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Strong convergence of shrinking projection methods for a family of pseudocontractive mappings in Hilbert spaces

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ABSTRACT

The shrinking projection method has gained more and more attention as a powerful tool for the approximation of a fixed point of nonlinear mappings. In this paper, we introduce a new shrinking projection method for the approximation of fixed points of a family of pseudocontractive mappings in a Hilbert space. Using this method, we also deal with the problem of finding a common zero of a family of monotone operators and obtain a strong convergence theorem.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let T be a self-mapping of C. We use $F(T)$ to denote the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$).

Definition 1.1 (*[1]*). A mapping $T : C \to C$ is said to be strict pseudo-contraction if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. If $k = 1$, then T is said to be pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \tag{1.2}$$

equivalent,

$$\langle (I - T)x - (I - T)y, x - y\rangle \ge 0, \tag{1.3}$$

for all $x, y \in C$.

A mapping T is said to be nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

It is obvious that all nonexpansive mappings and strictly pseudocontractive mappings are pseudocontractive mappings.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem [2–4], the split feasibility problem [5–7] and image denoising and deblurring [8–10]. However, the Picard sequence $\{T^n x\}_{n=0}^{\infty}$ often fails to converge even in the weak topology. Thus averaged iterations prevail. Mann's iteration is one of the type and is defined by:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{1.4}$$

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where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\} \subseteq [0, 1]$. Reich [11] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space (see e.g., [12]). From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [13]). Therefore, it is important to develop the theory of iterative methods for strict pseudo-contractions. Indeed, Browder and Petryshyn [1] proved that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad n \geq 0, \tag{1.5}$$

for any starting point $x_0 \in C$, α is a constant such that $k < \alpha < 1$, $\{x_n\}$ converges weakly to a fixed point of strict pseudo-contraction T. Marino and Xu [14] extended the result of Browder and Petryshyn [1] to Mann's iteration (1.4), they proved that $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$.

In 1974, Ishikawa [15] introduced a new iteration algorithm and proved the following theorem.

Theorem 1.1 (*[15]*). *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be a Lipschitzian pseudocontractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

(i) $\alpha_n \leq \beta_n,\ n \geq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.1. In order to approximate such a fixed point by virtue of the Mann iteration algorithm has been proved abortive. In fact, Chidume and Mutangadura [16] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point of Lipschitzian pseudocontractive mapping.

In an infinite dimensional Hilbert space, the Mann and Ishikawa iteration algorithm have only weak convergence, in general, even for nonexpansive mappings. In order to obtain the strong convergence theorem for the Mann iteration method (1.4) to nonexpansive mapping, in 2003, Nakajo and Takahashi [17] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 1.2 (*[17]*). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_0 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0 \end{cases} \tag{1.7}$$

where $\{\alpha_n\} \subseteq [0, 1]$ satisfies $\sup_{n\geq 0} \alpha_n < 1$ and $P_{C_n \bigcap Q_n} x_0$ is the metric projection of H onto $C_n \bigcap Q_n$. Then $\{x_n\}$ converges strongly to $Px_0 \in F(T)$.

Since then, the hybrid algorithm has been studied extensively by many authors (see, for example [18–26] and references therein). In 2007, Marino and Xu [14] extended the results of Nakajo and Takahashi [17] from nonexpansive mappings to strict pseudocontractive mappings. In 2008, Zhou [27] introduced a modification of Ishikawa iteration algorithm for Lipschitz pseudo-contractions which is based on the hybrid algorithm as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1-2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, \quad n \geq 0. \end{cases} \tag{1.8}$$

Under some mild assumptions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$, he proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a fixed point z of T, where $z = P_{F(T)}x_0$. Later, Zhang and Cheng [28] generalized the main results of Zhou [27] from a single Lipschitz pseudocontractive mapping T to a finite family of Lipschitz pseudocontractive mappings $\{T_i\}_{i=1}^N$.

We observe that the iterative algorithm (1.7)–(1.8) generate a sequence $\{x_n\}$ by projecting x_0 onto the intersection of the suitably constructed closed convex sets C_n and Q_n.

Motivated by [17], in 2008, Takahashi et al. [29] introduced the following iteration procedure which is usually called the shrinking projection method. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1 \end{cases} \tag{1.9}$$

where P_{C_n} is the metric projection of H onto C_n and $\{T_n\}$ is a family of nonexpansive mappings. They proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to $z = P_{F(T)}x_0$, where $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Aoyama et al. [30] proved a strong convergence theorem for a family of firmly nonexpansive mappings in Hilbert spaces by using the shrinking projection method; see also [31].

