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H_∞ control of singular time-delay systems via discretized Lyapunov functional

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Abstract

This paper addresses the problem of robust H_∞ control for uncertain continuous time singular systems with state delays. A new singular-type complete quadratic Lyapunov–Krasovskii functional (LKF) is introduced, which combines with the discretization LKF method to synthesis problems. An improved bounded real lemma (BRL) is presented to ensure the system to be regular, impulse free and stable with H_∞ performance condition. Based on the BRL, a memoryless state feedback controller is designed via linear matrix inequalities (LMIs), which greatly reduces the disturbance attenuation level. Numerical examples are given to illustrate improvements over some existing results.

1. Introduction

Singular systems have been extensively studied in the past years due to their extensive applications in circuits, economics, large-scale systems and other areas [1,2]. The singular systems contain three kinds of modes, that is, finite dynamic modes, infinite dynamic modes and non-dynamic modes, while the latter two do not appear in the state-space systems [3]. Therefore, the study for singular systems is much more complicated than that for state-space systems. Very recently, a great number of results based on the theory of

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state-space systems have been extended to the area of singular system, see e.g. [3–19], and the reference therein.

During the last decade, the problem of H_∞ control for singular systems with delay has received considerable attention since time delays appear in many practical systems such as regenerative chatter in metal cutting, internal combustion engine and delayed resonator, and so on. Depending on whether the existence condition of H_∞ controller includes the information of delay or not, the existing conditions can be classed into two types: delay-independent ones [9–11] and delay-dependent ones [12–19]. For the recent years, researchers have mainly focused on delay-dependent ones. There are two performance indexes which are used to judge the conservatism of the existing conditions. One is the H_∞ performance index while the other is admissible upper bound of the delay. For a given time-delay, the smaller the H_∞ performance index, the better the conditions. For a prescribed H_∞ performance level, the larger the admissible upper bound of the time-delay, the less conservative the conditions.

In the time-domain approach, the direct Lyapunov method is a powerful tool for studying the problems of the stability and H_∞ control for singular systems with delay. There are two different ideas how one can use this method. They are the Lyapunov–Krasovskii approach and the Lyapunov–Razumikhin approach. Both approaches can be used to handle systems with delay. The obtained results using Lyapunov–Krasovskii approach are usually less conservative than those using the Lyapunov–Razumikhin approach [20]. The choice of an appropriate Lyapunov–Krasovskii functional (LKF) is crucial for deriving stability criteria and for obtaining solutions to robust H_∞ control problems by Lyapunov–Krasovskii approach [20,21]. Special forms of LKFs were used to obtain delay-dependent bounded real lemmas (BRLs), which are based on model transformation and bounding techniques [12,13]. A tighter bound for cross terms can reduce the conservatism. However, there are no obvious ways to obtain less conservative results even if one is willing to commit more computational effort to the problem and to find a more tighter bound for cross terms. The free-weighting matrix methods, which are based on Leibniz–Newton formula, have been proposed to improve the delay-dependent results to a certain extent in [14–19]. All the conditions are based on the choice of the constant parameters of the LKFs, which considerably increase the conservatism.

In this paper, we introduce a singular-type complete quadratic LKF, which combines with the discretization LKF method to obtain a new BRL for singular time-delay system. This BRL guarantees singular time-delay system to be regular, impulse free and stable while satisfying a prescribed H_∞ performance level. Applying the BRL obtained, an H_∞ state feedback controller is designed by separating the decision variables. Examples are provided to demonstrate that the results in this paper are less conservative than the existing corresponding ones in the literature.

Notation: Throughout this paper, the superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. $L_2^n[0,\infty]$ is the space of square integrable functions $f:[0,\infty]\to\mathbb{R}^n$ with the norm $\|f\|_{L_2}=[\int_0^\infty \|f(t)\|^2\,dt]^{1/2}$. $x_t:=x(t+\theta),\theta\in[-r,0],t\geq 0$, denotes the function family defined on $[-r,0]$ which is generated by n-dimensional real vector valued continuous function $x(t),t\in[-r,+\infty)$. $sym\{\cdot\}$ is defined as $sym\{X\}=X+X^T$. $col\{\cdots\}$ and $diag\{\cdots\}$ represent, respectively, a column vector and a block diagonal matrix.

