(\tau)$ is the solution of the boundary-layer model (14). Moreover, given any $t_b > t_0$, there is $\epsilon^{**} \leq \epsilon^*$ such that

$$z(t, \epsilon) - h(t, \bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, \infty)$ whenever $\epsilon < \epsilon^{**}$.

We can see that if we employ the Tikhonov Theorem to solve a singular perturbation system, we can only derive the results on a finite time interval, when the reduced system does not satisfy the exponential stability, as well as the other strong conditions required by the Tikhonov Theorem for infinite time interval.

Therefore, we develop a new way to describe the behavior of the solution to the singular perturbation systems in infinite time interval, based on the criterion and boundedness proposition we proposed in the last section.

Proposition 3. Consider the singular perturbation problem of (9) and let $z = h(x, t)$ be an isolated root of (10). Assume that the following conditions are satisfied for all

$$[t, x, z - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$, in which D_x is convex and D_y contains the origin:

- (1) On any compact subset of $D_x \times D_y$, functions f, g , their first partial derivatives with respect to (x, y, ϵ) , and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ ;
- (2) The reduced problem (11) has a unique solution $\bar{x}(t) \in S$, for $t \in [t_0, \infty)$, where S is a compact subset of D_x , and

$$\alpha(F, \Omega_x, 0) < 0,$$

where $F(t, x) = f(t, x, h(t, x), 0)$ in $\Omega_x = \{x \in \mathbb{R}^n : \|x\| < b_x\} \subset S$;

- (3) The origin is an exponentially stable equilibrium point of the boundary-layer model (15), uniformly in (t, x) ; let $\mathcal{R}_y \subset D_y$ be the region of attraction of (14) and $\Omega_y = \{y \in \mathbb{R}^m : \|y\| \leq b_y, b_y > 0\}$ be a compact subset of \mathcal{R}_y .

Then there exists a positive constant ϵ^* , if the initial condition $x(0, \epsilon) = \xi(\epsilon) \in \Omega_x$ when $0 < \epsilon < \epsilon^*$, then for all $\eta(0) - h(t_0, \xi(0)) \in \Omega_y$ and $0 < \epsilon < \epsilon^*$, the singular perturbation problem of (9) has a unique solution $x(t, \epsilon), z(t, \epsilon)$ on $[t_0, \infty)$, and $x(t, \epsilon)$ converges to

$$\Gamma = \{x \in \mathbb{R}^n : \|x\| \leq -\alpha(F, \Omega_x, 0)^{-1}B_f\},$$

and

$$z(t, \epsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\epsilon) = O(\epsilon)$$

hold uniformly for $t \in [t_0, \infty)$, where $\bar{x}(t)$ and $\hat{y}(\tau)$ are the solutions of the reduced and boundary-layer models (11) and (14), and

$$B_f = \sup_{y \in \Omega_y, x \in \Omega_x, \epsilon \in (0, \epsilon^*), t \geq t_0} \|f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)\|.$$

Moreover, given any $t_b > t_0$, there is $\epsilon^{**} \leq \epsilon^*$ such that

$$z(t, \epsilon) - h(t, \bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, \infty)$ whenever $\epsilon < \epsilon^{**}$.

The proof of this proposition is similar to the proof of the Tikhonov Theorem for finite time interval shown in [3, Appendix, C17]. We just point out the main differences.

Proof. We can see that

$$\begin{aligned} \frac{dx(t, \epsilon)}{dt} &= f(t, x, y + h(t, x), \epsilon) \\ &= f(t, x, h(t, x), 0) + f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0) \\ &= F(t, x) + [f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)]. \end{aligned}$$

