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Approximation on attraction domain of Cohen–Grossberg neural networks

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ABSTRACT

In this paper, approximations of attraction domains of the asymptotically stable equilibrium points of some typical Cohen–Grossberg neural networks are achieved. Most Cohen–Grossberg neural networks are highly nonlinear systems which makes it difficult to approximate their attraction domain. Under some weak assumptions, we are allowed to employ the optimal Lyapunov method to obtain a Lyapunov function for asymptotically stable equilibrium points of a given Cohen–Grossberg neural network. With the help of this Lyapunov function, we approximate the corresponding attraction domain by the iterative expansion approach. Numerical simulations also illustrate that the approximation obtained is really part of the attraction domain.

1. Introduction

The Cohen–Grossberg neural networks were first presented by Cohen and Grossberg [1], including a lot of models derived from neurobiology, population biology and evolutionary theory [2]. Cohen–Grossberg neural networks play a role either as associative memories, i.e., pattern recognition, or optimization solvers in application. In this paper, we consider the model:

$$\frac{du_i(t)}{dt} = -a_i(u_i(t))\left[b_i(u_i(t)) - \sum_{j=1}^{n}\omega_{ij}g_j(u_j(t)) + I_i\right] \tag{1}$$

$i = 1,2,\ldots,n$, where $u_i(t)$ are the neural voltages, a_i the amplification functions, b_i the self-signal functions, ω_{ij} the connection weights, g_i the input–output connection functions and I_i the external inputs.

The attraction domain is the set of initial points from which the solution of a dynamical system can retrieve to corresponding asymptotically stable equilibrium points. In Cohen–Grossberg neural networks, attraction domains characterize their error correction or optimization solving capacity. Thus, it is naturally to estimate and to approximate the range of attraction domain. However, Cohen–Grossberg neural networks are highly nonlinear systems which makes it difficult to approximate the attraction domains of their asymptotically stable equilibrium points. To our best knowledge, there is little study focusing on estimating the attraction domain of Cohen–Grossberg neural networks. Most investigations on Cohen–Grossberg neural networks concern their stabilities, for instant [3–5].

Since 1985, some important approaches on attraction domain estimation were presented [6–8]. Under the assumption of diagonalizability of Jacobian matrix at the equilibrium point, the optimal Lyapunov function method was proposed by Kaslik and Balint [8], which provides a way in approximating the attraction domain. In 2009, an iterative expansion approach for improving the approximation of attraction domain was presented [9]. Employing this iterative expansion approach, a better approximation of the attraction domain is achieved.

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In this paper, to approximate the attraction domain for a class of Cohen–Grossberg neural networks, a proper Lyapunov function of some Cohen–Grossberg neural network is constructed by the optimal Lyapunov method. With the help of this Lyapunov function, an approximation of the attraction domain of the asymptotically stable equilibrium point for this Cohen–Grossberg neural network is achieved. Numerical examples are given to illustrate that the approximation obtained by our method is really part of corresponding attraction domain.

2. Approximating attraction domain by iterative expansion method

We consider the following Cohen–Grossberg neural networks

$$du_i(t)/dt = -a_i(u_i(t))\left[b_i(u_i(t)) - \sum_{j=1}^{n} \omega_{ij} g_j(u_j(t)) + I_i\right], \quad i = 1,2,\ldots,n. \tag{2}$$

Here $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$. Assume that $\mathbf{u}_0 = (u_1^0, u_2^0, \ldots, u_n^0)^T$ is an asymptotically stable equilibrium point of (2) and $f(\mathbf{u})$ is the vector field function, i.e., the right hand size of (2). Then

$$[\partial f(\mathbf{u}_0)/\partial \mathbf{u}] \tag{3}$$

is the Jacobian matrix of system (2) at $\mathbf{u}_0$. We make three assumptions on the Jacobian matrix (3):

1. $f(\mathbf{u})$ is $\mathbb{R}$-analytic;
2. The real part of the eigenvalues of the Jacobian matrix $[\partial f(\mathbf{u}_0)/\partial \mathbf{u}]$ are negative;
3. The Jacobian matrix $[\partial f(\mathbf{u}_0)/\partial \mathbf{u}]$ is diagonalizable.

The first and third assumptions are satisfied by a large class of Cohen–Grossberg neural networks. With the above assumptions, a Lyapunov function V_p can be obtained by the optimal Lyapunov method given in [8].