2. Preliminaries

Consider the following linear singular system with state delay and parameter uncertainties described by

$$\Sigma_1 : \begin{cases} E\dot{x}(t) = (A_0 + \Delta A_0)x(t) + (A_1 + \Delta A_1)x(t-r) + (B + \Delta B)u(t) + D\omega(t), \\ z(t) = (C_0 + \Delta C_0)x(t) + (C_1 + \Delta C_1)x(t-r) + (H + \Delta H)u(t), \\ x(t) = \phi(t), \quad t \in [-r, 0], \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input vector; $\omega(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L_2[0,\infty]$ and $z(t) \in \mathbb{R}^l$ is the controlled output. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular; we shall assume that $rank\, E = q \leq n$. The matrices A_0, A_1, B, C_0, C_1, D and H are known real constant matrices with appropriate dimensions. $r > 0$ is a constant time delay; $\phi(t)$ is a compatible vector valued initial function. $\Delta A_0, \Delta A_1, \Delta B, \Delta C_0, \Delta C_1$ and ΔH are time-invariant matrices representing norm-bounded parameter uncertainties and are assumed to be of the form

$$\begin{bmatrix} \Delta A_0 & \Delta A_1 & \Delta B \\ \Delta C_0 & \Delta C_1 & \Delta H \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F[N_1 \ \ N_2 \ \ N_3], \tag{1}$$

where M_1, M_2, N_1, N_2 and N_3 are constant matrices and F is an unknown real matrix satisfying

$$F^T F \leq I. \tag{2}$$

The parameter uncertainties $\Delta A_0, \Delta A_1, \Delta B, \Delta C_0, \Delta C_1$ and ΔH are said to be admissible if both Eqs. (1) and (2) hold.

The nominal singular time-delay system of Σ_1 can be written as

$$\Sigma_0 : \begin{cases} E\dot{x}(t) = A_0 x(t) + A_1 x(t-r) + Bu(t) + D\omega(t), \\ z(t) = C_0 x(t) + C_1 x(t-r) + Hu(t), \\ x(t) = \phi(t), \quad t \in [-r, 0]. \end{cases}$$

Throughout the paper, we will adopt the following definition.

Definition 1 (*Dai [1]*). (1) The pair (E, A_0) is said to be regular if $\det(sE - A_0)$ is not identically zero.

(2) The pair (E, A_0) is said to be impulse free if $deg(\det(sE - A_0)) = rank\, E$.

Lemma 1 (*Xu et al. [3]*). *Suppose the pair* (E, A_0) *is regular and impulsive free, then the solution to unforced system* Σ_0 *exists and is impulse free and unique on* $[0, \infty]$.

In view of this, we introduce the following definition for singular time-delay system Σ_0.

Definition 2 (*Xu et al. [3]*). (1) The singular time-delay system Σ_0 is said to be regular and impulse free, if the pair (E, A_0) is regular and impulse free.

(2) The singular time-delay system Σ_0 is said to be stable if for any $\varepsilon > 0$, there exist a scalar $\delta(\varepsilon) > 0$, such that for any compatible initial function $\phi(t)$ satisfies $\sup_{-r \leq t \leq 0} \|\phi(t)\| \leq \delta(\varepsilon)$, the solutions $x(t)$ of system Σ_0 satisfies $\|x(t)\| \leq \varepsilon$ for $t \geq 0$. Furthermore, $x(t) \to 0$ as $t \to \infty$.

Definition 3. The uncertain singular time-delay system Σ_1 is said to be robustly stable if the unforced system of Σ_1 is regular, impulse free and stable for all admissible uncertainties.

Definition 4. For a given scalar $\gamma > 0$, the uncertain singular time-delay system Σ_1 with $u(t) = 0$ is said to be robustly stable with H_∞ performance γ, if it is robustly stable and under zero initial condition $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$ is satisfied for any non-zero $\omega(t) \in L_2^p[0,\infty]$ and all admissible uncertainties.

The following lemmas will be used in the proof of our main results.

Lemma 2 (*Peterson [22]*). *For appropriate dimensional matrices Γ, Ξ, symmetric matrix Ω, all the F satisfied $F^T F \leq I$,*

$$\Omega + \Gamma F \Xi + \Xi^T F^T \Gamma^T < 0$$

if and only if there exists a constant $\varepsilon > 0$ such that

$$\Omega + \varepsilon \Gamma \Gamma^T + \varepsilon^{-1} \Xi^T \Xi < 0.$$

Lemma 3 (*Saadni et al. [23]*). *Let Φ, a and b be given matrices of appropriate dimension, then the two statements are equivalent*:

(1) *the following LMI*:

$$\begin{bmatrix} \Phi & a + bG^T \\ * & -G - G^T \end{bmatrix} = \begin{bmatrix} \Phi & a \\ * & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} G [\, b^T \quad -I \,] \right\} < 0$$

is feasible in the variable G.