We first prove that there exists $\epsilon^* > 0$ such that

$$-\alpha(F, \Omega_x, 0)^{-1}B_f < b_x.$$

First we can estimate the last item of the above, we have

$$\begin{aligned} \|f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)\| &\leq \|f(t, x, y + h(t, x), \epsilon) - f(t, x, y + h(t, x), 0)\| \\ &\quad + \|f(t, x, h(t, x), 0) - f(t, x, y + h(t, x), 0)\| \\ &\leq L_1\epsilon + L_2\|y\| \end{aligned}$$

where L_1 and L_2 are positive constants. According to [3, Appendix, C17, C.80], for $\epsilon < \epsilon_y^*$,

$$\|y(t, \epsilon)\| \leq k_1 \exp[-k_2(t - t_0)/\epsilon] + \epsilon\delta$$

where k_1, k_2 and δ are positive constants. Therefore

$$\begin{aligned} \|f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)\| &\leq L_1\epsilon + L_2\|y\| \\ &\leq \theta_1\epsilon + \theta_2 \exp[-k_2(t - t_0)/\epsilon] \end{aligned}$$

where $\theta_1 = L_1 + L_2\delta$ and $\theta_2 = L_2k_1$. Since Ω_x is an open ball and $x(t, \epsilon)$ is differentiable with respect to t with $x(0, \epsilon) = \xi(\epsilon) \in \Omega_x$, there exists a constant $\delta^x > 0$ such that $\forall t \in [0, \delta^x), x(t, \epsilon) \in D_x$. Let $\delta_m^x = \sup\{\delta^x > 0 : \forall t \in [0, \delta^x), x(t, \epsilon) \in D_x\}$. Let

$$\epsilon_x^* = \begin{cases} \min \left\{ -\frac{\alpha(F, \Omega_x, 0)b_x}{2\theta_1}, -\frac{k_2 \left(\frac{\delta_m^x - t_0}{2} \right)}{\ln \left(\frac{-\alpha(F, \Omega_x, 0)b_x}{2\theta_2} \right)} \right\}, & \text{if } \frac{-\alpha(F, \Omega_x, 0)b_x}{2\theta_2} < 1 \\ -\frac{\alpha(F, \Omega_x, 0)b_x}{2\theta_1}, & \text{else} \end{cases}$$

and $\epsilon^* = \min\{\epsilon_x^*, \epsilon_y^*\}$, then for $0 < \epsilon < \epsilon^*$,

$$B_f = \sup_{y \in \Omega_y, x \in \Omega_x, \epsilon \in (0, \epsilon^*), t \geq t_0} \|f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)\| \leq -\alpha(F, \Omega_x, 0)b_x.$$

By Proposition 2, as $\alpha(F, \Omega_x, 0) < 0$ and $-\alpha(F, \Omega_x, 0)^{-1}B_f < b_x$, then the solution $x(t, \epsilon)$ to (12a) exponentially converges to

$$\Gamma = \{x \in \mathbb{R}^n : \|x\| \leq -\alpha(F, \Omega_x, 0)^{-1}B_f\}$$

if $x(t_0) \in \Omega_x$. □

Corollary 1. Consider the singular perturbation problem (9). Let $z = h(x, t)$ be an isolated root of (10). Assume that the following conditions are satisfied for all

$$[t, x, z - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$, in which D_x is convex and D_y contains the origin:

- (1) On any compact subset of $D_x \times D_y$, functions f, g , their first partial derivatives with respect to (x, y, ϵ) , and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ ;
- (2) The reduced problem (11) has a unique solution $\bar{x}(t) \in S$, for $t \in [t_0, \infty)$, where S is a compact subset of D_x , and

$$\alpha(F, \Omega_x, 0) < 0,$$

where $F(t, x) = f(t, x, h(t, x), 0)$ in $\Omega_x = \{x \in \mathbb{R}^n : \|x\| < b_x\} \subset S$;

- (3) Let $G(t, y) = g(t, x, y, 0)$. Suppose that

$$\alpha(G, \tilde{\Omega}_y, 0) < 0,$$

are satisfied uniformly in (x, t) . $\tilde{\Omega}_y = \{y \in \mathbb{R}^m : \|y\|_n < b_y\}$, and $\Omega_y \subset D_y \cap \tilde{\Omega}_y$ is a compact set.

