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Exponential lag synchronization of fuzzy cellular neural networks with time-varying delays ☆

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Abstract

In this paper, we are concerned with the lag synchronization problem of fuzzy cellular neural networks (FCNNs) with time-varying delays. Some sufficient conditions on the exponential lag synchronization of the FCNNs are obtained using a nonlinear measure method. The exponential decay rate of synchronization error is estimated. We also show how to determine the controller gain matrix under this method. Finally, simulation examples are given to illustrate the effectiveness of our obtained results.

1. Introduction

The stability analysis of traditional cellular neural networks (CNNs) with or without delay has received much attention, and lots of results have been obtained both in theory and in practical applications since CNN was introduced by Chua and Yang in 1988 [1–8]. It is well known that for an artificial intelligent system, the key motivation is to make a brain model as the core of the system. A classical CNN universal machine cannot function as this core even though it had been proved to be a Turing machine (Turing machines cannot answer a simple question: “How are you feeling?”) and even though it can be used to explain lots of visual phenomena. However, in mathematical modeling of real world problems, the uncertainty or vagueness is unavoidable. In 1996, Yang and Yang proposed

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the FCNN which integrates fuzzy logic into the structure of a traditional CNN and maintains local connectedness among cells in order to take vagueness into consideration [9–11]. The FCNN structures are based on the uncertainties in human cognitive processes and in modeling neural systems, and provide an interface between the human expert and the classical CNNs.

Studies have shown the potential of FCNNs in pattern recognition and image producing. It can be used to realize most of the nonlinear filters in mathematical morphology and solve a global problem: Euclidean distance transformation [12,13]. For such applications, it is very important to ensure that the designed FCNNs are stable. Many results on stability have been derived for FCNN models with or without time delay, see [14–18] and the references therein.

In the implementation of neural networks, time delays, especially time-varying delays, are unavoidably encountered in the signal transmission among the neurons due to the finite switching speed of neurons and amplifiers. And it has been shown that the existence of time delays frequently causes oscillation, divergence, and even chaotic behavior if the networks' parameters and time delays are appropriately chosen [19,20]. Therefore, a natural question that follows is about how to control chaos. Since the drive-response concept was proposed by Pecora and Corroll for constructing the synchronization of coupled chaotic systems [21], chaotic synchronization has been extensively investigated. Synchronization dynamics has increasingly gained interest in applications to many fields including secure communication, parallel image processing, neural networks, biological systems and information science.

Recently, the synchronization and lag synchronization of the delayed FCNNs have been studied [22–28], and sufficient conditions for the synchronization of delayed FCNNs are obtained based on Lyapunov direct methods. For FCNNs with time-varying delay, the authors mentioned above have often assumed that the time-varying delay is a differential function with its derivative greater than 0 and less than 1. However, time delay often occurs in an irregular fashion, and sometimes the time-varying delay is not differentiable. Therefore, it is necessary to further investigate the synchronization problem of FCNNs with time-varying delays under milder assumptions. Moreover, controller design is very important factor for synchronization. How to design the controllers is also need to consider.

Motivated by the aforementioned discussion, in this paper we investigate the synchronization of coupled FCNNs with time-varying delay. With the help of the nonlinear measure method, we want to remove the requirement of derivability of a time-varying delay. We will also show how to determine the controller gain matrix under this method.

The organization of this paper is as follows: in Section 2, we state some definitions and lemmas needed in the later sections. In Section 3, we derive the sufficient conditions to ensure lag synchronization of the systems, and show how to determine the controller gain matrix. Some simulations are given in Section 4. Finally, concluding remarks are given in Section 5.