(2) *Φ, a and b satisfies $\Phi < 0$ and $\Phi + ab^T + ba^T < 0$.*

3. Main results

3.1. New delay-dependent bounded real lemma

At the first, a new delay-dependent bounded real lemma is presented which assures the singular time-delay system Σ_0 with $u(t) = 0$ to be regular, impulse free and stable with H_∞ performance γ. Next, the result on H_∞ performance analysis is extended to uncertain singular time-delay system Σ_1 with $u(t) = 0$. These results will play a key role in solving the H_∞ control problem.

Theorem 1. *For prescribed scalar $\gamma > 0$, singular time-delay system Σ_0 is regular, impulse free and stable with H_∞ performance γ, if there exist matrices $P, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0, 1, \ldots, N$ such that*

$$PE = E^T P^T \geq 0, \tag{3}$$

$$\begin{bmatrix} PE & \hat{Q} \\ * & \hat{R} + \hat{S} \end{bmatrix} \geq 0, \tag{4}$$

$$\Psi = \begin{bmatrix} \Delta & D^s & D^a & C^T \\ * & -R_d - S_d & 0 & 0 \\ * & * & -3S_d & 0 \\ * & * & * & -I_l \end{bmatrix} < 0 \tag{5}$$

are satisfied, where

$$\hat{Q} = [E^T Q_0 \ \ E^T Q_1 \ \ \cdots \ \ E^T Q_N],$$

$$\hat{S} = diag\left\{\frac{1}{h}S_0 \ \ \frac{1}{h}S_1 \ \ \cdots \ \ \frac{1}{h}S_N\right\},$$

$$\hat{R} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \Delta_{00} & \Delta_{01} & \Delta_{02} \\ * & \Delta_{11} & 0 \\ * & * & \Delta_{22} \end{bmatrix},$$

$$\Delta_{00} = PA_0 + A_0^T P^T + E^T Q_0 + Q_0^T E + S_0,$$

$$\Delta_{01} = PA_1 - E^T Q_N, \quad \Delta_{02} = PD,$$

$$\Delta_{11} = -S_N, \quad \Delta_{22} = -\gamma^2 I_p, \quad C = [C_0 \ \ C_1 \ \ 0],$$

$$S_d = diag\{S_{d1} \ \ S_{d2} \ \ \cdots \ \ S_{dN}\}, \quad S_{di} = S_{i-1} - S_i,$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}, \quad R_{dij} = h(R_{i-1,j-1} - R_{ij}),$$

$$D^s = [D_1^s \ \ D_2^s \ \ \cdots \ \ D_N^s], \quad D_i^s = \begin{bmatrix} D_{0i}^s \\ D_{1i}^s \\ D_{2i}^s \end{bmatrix},$$

$$D_{0i}^s = \frac{h}{2}A_0^T(Q_{i-1} + Q_i) + \frac{h}{2}(R_{0,i-1} + R_{0i}) - (E^T Q_{i-1} - E^T Q_i),$$

$$D_{1i}^s = \frac{h}{2}A_1^T(Q_{i-1} + Q_i) - \frac{h}{2}(R_{N,i-1} + R_{Ni}),$$

$$D_{2i}^s = \frac{h}{2}D^T(Q_{i-1} + Q_i),$$

$$D^a = [D_1^a \ D_2^a \ \cdots \ D_N^a], \quad D_i^a = \begin{bmatrix} D_{0i}^a \\ D_{1i}^a \\ D_{2i}^a \end{bmatrix},$$

$$D_{0i}^a = -\frac{h}{2}A_0^T(Q_{i-1}-Q_i)-\frac{h}{2}(R_{0,i-1}-R_{0i}),$$

$$D_{1i}^a = -\frac{h}{2}A_1^T(Q_{i-1}-Q_i)+\frac{h}{2}(R_{N,i-1}-R_{Ni}),$$

$$D_{2i}^a = -\frac{h}{2}D^T(Q_{i-1}-Q_i).$$

For the proof of Theorem 1, see the Appendix.