Then there exists a positive constant ϵ^* , if the initial condition $x(0, \epsilon) = \xi(\epsilon) \in \Omega_x$ when $0 < \epsilon < \epsilon^*$, then for all $\eta(0) - h(t_0, \xi(0)) \in \Omega_y$ and $0 < \epsilon < \epsilon^*$, the singular perturbation problem of (9) has a unique solution $x(t, \epsilon), z(t, \epsilon)$ on $[t_0, \infty)$, and $x(t, \epsilon)$ converges to

$$\Gamma = \{x \in \mathbb{R}^n : \|x\| \leq -\alpha(F, \Omega_x, 0)^{-1}B_f\},$$

and

$$z(t, \epsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\epsilon) = O(\epsilon)$$

hold uniformly for $t \in [t_0, \infty)$, where $\bar{x}(t)$ and $\hat{y}(\tau)$ are the solutions of the reduced and boundary-layer models (11) and (14),

$$B_f = \sup_{y \in \Omega_y, x \in \Omega_x, \epsilon \in (0, \epsilon^*), t \geq t_0} \|f(t, x, y + h(t, x), \epsilon) - f(t, x, h(t, x), 0)\|.$$

Moreover, given any $t_b > t_0$, there is $\epsilon^{**} \leq \epsilon^*$ such that

$$z(t, \epsilon) - h(t, \bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, \infty)$ whenever $\epsilon < \epsilon^{**}$.

Proof. Since $\alpha(G, \tilde{\Omega}_y, 0) < 0$, by Proposition 1, the origin is an exponentially stable equilibrium point of the boundary-layer model (15). Since every solution $y(\tau)$ starting in Ω_y converges to the origin, it is easy to see that Ω_y is part of the attraction domain of the origin. By Proposition 3, it is easy to obtain the above results. □

Corollary 2. Consider the autonomous system

$$\dot{x} = f(x, z, \epsilon), \quad x(t_0) = \xi(\epsilon), \tag{16a}$$

$$\epsilon \dot{z} = g(x, z, \epsilon), \quad z(t_0) = \eta(\epsilon) \tag{16b}$$

where $z = h(x)$ is an isolated root of

$$g(x, z, 0) = 0.$$

Assume that the following conditions are satisfied for all

$$[t, x, z - h(x), \epsilon] \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$, in which D_x is convex and D_y contains the origin:

- (1) On any compact subset of $D_x \times D_y$, functions f, g , their first partial derivatives with respect to (x, y, ϵ) , and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ ;
- (2) The reduced problem (11) has a unique solution $\bar{x}(t) \in S$, for $t \in [t_0, \infty)$, where S is a compact subset of D_x , and

$$\alpha(F, \Omega_x, 0) < 0,$$

where $F(t, x) = f(t, x, h(t, x), 0)$ in $\Omega_x = \{x \in \mathbb{R}^n : \|x\| < b_x\} \subset S$;

- (3) Let $G(t, y) = g(x, y, 0)$. Suppose that

$$\alpha(G, \tilde{\Omega}_y, 0) < 0,$$

are satisfied uniformly in x . $\tilde{\Omega}_y = \{y \in \mathbb{R}^n : \|y\|_n < b_y\}$, and $\Omega_y \subset D_y \cap \tilde{\Omega}_y$ is a compact set.

Then there exists a positive constant ϵ^* , if the initial condition $x(0, \epsilon) = \xi(\epsilon) \in \Omega_x$ when $0 < \epsilon < \epsilon^*$, then for all $\eta(0) - h(\xi(0)) \in \Omega_y$ and $0 < \epsilon < \epsilon^*$, the singular perturbation problem of (9) has a unique solution $x(t, \epsilon), z(t, \epsilon)$ on $[t_0, \infty)$, and $x(t, \epsilon)$ converges to

$$x(t, \epsilon) - \bar{x}(t) = O(\epsilon),$$

and

$$z(t, \epsilon) - h(\bar{x}(t)) - \hat{y}(t/\epsilon) = O(\epsilon)$$

hold uniformly for $t \in [t_0, \infty)$, where $\bar{x}(t)$ and $\hat{y}(\tau)$ are the solutions of the reduced system

$$\frac{dx}{dt} = f(x, h(x), 0), \quad x(t_0) = \xi(0) \tag{17}$$

and the boundary-layer system

$$\frac{dy}{d\tau} = g(x, y + h(x), 0), \quad y(t_0) = \eta(0) - h(\xi_0). \tag{18}$$