Recently, Yao et al. [32] have extended the shrinking projection method from nonexpansive mappings to pseudocontractive mappings. To be more precisely, they proved the following results.

Theorem 1.3 ([32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractive mapping such that $F(T) \ne \emptyset$. Assume the sequence $\{\alpha_n\} \in [a, b]$ for some $a, b \in \left(0, \frac{1}{1+L}\right)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n (I-T)y_n\|^2 \le 2\alpha_n \langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1. \end{cases} \tag{1.10}$$

Then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{F(T)}x_0$.

Tang et al. [33] generalized (1.10) to the Ishikawa iterative process and proved strong convergence of the scheme to a fixed point of Lipschitz pseudocontractive mapping in Hilbert space.

Motivated and inspired by the above works, in this paper, we first introduce the following shrinking projection method for a finite family of pseudocontractive mappings. Let $\{T\}_{i=1}^{N} : C \to C$ be a family of pseudocontractive mappings. Let $\{\alpha_n\}, \{\beta_n\}$ be a sequence in $[0, 1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ by the following:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_{[n]} z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T_{[n]} x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n (I-T_{[n]})y_n\|^2 \le 2\alpha_n \langle x_n - z, (I-T_{[n]})y_n\rangle \\ \qquad + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \cdot \|y_n - x_n + \alpha_n(I-T_{[n]})y_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1 \end{cases} \tag{1.11}$$

where $T_{[n]} = T_{n \,(\mathrm{mod}\, N)}$, with the mod N function taking values in the set $\{1, 2, \dots, N\}$. Furthermore, we prove strong convergence of the algorithm (1.11) to a common fixed point of a finite family of Lipschitz pseudocontractive mappings in Hilbert spaces. Our theorem extends the corresponding results of Yao et al. [32] and Tang et al. [33] to the approximation of a common fixed point of a finite family of Lipschitz pseudocontractive mappings. Moreover, we apply our main results to obtain strong convergence for a family of monotone operators in a Hilbert space.

2. Preliminaries

We use the following notions in the sequel:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

It is well known that a Hilbert space H satisfies the Opial condition [34], i.e., for each sequence $\{x_n\}$ in H which converges weakly to a point $x \in H$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in H$, $y \ne x$.

Recall that, given a closed convex subset of C of a real Hilbert space H, the nearest point projection P_C from H onto C assigns to each $x \in C$ its nearest point denoted $P_C x$ in C from x to C, that is, $P_C x$ is the unique point in X with the property

$$\|x - P_C x\| \le \|x - y\|, \quad \text{for all } y \in C.$$

The following two lemmas are well known.

Lemma 2.1. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \le 0, \quad \text{for all } y \in C.$$

Lemma 2.2. *Let H be a real Hilbert space, then for all $x, y \in H$*

$$\|x - y\|^2 = \|x\|^2 - \|y^2\| - 2\langle x - y, y\rangle.$$

The demiclosedness principle for pseudocontractive mappings play an important role in our proof in the subsequent section.

Lemma 2.3 ([28]). *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudocontractive mapping, then*

(i) *$F(T)$ is a closed convex subset of C;*
(ii) *$I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I - T)x_n \to 0$, then $(I - T)z = 0$.*

In our convergence argument, we need the following result.

Lemma 2.4 ([23]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \le \|u - q\|, \quad \forall n \ge 1.$$

Then $\{x_n\}$ converges strongly to q.

3. Main results

Theorem 3.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $\{T\}_{i=1}^N : C \to C$ be a family of pseudocontractive mappings with Lipschitz constants $L_i \ge 1$, $i = 1, 2, \ldots, N$ such that the common fixed point set $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying the following conditions:* (i) $b \le \alpha_n < \alpha_n(1+L)^2 < a$, *for some $a, b \in (0, 1)$;* (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{F(T)}x_0$, where $L = \max\{L_i : 1 \le i \le N\}$.*

Proof. It is obvious that C_n is a closed convex subset of C. Then $\{x_n\}$ is well defined. By Lemma 2.3(i), we can see that F is closed and convex. Hence, $P_{F(T)}$ is also well defined. Next, we show that $F \subseteq C_n$ for all n.