Remark 1. In the case of $r=0$, the singular time-delay system Σ_0 is reduced to the following singular system:

$$\Sigma : \begin{cases} E\dot{x}(t) = (A_0 + A_1)x(t) + Bu(t) + D\omega(t), \\ z(t) = (C_0 + C_1)x(t) + Hu(t), \\ x(t) = \phi(t), \quad t \in [-r, 0]. \end{cases}$$

Then Theorem 1 has the similar form of Theorem 5.1 in [4] by letting $Q_i = 0, R_{ij} = 0, S_0 = S_1 + \varepsilon I, S_k = 0, i,j = 0,1,\ldots,N, k = 2,3,\ldots,N.$

Theorem 2. *For prescribed scalar $\gamma>0$, the uncertain singular time-delay system Σ_1 with $u(t)=0$ is robustly stable with H_∞ performance γ for all admissible uncertainties, if there exist matrices $P, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0,1,\ldots,N$, and scalar $\lambda>0$ such that LMI* (3), (4) *and*

$$\begin{bmatrix} \Delta + \lambda\Delta_N & D^s & D^a & C^T & E_I \\ * & -R_d-S_d & 0 & 0 & E_s \\ * & * & -3S_d & 0 & E_a \\ * & * & * & -I_l & M_2 \\ * & * & * & * & -\lambda I_n \end{bmatrix} < 0 \tag{6}$$

are satisfied,

$$E_I = \begin{bmatrix} PM_1 \\ 0 \\ 0 \end{bmatrix}, \quad \Delta_N = \begin{bmatrix} N_1^TN_1 & N_1^TN_2 & 0 \\ * & N_2^TN_2 & 0 \\ * & * & 0 \end{bmatrix},$$

$$E_s = \begin{bmatrix} \frac{h}{2}(Q_0 + Q_1)^T M_1 \\ \frac{h}{2}(Q_1 + Q_2)^T M_1 \\ \vdots \\ \frac{h}{2}(Q_{N-1} + Q_N)^T M_1 \end{bmatrix}, \quad E_a = \begin{bmatrix} -\frac{h}{2}(Q_0-Q_1)^T M_1 \\ -\frac{h}{2}(Q_1-Q_2)^T M_1 \\ \vdots \\ -\frac{h}{2}(Q_{N-1}-Q_N)^T M_1 \end{bmatrix}.$$

Proof. Suppose that there exist matrices $P, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0, 1, \ldots, N$, and scalar $\lambda > 0$ such that LMI (3), (4) and (6) hold. Then, by Schur complement, it follows from Eq. (6) that

$$\Psi + \lambda \overline{N}^T \overline{N} + \frac{1}{\lambda} \overline{E}\, \overline{E}^T < 0,$$

where

$$\overline{E}^T = [E_I^T \;\; E_s^T \;\; E_a^T \;\; M_2^T], \quad \overline{N} = [N \;\; 0 \;\; 0 \;\; 0], \quad N = [N_1 \;\; N_2 \;\; 0].$$

By lemma 2,

$$\Psi + \overline{E} F \overline{N} + (\overline{E} F \overline{N})^T < 0,$$

which is in the form of Ψ by replacing A_0, A_1, C_0 and C_1 with $A_0 + M_1 F N_1, A_1 + M_1 F N_2, C_0 + M_2 F N_1$ and $C_1 + M_2 F N_2$ respectively. Thus, by Theorem 1, we have that the uncertain singular system Σ_1 with $u(t) = 0$ is robustly stable with H_∞ performance γ. □

3.2. Robust H_∞ controller design

In this subsection, we present the sufficient condition for the existence of state feedback controller in the form of

$$u(t) = Kx(t) \tag{7}$$

such that the resultant closed-loop system is robustly stable with H_∞ performance γ.

The resultant closed-loop system of Σ_0 can be written as

$$\Sigma_{0c}: \begin{cases} E\dot{x}(t) = (A_0 + BK)x(t) + A_1 x(t-r) + D\omega(t), \\ z(t) = (C_0 + HK)x(t) + C_1 x(t-r), \\ x(t) = \phi(t), \quad t \in [-r, 0]. \end{cases}$$

Based on the result of Theorem 1, H_∞ state feedback controller design methods are presented in the following theorems.

Theorem 3. *For prescribed scalar $\gamma > 0$, the singular time-delay system Σ_0 controlled by $u(t) = G^{-1} L x(t)$ is regular, impulse free and stable with H_∞ performance γ, if there exist matrices $P, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0, 1, \ldots, N$, and G, L such that LMI* (3), (4) *and*

$$\begin{bmatrix} \Delta & D^s & D^a & C^T & E_{lb} + \overline{L}^T \\ * & -R_d - S_d & 0 & 0 & E_{sb} \\ * & * & -3S_d & 0 & E_{ab} \\ * & * & * & -I_l & H \\ * & * & * & * & -G - G^T \end{bmatrix} < 0 \tag{8}$$

are satisfied, where $E_{lb}^T = [B^T P^T \ 0 \ 0]$, $\overline{L} = [L \ 0 \ 0]$,

$$E_{sb} = \begin{bmatrix} \frac{h}{2}(Q_0 + Q_1)^T B \\ \frac{h}{2}(Q_1 + Q_2)^T B \\ \vdots \\ \frac{h}{2}(Q_{N-1} + Q_N)^T B \end{bmatrix}, \quad E_{ab} = \begin{bmatrix} -\frac{h}{2}(Q_0 - Q_1)^T B \\ -\frac{h}{2}(Q_1 - Q_2)^T B \\ \vdots \\ -\frac{h}{2}(Q_{N-1} - Q_N)^T B \end{bmatrix}.$$