Moreover, given any $t_b > t_0$, there is $\epsilon^{**} \leq \epsilon^*$ such that

$$z(t, \epsilon) - h(\bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, \infty)$ whenever $\epsilon < \epsilon^{**}$.

Proof. Since $\alpha(G, \tilde{\Omega}_y, 0) < 0$, and $\alpha(F, \Omega_x, 0) < 0$, by Proposition 1, the origin is the exponentially stable equilibrium point of the reduced system (17) and the boundary-layer system (18). Since every solution $y(\tau)$ starting in Ω_y converges to the origin, it is easy to see that Ω_y is part of the attraction domain of the origin. For autonomous system, the Lyapunov function condition V satisfying condition (3) in Theorem 2 exists (see Lemma 4.5 in [3]). Then by Theorem 2, the above results can be obtained. \square

The condition $\alpha(H, \Omega, 0) < 0$ is not difficult to verify in many actual problems. We give an example to show this.

Example 1. Consider the singular perturbation system for the electric circuit (Example 11.3 in [3]):

$$\dot{x} = 1 - x - \frac{1}{2}(\psi(x+z) + \psi(x-z)), \quad x(0) = \xi_0 \tag{19}$$

$$\epsilon \dot{z} = -(\epsilon + 2)z - \frac{\epsilon}{2}(\psi(x+z) - \psi(x-z)), \quad z(0) = \eta_0, \tag{20}$$

in which $\psi(v) = a(\exp(\frac{v}{b}) - 1)$, $a > 0, b > 0$. Since $z = h(x) = 0$, the reduced model

$$\dot{x} = 1 - x - a(e^{\frac{x}{b}} - 1) = F(x) \tag{21}$$

has the unique equilibrium point x^* and by the simple calculation we have $0 < x^* < 1$, so we can shift the equilibrium point of the reduced model to the origin point by change of variables $\tilde{x} = x - x^*$. The reduced system is

$$\frac{d\tilde{x}}{dt} = -\tilde{x} - a(e^{\frac{\tilde{x}+x^*}{b}} - e^{\frac{x^*}{b}}).$$

Let $\tilde{F}(t, \tilde{x}) = -\tilde{x} - a \left(e^{\frac{\tilde{x}+x^*}{b}} - e^{\frac{x^*}{b}} \right)$. Taking advantage of the result of (7) with l^2 -norm in $\Omega_{\tilde{x}} = \{\tilde{x} : \|\tilde{x}\| = \|x - x^*\| < b_x, b_x > 0\}$ and $\Omega_x = \{x : \|x - x^*\| < b_x, b_x > 0\}$, we have

$$\begin{aligned} \alpha(F, \Omega_x, x^*) &= \alpha(\tilde{F}, \Omega_{\tilde{x}}, 0) \\ &= \sup_{\substack{\tilde{x} \in \Omega_{\tilde{x}}/0 \\ t \in [0, \infty)}} \lim_{s \rightarrow 0^+} \frac{\|\tilde{x} - 0 + sF(t, \tilde{x})\| - \|\tilde{x} - 0\|}{s\|\tilde{x} - 0\|} \\ &= \sup_{\substack{\tilde{x} \in \Omega_{\tilde{x}}/0 \\ t \in [0, \infty)}} \frac{\tilde{x} \left(-\tilde{x} - a \left(e^{\frac{\tilde{x}+x^*}{b}} - e^{\frac{x^*}{b}} \right) \right)}{\tilde{x}^2} \\ &= -1 + \frac{a}{b_x} \left(e^{-\frac{b_x}{b}} - 1 \right). \end{aligned}$$

Hence we have $\alpha(F, \Omega_x, x^*) = -1 + \frac{a}{b_x} \left(e^{-\frac{b_x}{b}} - 1 \right) < 0$ for any $b_x > 0$.