Let $S : \mathbf{C}^n \to \mathbf{C}^n$ be an linear isomorphism that

$$S^{-1}[\partial f/\partial \mathbf{u}(\mathbf{u}_0)]S = diag(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

$g = S^{-1} \circ f \circ S$ and $W = V \circ S$, where V is the optimal Lyapunov function. The Taylor's series of W at $\mathbf{u}_0$ is

$$W(\mathbf{u}) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1 j_2 \cdots j_n} (u_1 - u_1^0)^{j_1} (u_2 - u_2^0)^{j_2} \cdots (u_n - u_n^0)^{j_n} \tag{4}$$

and the Taylor's series of each components g_i of g at $\mathbf{u}_0$ is

$$g_i(\mathbf{u}) = \lambda_i(u_i - u_i^0) + \sum_{m=2}^{\infty} \sum_{|j|=m} b^i_{j_1 j_2 \cdots j_n} (u_1 - u_1^0)^{j_1} (u_2 - u_2^0)^{j_2} \cdots (u_n - u_n^0)^{j_n}, \tag{5}$$

where $i = 1,2,\ldots,n$, $j = \{j_1, j_2, \ldots, j_n\}$. j_i are nonnegative integers. Denote by $|j| = \sum_{i=1}^n j_i$ the sum of j_i. Then the coefficients $B_{j_1 j_2 \cdots j_n}$ of the series (4) are determined by the coefficients $b^i_{j_1 j_2 \cdots j_n}$ of (5) with the following recurrence formula:

$$B_{j_1 \cdots j_n} = \begin{cases} -1/(2\lambda_{i_0}) \sum\limits_{i=1}^{n} s_{ii_0}^2, \ \text{if } |j| = j_{i_0} = 2, \\ -2/(\lambda_p + \lambda_q) \sum\limits_{i=1}^{n} s_{ip} s_{iq}, \ \text{if } |j| = 2, j_p = j_q = 1, \\ -1/\left(\sum\limits_{i=1}^{n} j_i \lambda_i\right) \sum\limits_{p=2}^{|j|-1} \sum\limits_{|k|=p, k_i \leqslant j_i} \sum\limits_{i=1}^{n} [(j_i - k_i + 1) \\ \quad \times b^j_{k_1 \cdots k_n} B_{j_1 - k_1 \cdots j_i - k_i + 1 \cdots j_n - k_n}], \ \text{if } |j| \geqslant 3, \end{cases} \tag{6}$$

where s_{ij} are the elements of S. Let $p \geqslant 2$ be an integer. Suppose

$$V_p(\mathbf{u}) = \sum_{m=2}^{p} \sum_{|j|=m} A_{j_1 \cdots j_n} (u_1 - u_1^0)^{j_1} (u_2 - u_2^0)^{j_2} \cdots (u_n - u_n^0)^{j_n} \tag{7}$$

to be the pth order Taylor's expansion of the optimal Lyapunov function V. The coefficients $A_{j_1 j_2 \cdots j_n}$ are determined by the recurrence formula (6) with $V = W \circ S^{-1}$. $V_p(\mathbf{u})$ is proved to be a Lyapunov function of system (2) (see [8, Proposition 13]).

Suppose that

$$V_p(\mathbf{u}) > 0 \text{ and } \dot{V}_p(\mathbf{u}) =: \langle \nabla V_P(\mathbf{u}), f(\mathbf{u}) \rangle < 0 \tag{8}$$

are satisfied in $L_v = G_{V_P}(\mathbf{u}_0) \setminus U_1$. $G_{V_p}(\mathbf{u}_0)$ is an open region containing $\mathbf{u}_0$. $U_1 \subset G_{V_p}(\mathbf{u}_0)$ is a neighborhood of $\mathbf{u}_0$. In Propositions 1, 2 and 3, we show that all the orbicular regions involved in the iterative expansion approach can be replaced by other regular neighborhoods, which theoretically guarantees the validity of the iterative expansion approach.