Proof. Suppose that there exist matrices $P, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0,1,\ldots,N$, and G, L such that Eqs. (3), (4) and (8) hold. Then, it is easy to see that $-G - G^T < 0$, which implies that G is nonsingular. By Schur complement, it follows from Eq. (8) that

$$\begin{bmatrix} \Psi & U \\ U^T & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} G [G^{-1} V \ \ -I] \right\} < 0,$$

where

$$U^T = [E_{lb}^T \ \ E_{sb}^T \ \ E_{ab}^T \ \ H^T], \quad V^T = [\overline{L} \ \ 0 \ \ 0 \ \ 0].$$

By Lemma 3, it can be shown that

$$\Psi + UG^{-1}V + V^T G^{-T} U^T < 0,$$

which is in the form of Ψ by replacing A_0 and C_0 with $A_0 + BG^{-1}L$ and $C_0 + HG^{-1}L$ respectively. Then, by Theorem 1, the nominal singular time-delay system Σ_0 controlled by $u(t) = G^{-1}Lx(t)$ is regular, impulse free and stable with H_∞ performance γ. □

Remark 2. Notice that the feasibility of LMI (5) is a necessary condition of Theorem 3. Here, an additional initialization parameter K_0 will be introduced such that LMI (5) is satisfied when A_0 and C_0 is replaced by $A_0 + BK_0$ and $C_0 + BK_0$ respectively. The objective of this study is to develop a new stabilization method that provides much smaller H_∞ performance level.

Corollary 1. *For prescribed scalar* $\gamma > 0$, *the singular system* Σ *controlled by* $u(t) = G^{-1}Lx(t)$ *is regular, impulse free and stable with* H_∞ *performance* γ, *if there exist matrices* P, G, L and $S = S^T > 0$ *such that Eq.* (3) *and*

$$\begin{bmatrix} P(A_0 + A_1) + (A_0 + A_1)^T P^T + S & PD & (C_0 + C_1)^T & PB + L^T \\ * & -\gamma^2 I_p & 0 & 0 \\ * & * & -I_l & H \\ * & * & * & -G - G^T \end{bmatrix} < 0.$$

Now, we are in a position to present the result on the problem of delay-dependent robust H_∞ control for the uncertain singular time-delay system Σ_1.

Theorem 4. *For prescribed scalar* $\gamma > 0$, *the uncertain singular time-delay system* Σ_1 *controlled by* $u(t) = G^{-1}Lx(t)$ *is regular, impulse free and robustly stable with* H_∞ *performance* γ, *if there exist matrices* $P, G, L, Q_i, S_i = S_i^T, R_{ij} = R_{ji}^T, i,j = 0,1,\ldots,N$, *and scalar* $\lambda > 0$ *such that*

LMI (3), (4) *and*

$$\begin{bmatrix} \varDelta+\lambda\varDelta_N & D^s & D^a & C^T & E_{lb}+\overline{L}^T & E_I \\ * & -R_d-S_d & 0 & 0 & E_{sb} & E_s \\ * & * & -3S_d & 0 & E_{ab} & E_a \\ * & * & * & -I_l & H & M_2 \\ * & * & * & * & -G-G^T+\lambda N_3^T N_3 & 0 \\ * & * & * & * & * & -\lambda I_n \end{bmatrix} < 0 \tag{9}$$

are satisfied.

The proof can be carried out by resorting to Theorem 3 and following a similar line as in the proof of Theorem 2, and is thus omitted.

4. Numerical examples

In this section, two examples are given to show the effectiveness of the proposed method and improvements over the previous ones.

