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A necessary and sufficient condition for the strong convergence of Lipschitzian pseudocontractive mapping in Banach spaces ☆

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ABSTRACT

The purpose of this paper is to establish some necessary and sufficient conditions for the strong convergence of the Ishikawa iterative sequence and the Mann iterative sequence to a fixed point of pseudocontractive mapping in Banach spaces. Our results, to some extent, improve and extend the well-known result of [S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1) (1974) 147–150.] to Banach spaces.

1. Introduction and preliminaries

Let H be a Hilbert space and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be pseudocontractive mapping [1], if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$.

In 1974, Ishikawa [1] proved the following strong convergence theorem for Lipschitzian pseudocontractive mapping in Hilbert spaces.

Theorem 1.1 ([1]). *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be Lipschitzian pseudocontractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \ge 1, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

(i) $\alpha_n \le \beta_n, n \ge 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

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Remark 1.2. (i) Since $0 \le \alpha_n \le \beta_n \le 1$, $n \ge 1$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, the iterative sequence (1.2) could not be reduced to the Mann iterative sequence by setting $\beta_n = 0$. The Mann iterative sequence [2] is defined by the following

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \ge 1, \tag{1.3}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$. Therefore, the iterative sequence (1.2) has its own right.

(ii) The iterative sequence (1.2) is usually called the Ishikawa iterative sequence.

(iii) Chidume and Mutangadura [3] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point of Lipschitzian pseudocontractive mapping.

Since Theorem 1.1 was published, it has been studied by many authors. The improvement of Theorem 1.1 usually diverges into two directions: one is general Banach spaces, but with strong assumption on the mapping, such as strictly pseudocontractive mapping (see e.g., [4–6] etc.); the other is Hilbert spaces, but with more general type of mappings, such as hemicontractive mapping (see e.g., [7] and others).

We would like to quote some sentences to highlight the significance of our work. In [8], Berinde said that "Since its publication in 1974, as far as we know, Theorem 1.1 has never been extended to more general Banach spaces in its original formulation". As mentioned in a recent book written by Chidume [9] that "It is still an open question whether or not Theorem 1.1 can be extended to Banach spaces more general than Hilbert spaces".

Let us recall the concept of pseudocontractive mapping in Banach spaces.

Let E be a Banach space, E^* its dual space, $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in E.$$

It is well known that if E is smooth Banach space, then J is single-valued.

Definition 1.1 (*[8]*)**.** Let E be a Banach space and C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is called pseudocontractive mapping, if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \tag{1.4}$$

for all $x, y \in C$.

Remark 1.3. (i) If $E = H$, then $j = I$, where I denotes the identity operator. It is easy to prove that (1.4) is equivalent to (1.1).

(ii) If T is pseudocontractive mapping, then the following inequality holds (see also [8]):

$$\|x-y\| \le \|x-y+r[(I-T)x-(I-T)y]\|, \tag{1.5}$$

for all $x, y \in C$ and $r > 0$.

In this paper, we give a necessary and sufficient condition for the strong convergence of the Ishikawa iterative sequence and the Mann iterative sequence to a fixed point of pseudocontractive mapping in Banach spaces. Our results, to some extent, improve and extend the corresponding results of Ishikawa [1] to Banach spaces.

In the sequel, we shall need the following lemma.

Lemma 1.4 (*[10]*)**.** *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+c_n)a_n + b_n, \quad n \ge 1.$$

If $\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$, then (i) *$\lim_{n\to\infty} a_n$ exists.* (ii) *In particular, if $\liminf_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} a_n = 0$.*

2. Main results

We prove the following lemma first.

Lemma 2.1. *Let C be a nonempty convex subset of Banach space E. Let $T : C \to C$ be Lipschitzian pseudocontractive mapping with Lipschitz constant $L > 1$ and $F(T) \ne \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (1.2) satisfying the conditions $\sum_{n=1}^{\infty} \alpha_n \beta_n < +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. Then*

(i) *there exists a sequence $\{r_n\} \subseteq (0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < +\infty$ and*

$$\|x_{n+1} - p\| \le (1 + r_n)\|x_n - p\|,$$

for all $p \in F(T)$ and $n \ge 1$.

(ii) *there exists a constant $M > 1$, for all integer $m \ge 1$ such that*

$$\|x_{n+m} - p\| \le M\|x_n - p\|,$$

for all $p \in F(T)$.

Proof. (i) Let $p \in F(T)$. By (1.2), we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n \\ &= x_{n+1} + \alpha_n (I - T) x_{n+1} + \alpha_n^2 (x_n - T y_n) + \alpha_n (T x_{n+1} - T y_n). \end{aligned} \tag{2.1}$$

Observe that

$$p = p + \alpha_n (I - T) p. \tag{2.2}$$

Together with (2.1) and (2.2), we obtain

$$x_n - p = x_{n+1} - p + \alpha_n [(I - T) x_{n+1} - (I - T) p] + \alpha_n^2 (x_n - T y_n) + \alpha_n (T x_{n+1} - T y_n). \tag{2.3}$$