Proposition 1. *Suppose that the field function $f \in C^2(\mathbb{R}^n)$. K is a compact subset of $G_{V_p}(\mathbf{u}_0)$ satisfying $K \bigcap \partial G_{V_p}(\mathbf{u}_0) = \phi$ and $\overline{U}_1 \subset intK$. Define function ${V_p}^d$ by*

$$V_p^d(\mathbf{u}) = V_p(\mathbf{u} + df(\mathbf{u})).$$

For any given neighborhood U of $\mathbf{u}_0$, if U satisfies

$$U_1 \subset U$$

and

$$\overline{U} \subset intK$$

then there exists a constant $d^* > 0$ that for any $d \in (0, d^*)$, the function ${V_p}^d$ satisfies

$$V_p^d(\mathbf{u}) > 0, \quad \dot{V_p}^d(\mathbf{u}) = \langle \nabla V_p^d(\mathbf{u}), \quad f(\mathbf{u})\rangle < 0 \tag{9}$$

in $K \backslash U$. Here $\overline{U}_1$ and $\overline{U}$ are the closures of U_1 and U, respectively.

Proof. The derivative of V_p^d along the solution of system (2) for any $d > 0$ is given by [10]

$$\dot{V_p}^d(\mathbf{u}) = \langle \nabla V_p(\mathbf{u} + df(\mathbf{u})), f(\mathbf{u}) + d[\partial f(\mathbf{u})/\partial \mathbf{u}]f(\mathbf{u})\rangle, \tag{10}$$

where $[\partial f(\mathbf{u})/\partial \mathbf{u}]$ is the Jacobian matrix of $f(\mathbf{u})$.

For the neighborhood U_1 of $\mathbf{u}_0$, since $G_{V_p}(\mathbf{u}_0)$ is open and K is compact, the neighborhood U_2 satisfying

$$U_1 \subset U_2 \subset U$$

is a subset of the open region

$$U_G = \{\mathbf{u} \in G_{V_p}(\mathbf{u}_0) : dist(\mathbf{u}, \partial G_{V_p}(\mathbf{u}_0)) > dist(\partial G_{V_p}(\mathbf{u}_0), K)/2\}.$$

Therefore, the continuity of f and $\nabla V_p(\mathbf{u})$ implies that for some $\delta > 0$,

$$\dot{V}_p(\mathbf{u}) = \langle \nabla V_p(\mathbf{u}), f(\mathbf{u})\rangle < -2\delta, \ \forall \mathbf{u} \in \overline{U}_G \setminus U_2. \tag{11}$$

Moreover, it is easy to see that

$$K \setminus U \subset \overline{U}_G \setminus U_2.$$

Hence, a constant $d_1 > 0$ can be found such that $\forall d \in [0, d_1]$ and $\forall \mathbf{u} \in K \backslash U$,

$$\mathbf{u} + df(\mathbf{u}) \in \overline{U}_G \setminus U_2,$$

which implies that $V_p^d(\mathbf{u}) > 0$ and

$$\langle \nabla V_p(\mathbf{u} + df(\mathbf{u})), f(\mathbf{u} + df(\mathbf{u}))\rangle < -2\delta. \tag{12}$$

Let

$$K_1 = \{\mathbf{y} = \mathbf{u} + df(\mathbf{u}) : \mathbf{u} \in K \setminus U, d \in [0, d_1]\}.$$

Then the compactness of $K\backslash U$, associated with the continuity of f, guarantees that K_1 is a compact set and hence f is bounded in K_1, i.e., $f(\mathbf{u} + df(\mathbf{u}))$ is bounded in $K\backslash U$ for any $d \in (0, d_1)$. Now, we consider the second-order Taylor's expansion of $f(\mathbf{u} + d f(\mathbf{u}))$

$$f(\mathbf{u} + df(\mathbf{u})) = f(\mathbf{u}) + d[\partial f/\partial \mathbf{u}(\mathbf{u})]f(\mathbf{u}) + d^2 h(\mathbf{u}, d) \tag{13}$$

with respect to d at 0, where

$$h(\mathbf{u}, d) = \frac{1}{2}\langle Hessef(\mathbf{u} + \xi f(\mathbf{u})) \cdot f(\mathbf{u}), f(\mathbf{u})\rangle$$

for some $\xi \in (0, d)$, *Hesse f* denoting the Hessian functional of f. Obviously, $h(\mathbf{u}, d)$ is bounded in $[K\setminus U] \times [0, d_1]$, that is, there exists $M_1 > 0$ such that for any $\mathbf{u} \in K\backslash U$ and $0 < d < d_1$,

$$\|h(\mathbf{u}, d)\| < M_1.$$

Therefore, letting $d_2 = \min\{d_1, 1/M_1\}$, for any $\mathbf{u} \in K\backslash U$ and $0 < d < d_2$, we have

$$\|f(\mathbf{u} + df(\mathbf{u})) - f(\mathbf{u}) - d[\partial f/\partial \mathbf{u}]f(\mathbf{u}\| = \|d^2 h(\mathbf{u}, d)\| < d. \tag{14}$$