Example 1. Consider a singular time-delay system in the form of Σ_0 with

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad A_0=\begin{bmatrix}-0.3012 & 0.1257\\ 0.2351 & -2.5652\end{bmatrix},\quad A_1=\begin{bmatrix}-0.5124 & 0.9648\\ 0.1023 & 0.8197\end{bmatrix},$$

$$D=\begin{bmatrix}0.2102\\ -0.8152\end{bmatrix},\quad C_0=[1.2321\ \ 0.3185],\quad C_1=[0.8765\ \ 0.8231].$$

For given $\gamma>0$, we can calculate the maximum allowed delay $r^\star$ satisfying the LMIs in Theorem 1. To show the low conservativeness of the result, we compare ours with the criterion of [16] in Table 1. It is clear that the characterization of bounded realness in this paper is an improvement over the previous one.

Example 2. We consider the problem of H_∞ control for the singular time-delay system Σ_0 with the following parameters:

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad A_0=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\quad A_1=\begin{bmatrix}-1 & 0\\ 1 & -1\end{bmatrix},\quad B=\begin{bmatrix}-0.5\\ 1\end{bmatrix},$$

$$D=\begin{bmatrix}1\\ 1\end{bmatrix},\quad C_0=[1\ \ 0.2],\quad C_1=[0\ \ 0],\quad H=0.1.$$

Table 1
Comparisons of maximum allowed delay $r^\star$ for Example 1.

γ	2	2.5	3	3.5
$r^\star$ by [16]	2.2761	2.6131	2.8739	3.0855
$r^\star$ by Theorem 1 ($N=1$)	3.0710	3.4810	3.8030	4.0640

Table 2
Comparisons of minimum H_∞ performance index $\gamma^\star$ for Example 2.

Methods	[12]	[16]	[17]	[18]	Theorem 3
$\gamma^\star$	21	15.0268	9.9514	9.6754	6.61

Using the results of [17], we prescribe the initialization parameters $K_0=[1.2128\ \ -1.5255]$, for a given delay $r=1.2$, the minimum $\gamma=6.61$ can be calculated by Theorem 3. The state feedback controller achieving the minimum H_∞ performance level can be obtained as

$$u(t)=[-4.4673\ \ 4.9357]x(t).$$

Table 2 provides the comparison results on the minimum H_∞ performance index for given delay via the methods in [12,16–18] and Theorem 3 in this paper, which shows Theorem 3 in this paper can lower the H_∞ performance index.

5. Conclusions

The problem of delay-dependent robust H_∞ control for singular time-delay system with admissible uncertainties has been investigated by discretized LKF method. An improved bounded real lemma is presented to ensure the system to be regular, impulse free and stable with H_∞ performance condition. A memoryless state feedback H_∞ controller law is designed to reduce the disturbance attenuation level. Two examples are given to illustrate the effectiveness of our method and the improvement over some existing ones.

Appendix A. The proof of Theorem 1

The proof of Theorem 1 is divided into three parts. First of all, we show the singular time-delay system is regular and impulse free under the condition of theorem.

Since $rank(E)=q\leq n$, there exist two nonsingular matrices $\tilde{M}$ and $\tilde{N}$ such that $\hat{E}=\tilde{M}E\tilde{N}=\left[\begin{smallmatrix} I_q & 0 \\ 0 & 0\end{smallmatrix}\right]$, define

$$\overline{A}_0=\tilde{M}A_0\tilde{N}=\begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04}\end{bmatrix},\quad \overline{A}_1=\tilde{M}A_1\tilde{N}=\begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14}\end{bmatrix},$$

$$\overline{C}_0=C_0\tilde{N}=[C_{01}\ \ C_{02}],\quad \overline{C}_1=C_1\tilde{N}=[C_{11}\ \ C_{12}],\quad \overline{D}=\tilde{M}D,$$

$$\overline{Q}_i=\tilde{M}^{-T}Q_i\tilde{N},\quad \overline{S}_i=\tilde{N}^TS_i\tilde{N},\quad \overline{R}_{ij}=\tilde{N}^TR_{ij}\tilde{N}\ \ (i,j=0,1,\ldots,N),$$

$$\overline{P}=\tilde{N}^TP\tilde{M}^{-1}=\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4\end{bmatrix},\quad \overline{Q}_0=\begin{bmatrix} Q_{01} & Q_{02} \\ Q_{03} & Q_{04}\end{bmatrix},\quad \overline{Q}_N=\begin{bmatrix} Q_{N1} & Q_{N2} \\ Q_{N3} & Q_{N4}\end{bmatrix},$$

$$\overline{S}_0=\begin{bmatrix} S_{01} & S_{02} \\ S_{02}^T & S_{03}\end{bmatrix},\quad \overline{S}_N=\begin{bmatrix} S_{N1} & S_{N2} \\ S_{N2}^T & S_{N3}\end{bmatrix}.$$