Let $p \in F$, from the definition of pseudocontractive mappings, and by Lemma 2.2 and (1.3), we have

$$\begin{aligned}
\|x_n - p - \alpha_n(I - T_{[n]})y_n\|^2 &= \|x_n - p\|^2 - 2\alpha_n\langle (I - T_{[n]})y_n, x_n - p - \alpha_n(I - T_{[n]})y_n\rangle - \|\alpha_n(I - T_{[n]})y_n\|^2 \\
&= \|x_n - p\|^2 - \|\alpha_n(I - T_{[n]})y_n\|^2 - 2\alpha_n\langle (I - T_{[n]})y_n - (I - T_{[n]})p, y_n - p\rangle \\
&\quad - 2\alpha_n\langle (I - T_{[n]})y_n, x_n - y_n - \alpha_n(I - T_{[n]})y_n\rangle \\
&\le \|x_n - p\|^2 - \|\alpha_n(I - T_{[n]})y_n\|^2 - 2\alpha_n\langle (I - T_{[n]})y_n, x_n - y_n - \alpha_n(I - T_{[n]})y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \alpha_n(I - T_{[n]})y_n\|^2 \\
&\quad - 2\alpha_n\langle (I - T_{[n]})y_n, x_n - y_n - \alpha_n(I - T_{[n]})y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|^2 \\
&\quad - 2\langle x_n - y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle \\
&\quad + 2\alpha_n\langle (I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle \\
&\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|^2 \\
&\quad + 2|\langle x_n - y_n - \alpha_n(I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle|. \qquad (3.1)
\end{aligned}$$

For the last item of (3.1), we obtain

$$\begin{aligned}
&|\langle x_n - y_n - \alpha_n(I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle| \\
&= \alpha_n|\langle x_n - T_{[n]}z_n - (I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle| \\
&= \alpha_n|\langle x_n - T_{[n]}x_n + T_{[n]}x_n - T_{[n]}z_n - (I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle| \\
&= \alpha_n|\langle (I - T_{[n]})x_n - (I - T_{[n]})y_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle + \langle T_{[n]}x_n - T_{[n]}z_n, y_n - x_n + \alpha_n(I - T_{[n]})y_n\rangle| \\
&\le \alpha_n(L + 1)\|x_n - y_n\|\,\|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| + \alpha_n L\|x_n - z_n\|\,\|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| \\
&= \alpha_n(L + 1)\|x_n - y_n\|\,\|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| + \alpha_n\beta_n L\|x_n - T_{[n]}x_n\|\,\|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| \\
&\le \frac{\alpha_n(L + 1)}{2}\left(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|^2\right) + \alpha_n\beta_n L\|x_n - T_{[n]}x_n\|\,\|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|. \qquad (3.2)
\end{aligned}$$

Substituting (3.2) into (3.1), we have that

$$\begin{aligned}\|x_n - p - \alpha_n(I - T_{[n]})y_n\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \alpha_n(I - T_{[n]})y\|^2 \\ &\quad + \alpha_n(L+1)(\|x_n - y_n\|^2 + \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|^2) \\ &\quad + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \, \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| \\ &\le \|x_n - p\|^2 + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \, \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|. \end{aligned} \tag{3.3}$$

Observe that

$$\|x_n - p - \alpha_n(I - T_{[n]})y_n\|^2 = \|x_n - p\|^2 - 2\alpha_n\langle x_n - p, (I - T_{[n]})y_n\rangle + \|\alpha_n(I - T_{[n]})y_n\|^2. \tag{3.4}$$

Therefore, by (3.3) and (3.4), we get

$$\|\alpha_n(I - T_{[n]})y_n\|^2 \le 2\alpha_n\langle x_n - p, (I - T_{[n]})y_n\rangle + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \, \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\|.$$

This implies that $p \in C_{n+1}$ if $p \in C_n$. From induction, we have that $F \subseteq C_n$ for all n.

Since $x_n = P_{C_n}x_0$ and $F \subseteq C_n$, we have that $\|x_n - x_0\| \le \|p - x_0\|$, for any $p \in F$. In particular,

$$\|x_n - x_0\| \le \|q - x_0\|, \quad \text{where } q = P_F x_0. \tag{3.5}$$

Hence $\{x_n\}$ is bounded. Since $\{T\}_{i=1}^N$ are Lipschitz continuous, then $\{y_n\}$, $\{T_i x_n\}$ and $\{T_i y_n\}(i = 1, 2, \ldots, N)$ are all bounded too.