As a result, if

$$M_2 = \sup_{\mathbf{u} \in K\setminus U, d \in [0, d_2]} \|\nabla V_p(\mathbf{u} + df(\mathbf{u}))\|$$

we have

$$\langle \nabla V_p(\mathbf{u}+df(\mathbf{u})), f(\mathbf{u}) + d[\partial f/\partial \mathbf{u}]f(\mathbf{u}) - f(\mathbf{u}+df(\mathbf{u}))\rangle < M_2 d. \tag{15}$$

Let $M = \max\{M_1, M_2\}$. Denote

$$d^* = \min\{d_2, \delta/M\}.$$

When $0 < d < d^*$, we have

$$\dot{V}_p^d(\mathbf{u}) = \langle \nabla V_p(\mathbf{u}+df(\mathbf{u})), f(\mathbf{u}) + d[\partial f/\partial \mathbf{u}]f(\mathbf{u})\rangle < \langle \nabla V_p(\mathbf{u}+df(\mathbf{u})), f(\mathbf{u}+df(\mathbf{u}))\rangle + Md < -2\delta + Md < -\delta \tag{16}$$

for any $\mathbf{u} \in K\backslash U$. □

Proposition 2. *For any neighborhood U which satisfies $\overline{U} \subset intK$, there exists a constant $d^{**} \in (0, d^*)$ that when $0 < d < d^{**}$, the function $V_p{}^d(\mathbf{u})$ satisfies that*

$$V_p^d(\mathbf{u}) < V_p(\mathbf{u}) \tag{17}$$

for any $\mathbf{u} \in K\backslash U$.

Proof. For any $\mathbf{u} \in K\backslash U$, the Taylor expansion of the real function $\mathbf{y}(d) = V_p(\mathbf{u} + df(\mathbf{u}))$ respect to d at 0 is

$$V_p^d(\mathbf{u}) = V_p(\mathbf{u}+df(\mathbf{u})) = V_p(\mathbf{u}) + d\langle \nabla V_p(\mathbf{u}), f(\mathbf{u})\rangle + d^2 h_v(\mathbf{u}, d). \tag{18}$$

The function $h_v(\mathbf{u},d)$ is given by

$$h_v(\mathbf{u}, d) = f^T(\mathbf{u}) Hesse V_p(\mathbf{u} + \xi f(\mathbf{u})) f(\mathbf{u})/2$$

for some $\xi \in (0, d)$. Since $V_p \in C^2(\mathbb{R}^n)$ and f is continuous, $h_v(\mathbf{u},d)$ is bounded in the compact set $K\backslash U$ for any $d \in (0, d^*)$.

For a given $\delta_v > 0$, we have

$$\dot{V}_p(\mathbf{u}) = \langle \nabla V_p(\mathbf{u}), f(\mathbf{u})\rangle < -\delta_v \tag{19}$$

for any $\mathbf{u} \in K\backslash U$. Let

$$M_v = \sup\{\|h_v(\mathbf{u}, d)\| : \mathbf{u} \in K \setminus U, 0 < d < d^*\}.$$

For any $\mathbf{u} \in K\backslash U$ and $0 < d < d^*$,

$$V_p^d(\mathbf{u}) - V_p(\mathbf{u}) = d\langle \nabla V_p(\mathbf{u}), f(\mathbf{u})\rangle + d^2 h_v(\mathbf{u}, d) < -\delta_v d + M_v d^2. \tag{20}$$

Let $d^{**} = \min\{d^*, \delta_v/M_v\}$. We have

$$V_p^d(\mathbf{u}) < V_p(\mathbf{u}), \quad \forall \mathbf{u} \in K \setminus U \tag{21}$$

for any $d \in (0, d < d^{**})$. □

We further suppose that $\partial U_1 \subset N_{V_p}^{c_\rho}$ where

$$N_{V_p}^{c_\rho} := \{\mathbf{u} \in U_3 : V_p(\mathbf{u}) \leqslant c_\rho\}.$$