2. Preliminaries

A FCNN can be represented as follows:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} v_j + \bigwedge_{j=1}^{n} \alpha_{ij} g_j(x_j(t-\tau_j(t)))$$

$$+\bigwedge_{j=1}^{n} T_{ij}v_j+\bigvee_{j=1}^{n} \beta_{ij}g_j(x_j(t-\tau_j(t)))+\bigvee_{j=1}^{n} S_{ij}v_j+I_i, \tag{1}$$

where x_i,v_i and I_i denote state, input and bias of the i-th neurons, $i=1,2,\ldots,n$, respectively; α_{ij}, β_{ij}, T_{ij} and S_{ij} are elements of fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; a_{ij} and b_{ij} are elements of feedback template and feed-forward template; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively; f_i, g_i are output transfer functions; and $\tau_i(t)$ are bounded time-varying delays that satisfy $0\le\tau_i(t)\le\tau_0$, $i=1,2,\ldots,n$.

The initial conditions of system (1) are of the form

$$x_i(\theta)=\phi_i(\theta)\in\mathcal{C}([-\tau_0,0],\mathbb{R}),\quad \theta\in[-\tau_0,0],\ i=1,2,\ldots,n.$$

Consider the response system that is driven by system (1) as follows:

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t}=-c_iy_i(t)+\sum_{j=1}^{n} a_{ij}f_j(y_j(t))+\sum_{j=1}^{n} b_{ij}v_j+\bigwedge_{j=1}^{n} \alpha_{ij}g_j(y_j(t-\tau_j(t)))$$

$$+\bigwedge_{j=1}^{n} T_{ij}v_j+\bigvee_{j=1}^{n} S_{ij}v_j+\bigvee_{j=1}^{n} \beta_{ij}g_j(y_j(t-\tau_j(t)))+I_i+u_i(t), \tag{2}$$

where $u_i(t)$ is a controller defined as

$$u_i(t)=\sum_{j=1}^{n} w_{ij}(f_j(y_j(t))-f_j(x_j(t-\sigma))),\quad i=1,\ldots,n, \tag{3}$$

and $W=(w_{ij})_{n\times n}$ denotes the controller gain matrix, $\sigma>0$ is the lag delay. The initial conditions of system (2) are of the form $y_i^{\sigma}(\theta)=\varphi_i(\theta)\in\mathcal{C}([-\tau_0,0],\mathbb{R})$, $\theta\in[-\tau_0,0]$, $i=1,2,\ldots,n$, where $y_i^{\sigma}(s)=y_i(s+\sigma)$ for all $s\in[-\tau_0,0]$.

Now, some definitions which will be used later are given.

Definition 1. Systems (1) and (2) are said to be exponentially lag synchronized if there exist constants $M>0,\eta>0$ such that

$$\|y(t)-x(t-\sigma)\|_1\le M\|\varphi-\phi\|_{\tau}e^{-\eta(t-\sigma)},\quad t>\sigma,$$

where $\|\varphi-\phi\|_{\tau_0}=\sup_{s\in[-\tau,0]}\|\varphi(s)-\phi(s-\sigma)\|_1$ and $\|\cdot\|_1$ defines the 1-norm of $\mathbb{R}^n$, i.e., $\|x\|_1=\sum_{i=1}^{n}|x_i|$.

Lemma 1 (*Yang and Yang [9]*). *Let x and y be two states of system* (1), *then we have*

$$\left|\bigwedge_{j=1}^{n}\alpha_{ij}g_j(x_j)-\bigwedge_{j=1}^{n}\alpha_{ij}g_j(y_j)\right|\le\sum_{j=1}^{n}|\alpha_{ij}||g_j(x_j)-g_j(y_j)|,$$

$$\left|\bigvee_{j=1}^{n}\beta_{ij}g_j(x_j)-\bigvee_{j=1}^{n}\beta_{ij}g_j(y_j)\right|\le\sum_{j=1}^{n}|\beta_{ij}||g_j(x_j)-g_j(y_j)|.$$