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we have

$$\langle x_n - x_0, x_{n+1} - x_n\rangle \ge 0. \tag{3.6}$$

By Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0 - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \end{aligned} \tag{3.7}$$

which implies that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|,$$

for all n. Then $\{\|x_n - x_0\|\}$ is a nondecreasing sequence, and notice that $\{\|x_n - x_0\|\}$ is also bounded. Hence, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. At the same time, letting $n \to \infty$ in the right side of inequality (3.7), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} \in C_{n+1} \subseteq C_n$, we have

$$\begin{aligned}\|\alpha_n(I - T_{[n]})y_n\|^2 &\le 2\alpha_n\langle x_n - x_{n+1}, (I - T_{[n]})y_n\rangle + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \, \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| \\ &\le 2\alpha_n\|x_n - x_{n+1}\| \, \|y_n - T_{[n]}y_n\| + 2\alpha_n\beta_n L\|x_n - T_{[n]}x_n\| \, \|y_n - x_n + \alpha_n(I - T_{[n]})y_n\| \\ &\longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned}\|x_n - T_{[n]}x_n\| &\le \|x_n - y_n\| + \|y_n - T_{[n]}y_n\| + \|T_{[n]}y_n - T_{[n]}x_n\| \\ &\le (L+1)\|x_n - y_n\| + \|y_n - T_{[n]}y_n\| \\ &\le \alpha_n(L+1)\|x_n - T_{[n]}x_n\| + \alpha_n L(L+1)\|x_n - z_n\| + \|y_n - T_{[n]}y_n\| \\ &= \alpha_n(L+1)\|x_n - T_{[n]}x_n\| + \alpha_n\beta_n L(L+1)\|x_n - T_{[n]}x_n\| + \|y_n - T_{[n]}y_n\|. \end{aligned}$$

Hence

$$\|x_n - T_{[n]}x_n\| \le \frac{1}{1 - \alpha_n(L+1)^2}\|y_n - T_{[n]}y_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

By $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have $\lim_{n\to\infty}\|x_{n+i} - x_n\| = 0$, for all $i = 1, 2, \ldots, N$. Then, together with (3.8), we get

$$\begin{aligned}\|x_n - T_{[n+i]}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{[n+i]}x_{n+i}\| + \|T_{[n+i]}x_{n+i} - T_{[n+i]}x_n\| \\ &\le (1+L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{[n+i]}x_{n+i}\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.9}$$

It is clear that for any $l = 1, 2, \ldots, N$, there exists i such that $l = (n+i) \bmod N$. Consequently,

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = \lim_{n\to\infty}\|x_n - T_{[n+i]}x_n\| = 0.$$

By Lemma 2.3(ii), $I - T_i$ $(i = 1, 2, \ldots, N)$ is demiclosed at zero. This together with the fact that $\{x_n\}$ is bounded guarantees that every weak limit point of $\{x_n\}$ is a fixed point of $T_i(i = 1, 2, \ldots, N)$. That is, $\omega_w(x_n) \subseteq F$. Therefore, by inequality (3.5) and Lemma 2.4, we have that $\{x_n\}$ converges strongly to $q = P_F x_0$. This completes the proof. □

Remark 3.1. Theorem 3.1 extends the main results of Yao et al. [32] and Tang et al. [33] from one single Lipschitz pseudocontractive mapping to a finite family of Lipschitz pseudocontractive mappings.

In the following, we consider the problem of finding a common zero of a family of monotone operators, which can be applied to various kinds of problems such as equilibrium problems, variational inequalities, convex minimization problems, and others.

Recall that a mapping A is said to be monotone, if $\langle Ax - Ay, x - y\rangle \geq 0$, for all $x, y \in H$. The pseudocontractive mapping is strongly related to the monotone mapping. It is well known that A is monotone mapping if and only if $(I - A)$ is pseudocontractive mapping. Hence, the fixed points of pseudocontractive mapping actually is the zero of monotone mapping.

Theorem 3.2. *Let* $\{A\}_{i=1}^{N} : H \to H$ *be a family of monotone mappings with Lipschitz constants* $L_i (i = 1, 2, \dots, N)$ *for which* $\Omega := \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$. *Assume the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0, 1)$ *satisfying:* (i) $b \leq \alpha_n < \alpha_n(L+1)^2 < a < 1$, *for some* $a, b \in (0, 1)$; (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence* $\{x_n\}$ *generated by the following:*

$$\begin{cases} y_n = x_n - \alpha_n A_{[n]} z_n, \\ z_n = x_n - \beta_n A_{[n]} x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n A_{[n]} y_n\|^2 \leq 2\alpha_n \langle x_n - z, A_{[n]} y_n\rangle + 2\alpha_n\beta_n L\|A_{[n]} x_n\| \cdot \|y_n - x_n + \alpha_n A_{[n]} y_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

converges strongly to $P_\Omega x_0$.

Proof. Set $T_i = I - A_i,\ i = 1, 2, \dots, N$, then T_i is pseudocontractive mapping with nonempty common fixed point set $\bigcap_{i=1}^{N} F(T_i)$. By Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$. □

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