U_3 is a neighborhood of the $\mathbf{u}_0$ that $U_1 \subset U_3$,

$$U_3 \subset G_{V_p}(\mathbf{u}_0)$$

and for a certain

$$c_\rho < \inf_{\|\mathbf{u}\| \in \partial U_3} \{V_p(\mathbf{u})\}.$$

Obviously, $N_{V_p}^{c_\rho}$ is a compact subset of $G_{V_p}(\mathbf{u}_0)$. Let U_4 be an open region that $U_1 \subset U_4$ and

$$\partial U_4 \subset int N_{V_p}^{c_\rho}.$$

Then there exists a constant $d^{**} > 0$ corresponding to U_4 that

$$V_p(\mathbf{u}) > V_p^d(\mathbf{u}) > 0$$

and

$$\langle \nabla V_p^d(\mathbf{u}), f(\mathbf{u})\rangle < 0$$

in $N_{V_p}^{c_\rho} \setminus U_4$ when $0 < d < d^{**}$. For each $d \in (0, d^{**})$, let

$$G_{V_p^d}(\mathbf{u}_0) := \left\{\mathbf{u} \in \mathbb{R}^n : V_p^d(\mathbf{u}) > 0, \langle \nabla V_p^d(\mathbf{u}), f(\mathbf{u})\rangle < 0\right\} \bigcup U_4. \tag{22}$$

It follows that

$$\partial U_4 \subset \{\mathbf{u} \in \mathbb{R}^n : V_p^d(\mathbf{u}) > 0, \langle \nabla V_p^d(\mathbf{u}), f(\mathbf{u})\rangle < 0\}. \tag{23}$$

We have the following results:

Proposition 3. *For a given $d \in (0, d^{**})$ satisfying that*

$$U_3 \subset G_{V_p^d}(\mathbf{u}_0)$$

if there exists an open region U_5 that

$$U_3 \subset U_5 \subset G_{V_p^d}(\mathbf{u}_0)$$

such that

$$c_d := \inf_{\|\mathbf{u}\| \in \partial U_5} \{V_p^d(\mathbf{u})\} > c_\rho$$

then for any solution $\mathbf{u}(t)$ to system (2) starting in $D_{V_p^d}^{c_\rho} \setminus U_4$, there exists a constant $T \geqslant 0$ that $\mathbf{u}(t)$ goes into $N_{V_p}^{c_\rho}$ at $t > T$, where

$$N_{V_p}^{c_\rho} \subset D_{V_p^d}^{c_\rho} \bigcup U_4$$

and the compact set

$$D_{V_p^d}^{c_\rho} = \{\mathbf{u} \in U_5 : V_p^d(\mathbf{u}) \leqslant c_\rho\}.$$

Proof. For any $\mathbf{u} \in N_{V_p}^{c_\rho} \setminus U_4$, we have $V_p^d(\mathbf{u}) < V_p(\mathbf{u})$. It follows that

$$V_p^d(\mathbf{u}) < c_\rho$$

for all $\mathbf{u} \in N_{V_p}^{c_\rho} \setminus U_4$. Therefore, since $N_{V_p}^{c_\rho} \subset U_3 \subset U_5$, we have $N_{V_p}^{c_\rho} \setminus U_4 \subset D_{V_p^d}^{c_\rho}$. Moreover, since $\partial U_4 \subset intN_{V_p}^{c_\rho}$, it follows that $\partial U_4 \subset D_{V_p^d}^{c_\rho}$ and thus $N_{V_p}^{c_\rho} \subset D_{V_p^d}^{c_\rho} \bigcup U_4$. It can be shown that $D_{V_p^d}^{c_\rho} \bigcup U_4$ is a positive invariant set. In fact, for any solution $\mathbf{u}(t)$ of (2) starting in $D_{V_p^d}^{c_\rho}$, since $\dot{V}_p^d(\mathbf{u}(t)) < 0$ whenever $\mathbf{u}(t)$ in $D_{V_p^d}^{c_\rho}$, it is trivial to prove that

$$V_p^d(\mathbf{u}(t)) \leqslant V_p^d(\mathbf{u}(0)) < c_\rho, \quad \forall t \geqslant 0 \tag{24}$$

which, together with the fact that

$$c_\rho < c_d$$

and

$$D_{V_p^d}^{c_\rho} \subset U_5 \subset G_{V_p^d}(\mathbf{u}_0)$$

implies that $\mathbf{u}(t)$ stays in $D_{V_p^d}^{c_\rho} \bigcup U_4$ for all $t > 0$. In the same way, it can be proved that any solution $\mathbf{u}(t)$ of (2) starting in U_4 stays in $D_{V_p^d}^{c_\rho} \bigcup U_4$.