It follows from (1.5) and (2.3) that

$$\begin{aligned} \|x_n - p\| &\geq \|x_{n+1} - p + \alpha_n [(I - T) x_{n+1} - (I - T) p]\| - \alpha_n^2 \|x_n - T y_n\| - \alpha_n \|T x_{n+1} - T y_n\| \\ &\geq \|x_{n+1} - p\| - \alpha_n^2 \|x_n - T y_n\| - \alpha_n \|T x_{n+1} - T y_n\|. \end{aligned}$$

This implies that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_n^2 \|x_n - T y_n\| + \alpha_n \|T x_{n+1} - T y_n\|. \tag{2.4}$$

Next, we make the following estimations:

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)(x_n - p) + \beta_n (T x_n - p)\| \\ &\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|T x_n - p\| \\ &\leq [1 + (L - 1)\beta_n] \|x_n - p\| \leq L \|x_n - p\|. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|x_n - T y_n\| &\leq \|x_n - p\| + \|p - T y_n\| \\ &\leq \|x_n - p\| + L \|y_n - p\| \\ &\leq (1 + L^2) \|x_n - p\|. \end{aligned} \tag{2.6}$$

$$\begin{aligned} \|T x_{n+1} - T y_n\| &\leq L \|x_{n+1} - y_n\| \\ &= L \|x_n - y_n + \alpha_n (T y_n - x_n)\| \\ &\leq L \|x_n - y_n\| + L \alpha_n \|T y_n - x_n\| \\ &\leq L(L + 1)\beta_n \|x_n - p\| + L(1 + L^2)\alpha_n \|x_n - p\|. \end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.4), we get

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|x_n - p\| + (1 + L^2)\alpha_n^2 \|x_n - p\| + L(L + 1)\alpha_n \beta_n \|x_n - p\| + L(1 + L^2)\alpha_n^2 \|x_n - p\| \\ &= (1 + r_n) \|x_n - p\|, \end{aligned}$$

where $r_n = (1+L^2)\alpha_n^2 + L(L+1)\alpha_n\beta_n + L(1+L^2)\alpha_n^2$. Since $\sum_{n=1}^{\infty} \alpha_n \beta_n < +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$, then $\sum_{n=1}^{\infty} r_n < +\infty$. By Lemma 1.4, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof of part (i).

(ii) By virtue of the inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq (1 + r_{n+m-1}) \|x_{n+m-1} - p\| \\ &\leq e^{r_{n+m-1}} \|x_{n+m-1} - p\| \\ &\leq e^{r_{n+m-1}} e^{r_{n+m-2}} \|x_{n+m-2} - p\| \cdots \\ &\leq e^{\sum_{k=n}^{n+m-1} r_k} \|x_n - p\| \\ &\leq e^{\sum_{n=1}^{\infty} r_n} \|x_n - p\| = M \|x_n - p\|, \end{aligned}$$

where $M = e^{\sum_{n=1}^{\infty} r_n}$. This completes the proof. □

Theorem 2.2. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant $L > 1$ and $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (1.2) satisfying the conditions: $\sum_{n=1}^{\infty} \alpha_n \beta_n < +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. Then $\{x_n\}$ converges strongly to a fixed point T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{q \in F(T)} \|x - q\|$.*

Proof. The necessity of Theorem 2.2 is obvious. We just need to prove the sufficiency. From Lemma 2.1(i), we have

$$d(x_{n+1}, F(T)) \leq (1 + r_n) d(x_n, F(T)).$$

By Lemma 1.4 and notice the condition $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, then $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Indeed, for any $\varepsilon > 0$, there exists an integer $n_1 > 0$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4M}, \quad \text{for all } n \geq n_1.$$

In particular, there exists $p_1 \in F(T)$ and a constant $n_2 > n_1$ such that

$$\|x_{n_2} - p_1\| < \frac{\varepsilon}{2M}. \tag{2.8}$$

Using Lemma 2.1(ii) and (2.8), for all $n \geq n_2$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \\ &\leq 2M\|x_{n_2} - p_1\| < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since C is a nonempty closed convex subset of E, there exists a $q \in C$ such that $x_n \to q$ as $n \to \infty$. Finally, we prove that $q \in F(T)$. In fact, note that $d(q, F(T)) = 0$. Therefore, for any $\varepsilon_1 > 0$, there exists a $p_2 \in F(T)$ such that $\|p_2 - q\| < \varepsilon_1$. Then, we have

$$\begin{aligned}\|Tq - q\| &\leq \|Tq - p_2\| + \|p_2 - q\| \\ &\leq (1 + L)\|p_2 - q\| < (1 + L)\varepsilon_1.\end{aligned}$$

By the arbitrary of ε_1, we know that $Tq = q$, i.e., $q \in F(T)$. □

Corollary 2.3. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant $L > 1$ and $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (1.3) satisfying $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. Then $\{x_n\}$ converges strongly to a fixed point T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{q \in F(T)} \|x - q\|$.*

Proof. Let $\beta_n = 0$, for all $n \geq 1$. Then the conclusion of Corollary 2.3 follows from Theorem 2.2 immediately. □

Remark 2.4. It is obvious that Theorem 1.1 of Ishikawa [1] cannot be extended to the Mann iterative sequence directly (see Remark 1.2). However, our results hold true both for the Ishikawa iterative sequence and the Mann iterative sequence.

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