The boundary-layer model is

$$\frac{dy}{d\tau} = -2y = G(y).$$

$x(t, \epsilon)$ to system (19). It is easy to see that $\alpha(G, \Omega_y, 0) < 0$, where Ω_y is any bounded set in \mathbb{R} . In fact, the origin is a global exponentially stable equilibrium point of this boundary-layer system. Next, we evaluate B_f .

$$\begin{aligned} \|\tilde{f}(t, \tilde{x}, y + h(t, \tilde{x}), \epsilon) - \tilde{f}(t, \tilde{x}, h(t, \tilde{x}), 0)\| &= \left\| a e^{\frac{\tilde{x}+x^*}{b}} \left(1 - \frac{1}{2} \left(e^{\frac{y}{b}} + e^{-\frac{y}{b}} \right) \right) \right\| \\ &\leq \frac{a}{2b} e^{\frac{b_x}{b}} \left(e^{\frac{b_y}{b}} + e^{-\frac{b_y}{b}} \right) \|y\| \\ &\leq L_2 \|y\|, \end{aligned}$$

where $L_2 = \frac{a}{2b} e^{\frac{b_x}{b}} \left(e^{\frac{b_y}{b}} + e^{-\frac{b_y}{b}} \right)$. The solution of the boundary-layer model

$$\epsilon \frac{dy}{dt} = -2y, \quad y(0) = \eta(\epsilon) - h(0, \xi(\epsilon)) = \eta_0$$

satisfies

$$\begin{aligned} \|y(t, \epsilon)\| &= \left\| \eta(\epsilon) e^{-\frac{2(t-t_0)}{\epsilon}} \right\| \\ &= \left\| \eta_0 e^{-\frac{2(t-t_0)}{\epsilon}} \right\| \\ &\leq \|\eta_0\|. \end{aligned}$$

Then we have $b_y \geq \|\eta_0\|$ and $B_f = L_2 \|\eta_0\|$.

For numerical simulation, we let $a = 1, b = 100$. Then the equilibrium point is about (0.9901, 0.0002451). Select $\epsilon = 0.001$. By randomly choosing 10 initial points in $[-10, 10] \times [-10, 10]$, then $b_x = b_y = 10, B_f = 0.111$ and $\alpha(F, \Omega_x, x^*) = -1.0095$. So the set $\Gamma = \{x(t, \epsilon) \in \mathbb{R} : \|x(t, \epsilon) - 0.9901\| = \|\tilde{x}(t)\| \leq 0.1109\}$. The solutions $x(t, \epsilon)$ for system (19) are portrayed in Fig. 1, which shows that all the solutions $x(t, \epsilon)$ converge to Γ .

The approximating errors

$$x(t, \epsilon) - \bar{x}(t)$$

and

$$z(t, \epsilon) - h(\bar{x}(t)) - \hat{y}(t/\epsilon)$$

are portrayed in Figs. 2 and 3, where all the errors are $O(10^{-3})$. They show that $\bar{x}(t)$ and $h(t, \bar{x}(t)) - \hat{y}(t/\epsilon)$ are good approximations of the solutions $x(t, \epsilon)$ and $z(t, \epsilon)$.