With the definition of $D_{V_p^d}^{c_\rho}$, it is clear to see that $D_{V_p^d}^{c_\rho}$ is compact. Therefore, since

$$\dot{V}_p^d(\mathbf{u}) < 0$$

for all $\mathbf{u} \in D_{V_p^d}^{c_\rho} \setminus U_4$, then

$$-\gamma := \max_{\mathbf{u} \in D_{V_p^d}^{c_\rho} \setminus U_4} \dot{V}_p^d(\mathbf{u})$$

is negative. Hence, for any solution $\mathbf{u}(t)$ of (2) starting in $D_{V_p^d}^{c_\rho} \setminus U_4$, there holds that

$$V_p^d(\mathbf{u}(t)) = V_p^d(\mathbf{u}(0)) + \int_0^t \dot{V}_p^d(\mathbf{u}(\tau))d\tau \leqslant V_p^d(\mathbf{u}(0)) - \gamma t \tag{25}$$

as long as $\mathbf{u}(t)$ remains in $D_{V_p^d}^{c_\rho} \setminus U_4$. Obviously, $\mathbf{u}(t)$ is not able to stay in $D_{V_p^d}^{c_\rho} \setminus U_4$ for all $t > 0$, otherwise,

$$V_p^d(\mathbf{u}(t)) < 0$$

when t is sufficiently large, which contradicts that

$$V_p^d(\mathbf{u}) > 0$$

for all $\mathbf{u} \in D^{c_\rho}_{V_p^d} \setminus U_4$. Therefore, since $D^{c_\rho}_{V_p^d} \bigcup U_4$ is positive invariant, it follows that

$$\mathbf{u}(T) \in \partial U_4 \subset N^{c_\rho}_{V_p}$$

for some $T > 0$. □

In the above propositions, we prove that all the balls involved in the iterative expansion approach can be replaced by some regular neighborhoods of the equilibrium point, which theoretically guarantees the validity of the iterative expansion approach as well as its application scope.

Example 1. Consider the following Cohen–Grossberg neural networks:

$$\begin{cases} dx/dt = -(2+\sin(x)/3)(x - 17\ln 4/15\tanh(y)), \\ dy/dt = -(2+\cos(x)/3)(y - 17\ln 4/15\tanh(x)). \end{cases} \tag{26}$$

It is clear that there are three equilibrium points of Cohen–Grossberg neural network system (26): the unstable equilibrium point $O = (0,0)^T$ and the locally exponentially stable equilibrium points $A = (\ln 4, \ln 4)^T$ and $B = (-\ln 4, -\ln 4)^T$.

We compute the Lyapunov function V_6 for system (26) by the optimal Lyapunov method. The boundaries of attraction domains of $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, derived by iterating two steps with iterative expansion approach are represented in Fig. 1, where the step length $d = 0.05$, $c_\rho = 3.45$. Denote these approximation regions of the attraction domains of (ln4,

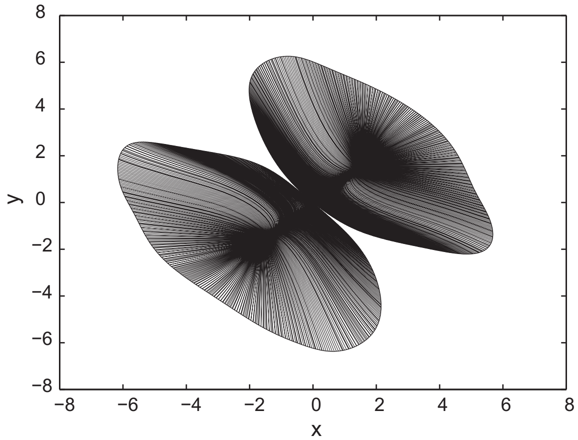

Fig. 1. $\Omega^2_{A,0.5}$ and $\Omega^2_{B,0.5}$ derived by iterating the Lyapunov function V_6 two steps with step length $d = 0.5$ and $c_\rho = 3.45$. V_6 is derived by the optimal Lyapunov method. They are the approximations of the attraction domains of $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$. Numerical solution traces starting at 643 and 539 initial points on the boundaries of $\Omega^2_{A,0.5}$ and $\Omega^2_{B,0.5}$ converge to $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, respectively.