Then, from Eqs. (3) and (5), we have

$$\overline{P}\hat{E} = \hat{E}^T\overline{P}^T \geq 0, \tag{A.1}$$

$$\begin{bmatrix} \overline{P}\,\overline{A}_0 + \overline{A}_0^T\overline{P}^T + \hat{E}^T\overline{Q}_0 + \overline{Q}_0^T\hat{E} + \overline{S}_0 & \overline{P}\,\overline{A}_1 - \hat{E}^T\overline{Q}_N \\ * & -\overline{S}_N \end{bmatrix} < 0. \tag{A.2}$$

We can deduce from Eq. (A.1) that $P_1 = P_1^T \geq 0$ and $P_3 = 0$, therefore $\overline{P}$ reduces to $\overline{P} = [\begin{smallmatrix} P_1 & P_2 \\ 0 & P_4 \end{smallmatrix}]$. Substituting the expression of $\hat{E}, \overline{P}, \overline{Q_0}, \overline{Q}_N, \overline{S_0}, \overline{S_N}$ into Eq. (A.2), we eventually get

$$\begin{bmatrix} \star & \star & \star & \star \\ * & P_4A_{04} + A_{04}^TP_4^T + S_{03} & \star & P_4A_{14} \\ * & * & \star & \star \\ * & * & * & -S_{N3} \end{bmatrix} < 0, \tag{A.3}$$

here, the terms denoted $\star$ are irrelevant to the results of the following discussion, the real expression of these variables are omitted. Since $S_{03} > 0$, the inequality $P_4A_{04} + A_{04}^TP_4^T + S_{03} < 0$ implies that A_{04} is nonsingular. Thus system Σ_0 is regular and impulse free. Similar to [3,14], it follows that

$$\begin{bmatrix} P_4A_{04} + A_{04}^TP_4^T + S_{03} & P_4A_{14} \\ * & -S_{N3} \end{bmatrix} < 0,$$

we can conclude $\rho(A_{04}^{-1}A_{14}) < 1$, which is a necessary condition for the singular time-delay systems Σ_0 to be stable [24].

Next, we will show the singular time-delay system Σ_0 is stable. Let us introduce a singular-type complete quadratic LKF

$$\begin{aligned} V(t,x_t) &= x^T(t)PEx(t) + 2x^T(t)E^T\int_{-r}^{0} Q(\xi)x(t+\xi)\,d\xi \\ &\quad + \int_{-r}^{0}\int_{-r}^{0} x^T(t+\xi)R(\xi,\eta)x(t+\eta)\,d\xi\,d\eta + \int_{-r}^{0} x^T(t+\xi)S(\xi)x(t+\xi)\,d\xi, \end{aligned}$$

where $Q(\xi)$, $S(\xi) = S^T(\xi)$ and $R(\xi,\eta) = R^T(\eta,\xi)$ are continuous functions. The derivation of $V(t,x_t)$ along system Σ_0 gives

$$\begin{aligned} \dot{V}(t,x_t) &= x^T(t)[PA_0 + A_0^TP^T + E^TQ(0) + Q^T(0)E + S(0)]x(t) \\ &\quad + 2x^T(t)[PA_1 - E^TQ(-r)]x(t-r) - x^T(t-r)S(-r)x(t-r) \\ &\quad + 2x^T(t)\int_{-r}^{0}[A_0^TQ(\xi) - E^T\dot{Q}(\xi) + R(0,\xi)]x(t+\xi)\,d\xi \\ &\quad + 2x^T(t-r)\int_{-r}^{0}[A_1^TQ(\xi) - R(-r,\xi)]x(t+\xi)\,d\xi \\ &\quad - \int_{-r}^{0}\int_{-r}^{0} x^T(t+\xi)\left(\frac{\partial}{\partial\xi}R(\xi,\eta) + \frac{\partial}{\partial\eta}R(\xi,\eta)\right)x(t+\eta)\,d\xi\,d\eta \\ &\quad - \int_{-r}^{0} x^T(t+\xi)\dot{S}(\xi)x(t+\xi)\,d\xi. \end{aligned}$$

We apply the discretization of Gu [20]: divide the delay interval $[-r,0]$ into N segments $[\theta_i,\theta_{i-1}]$ of equal length $h=r/N$, where $\theta_i=-ih(i=1,2,\dots,N)$. This divides the square $[-r,0]\times[-r,0]$ into $N\times N$ small squares $[\theta_i,\theta_{i-1}]\times[\theta_j,\theta_{j-1}]$, each small square is further divided into two triangles. The continuous matrix function $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment and the continuous matrix $R(\xi,\eta)$ is chosen to be linear within each triangular region. They can be expressed in terms of their values at the dividing points using a linear interpolation formula, i.e.