4. Conclusion

Through the definition of a new stability index $\alpha(H, \Omega, e)$ we develop a new method to determine the exponential stability and boundedness of nonlinear systems. We give a new criterion of exponential stability of nonlinear systems in a general form. In this criterion, we introduce a new kind of characteristic value, which makes the exponential stability measurable, such that the exponential decay of solution to the equilibrium point can be described clearly. We also derive

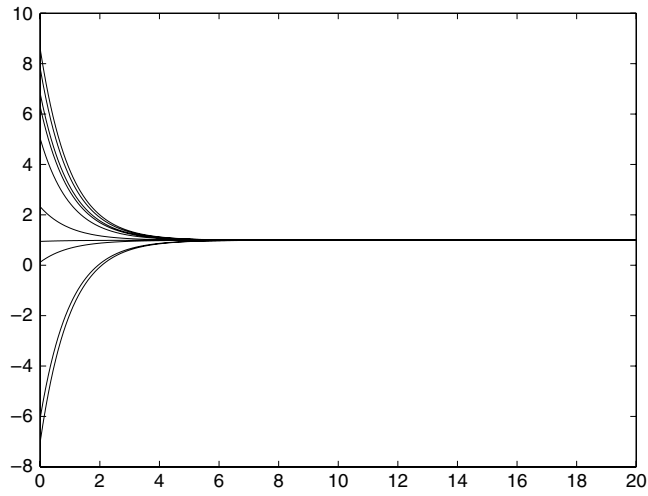


Fig. 1. The solutions $x(t, \epsilon)$ for the system (19).

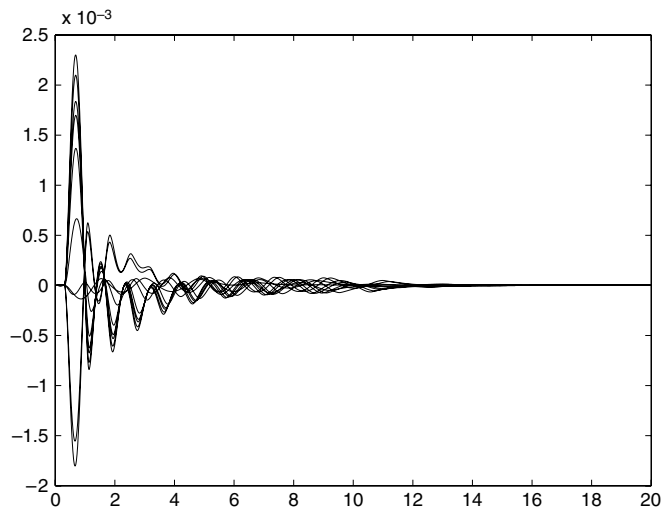


Fig. 2. The errors $x(t, \epsilon) - \bar{x}(t)$.

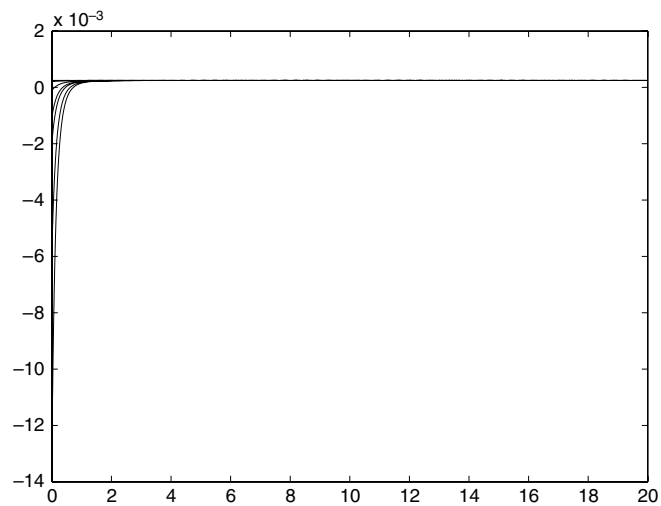


Fig. 3. The errors $z(t, \epsilon) - h(\bar{x}(t)) - \hat{y}(t/\epsilon)$.

some new results relevant to the ultimate uniform boundedness of the solutions to the nonlinear systems. With this new approach, we develop a new way to describe the behavior of the solution to the singular perturbation systems in infinite time interval, based on the criterion and boundedness proposition we proposed.

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