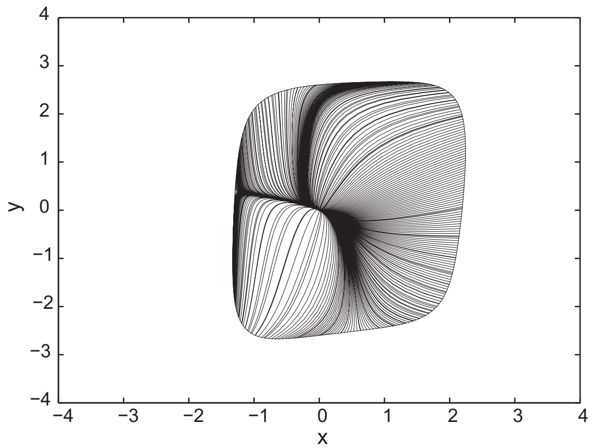

Fig. 2. $\Omega^2_{0.5}$ derived by iterating the Lyapunov function V_6 two steps with step length $d = 0.5$. V_6 is derived by the optimal Lyapunov method. It is the approximation of the attraction domain of $O = (0,0)^T$. Numerical solution traces starting at 353 initial points on the boundary of $\Omega^2_{0.5}$ converge to $O = (0,0)^T$.

$\ln 4)^T$ and $(-\ln 4, -\ln 4)^T$ by $\Omega^2_{A,0.05}$ and $\Omega^2_{B,0.05}$, respectively. To show the regions $\Omega^2_{A,0.05}$ and $\Omega^2_{B,0.05}$ are really parts of attraction domains of the equilibrium points $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, we choose 643 and 539 points on the boundaries of $\Omega^2_{A,0.05}$ and $\Omega^2_{B,0.05}$, respectively as initial points. As shown in Fig. 1, all the solution traces starting on the boundary of $\Omega^2_{A,0.05}$ converge to $(\ln 4, \ln 4)^T$, when the others starting on the boundary of $\Omega^2_{B,0.05}$ converge to $(-\ln 4, -\ln 4)^T$. Therefore, $\Omega^2_{A,0.05}$ and $\Omega^2_{B,0.05}$ are parts of the attraction domains of $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, respectively.

Example 2. Consider the following Cohen–Grossberg neural networks:

$$\begin{cases} dx/dt = (1 + sin(x))(-x + 1/(1 + \exp(-x)) - 1/(1 + \exp(-y))), \\ dy/dt = (1 + \cos(y))(-y - 1/(1 + \exp(-x)) + 1/(1 + \exp(-y))). \end{cases} \tag{27}$$

It is clear that $O = (0,0)^T$ is an asymptotically stable equilibrium point of (27).

We compute the Lyapunov function V_6 Cohen–Grossberg neural network system (27) by the optimal Lyapunov method. The boundary of attraction domain of $O = (0,0)^T$, derived by iterating two steps with iterative expansion approach is represented in Fig. 2, where the step length $d = 0.05$, $c_\rho = 5.5$. Denote the approximation region of the attraction domain of $O = (0,0)^T$ by $\Omega^2_{0.05}$. To show the region that we find is really parts of attraction domain of $O = (0,0)^T$, we choose 353 points on the boundary of $\Omega^2_{0.05}$ as the initial points. As shown in Fig. 2, all the solution traces starting at these points converge to $O = (0,0)^T$. It shows that $\Omega^2_{0.05}$ is part of the attraction domain of $O = (0,0)^T$.

3. Conclusion

In this paper, approximations of attraction domains of the asymptotically stable equilibrium points of some typical Cohen–Grossberg neural networks are achieved. By employing the optimal Lyapunov method, a proper Lyapunov function is obtained for a given Cohen–Grossberg neural network. Using this Lyapunov function, the approximation of the attraction domain of an asymptotically stable equilibrium point can be obtained by the iterative expansion approach. Numerical simulations show that our approximation approach is feasible and efficient.

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