$$Q(\theta_i+\alpha h)=(1-\alpha)Q_i+\alpha Q_{i-1},\quad S(\theta_i+\alpha h)=(1-\alpha)S_i+\alpha S_{i-1},$$

$$R(\theta_i+\alpha h,\theta_j+\beta h)=\begin{cases}(1-\alpha)R_{i,j}+\beta R_{i-1,j-1}+(\alpha-\beta)R_{i-1,j}, & \alpha\geq\beta,\\ (1-\beta)R_{i,j}+\alpha R_{i-1,j-1}+(\beta-\alpha)R_{i,j-1}, & \alpha<\beta\end{cases}$$

for $0\leq\alpha\leq 1,\ 0\leq\beta\leq 1,\ i,j=1,2,\dots,N$. Thus the LKF is completely determined by $P,Q_i,S_i,R_{i,j},i,j=1,2,\dots,N$. Similar to [20], the LKF condition $V(t,x_t)\geq\mu\|x_1(t)\|^2$ is satisfied for $\mu=\lambda_{min}(P_1)>0$, if $S_i>0,\ i=0,1,\dots,N$ and

$$\begin{bmatrix} PE & \hat{Q}\\ * & \hat{R}+\hat{S}\end{bmatrix}\geq 0.$$

We note that for $\theta_i<\xi<\theta_{i-1},\theta_j<\eta<\theta_{j-1}$,

$$\dot{Q}(\xi)=\frac{1}{h}(Q_{i-1}-Q_i),\quad \dot{S}(\xi)=\frac{1}{h}(S_{i-1}-S_i),$$

$$\frac{\partial}{\partial\xi}R(\xi,\eta)+\frac{\partial}{\partial\eta}R(\xi,\eta)=\frac{1}{h}(R_{i-1,j-1}-R_{i,j}).$$

Thus

$$\begin{aligned}\dot{V}(t,x_t)&=\zeta^T(t)\overline{\Delta}\zeta(t)+2\zeta^T(t)\sum_{i=1}^{N}\int_0^1[\overline{D}_i^s+(1-2\alpha)\overline{D}_i^a]x(t+\theta_i+\alpha h)\,d\alpha\\ &\quad-\sum_{i=1}^{N}\int_0^1 x^T(t+\theta_i+\alpha h)S_{di}x(t+\theta_i+\alpha h)\,d\alpha\\ &\quad-\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^1\left[\int_0^1 x^T(t+\theta_i+\alpha h)R_{dij}x(t+\theta_j+\beta h)\,d\alpha\right]d\beta,\end{aligned}$$

where $\zeta^T(t)=[x^T(t)\ x^T(t-r)]$, $\overline{\Delta}=\begin{bmatrix}\Delta_{00} & \Delta_{01}\\ * & \Delta_{11}\end{bmatrix},\overline{D}_i^s=\begin{bmatrix}D_{0i}^s\\ D_{1i}^s\end{bmatrix},\overline{D}_i^a=\begin{bmatrix}D_{0i}^a\\ D_{1i}^a\end{bmatrix}$ and denote $\overline{D}^s=\begin{bmatrix}\overline{D}_1^s & \overline{D}_2^s & \cdots & \overline{D}_N^s\end{bmatrix},\overline{D}^a=\begin{bmatrix}\overline{D}_1^a & \overline{D}_2^a & \cdots & \overline{D}_N^a\end{bmatrix}$. As (5) implies that

$$\begin{bmatrix}\overline{\Delta} & \overline{D}^s & \overline{D}^a\\ * & -R_d-S_d & 0\\ * & * & -3S_d\end{bmatrix}<0,$$

we conclude that $\dot{V}(t,x_t)<0$. By Lemma 1 in [24], the singular time-delay system Σ_0 is stable.

Finally, we analyse the H_∞ performance of the system. Let us consider the following index:

$$J_{z\omega} = \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))\,dt.$$

Noting that we have

$$\begin{aligned} J_{z\omega} &= \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t,x_t))\,dt - \int_0^\infty \dot{V}(t,x_t))\,dt \\ &= \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t,x_t))\,dt - \lim_{t\to\infty} V(t,x(t)) + V(x_0). \end{aligned}$$

Under zero initial condition and $\lim_{t\to\infty} V(t,x_t) \geq 0$, Eq. (5) implies that $J_{z\omega} < 0$, which gives the desired results

$$\|z(t)\|_2 < \gamma\|\omega(t)\|_2.$$

This completes the proof.

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