For vector norm $\|\cdot\|_1$ of $\mathbb{R}^n$,

$$\|x\|_1=\langle x,\mathrm{sign}(x)\rangle\quad\text{and}\quad\|x\|_1\ge\langle x,\mathrm{sign}(y)\rangle \tag{4}$$

hold for any $x,y \in \mathbb{R}^n$. Then for a matrix A, the definition of norm $\|A\|_1$ and matrix measure $\mu(A)$ induced by vector norm $\|\cdot\|_1$ are

$$\|A\|_1 = \sup_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} \quad \text{and} \quad \mu(A) = \lim_{s\to 0}\frac{\|I+sA\|_1 - 1}{s},$$

where I denotes the identity matrix. Then $\mu(A)$ can be equivalently calculated by

$$\mu(A) = \sup_{x\in\Omega, x\neq 0}\frac{\langle Ax, \operatorname{sign}(x)\rangle}{\|x\|_1}.$$

In the light of the matrix measure, nonlinear measure of an operator F is given in [30].

Definition 2 (*Qiao et al. [30]*). Let F be an operator on a subset Ω of $\mathbb{R}^n$, the constant

$$m(F) = \sup_{x,y\in\Omega, x\neq y}\frac{\langle F(x) - F(y), \operatorname{sign}(x-y)\rangle}{\|x-y\|_1}$$

is called the nonlinear measure of operator F on Ω.

And in functional analysis terms, a function f from an open set $\Omega \in \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be a Lipschitz operator on Ω whenever there exists a nonnegative constant L such that, for any $x, y \in \mathbb{R}^n$, $\|f(x) - f(y)\|_1 \le L\|x-y\|_1$, where L is called the Lipschitz constant of f. Furthermore, the constant defined by Soderlind [29]

$$L(f) = \sup_{x,y\in\Omega, x\neq y}\frac{\|f(x) - f(y)\|_1}{\|x-y\|_1} \tag{5}$$

is called the minimal Lipschitz constant of f on Ω.

Throughout the paper, we assume that

(H) There exist constants $L_i^f, L_i^g > 0$ such that

$$0 \le (f_i(\xi_1) - f_i(\xi_2))/(\xi_1 - \xi_2) \le L_i^f,$$

$$0 \le (g_i(\xi_1) - g_i(\xi_2))/(\xi_1 - \xi_2) \le L_i^g$$

hold for any $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$, $i = 1, 2, \ldots, n$.

And for the sake of convenience, we always use the following notations:

$$C = \operatorname{diag}\{c_1, c_2, \ldots, c_n\}, \quad A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n},$$

$$\alpha = (\alpha_{ij})_{n\times n}, \quad \beta = (\beta_{ij})_{n\times n}, \quad W = (w_{ij})_{n\times n},$$

$$T = (T_{ij})_{n\times n}, \quad S = (S_{ij})_{n\times n}, \quad I = (I_1, I_2, \ldots, I_n)^T.$$

3. Main results

In this section, we will give some sufficient conditions for the exponential lag synchronization of systems (1) and (2) by the nonlinear measure method.

For the lag synchronization problem, we define the error between systems (1) and (2) as $e(t) = [e_1(t), \ldots, e_n(t)]^T$ with $e_i(t) = y_i(t) - x_i(t-\sigma)$. Given any diagonal positive-definite matrix $R=\mathrm{diag}\{r_1,r_2,\ldots,r_n\}$, then $\|e(t)\|_1\to 0(t\to\infty)$ if and only if $\|R^{-1}e(t)\|_1\to 0(t\to\infty)$. Let $e(t)=R^{-1}e(t)$, then we have

$$\frac{\mathrm{d}e_i(t)}{\mathrm{d}t}=-c_ie_i(t)+r_i^{-1}\sum_{j=1}^{n}(a_{ij}+w_{ij})(f_j(r_jy_j(t))-f_j(r_jx_j(t-\sigma)))$$
$$+r_i^{-1}\bigwedge_{j=1}^{n}\alpha_{ij}g_j(r_jy_j(t-\tau_j(t)))-r_i^{-1}\bigwedge_{j=1}^{n}\alpha_{ij}g_j(r_jx_j(t-\tau_j(t)-\sigma))$$
$$+r_i^{-1}\bigvee_{j=1}^{n}\beta_{ij}g_j(r_jy_j(t-\tau_j(t)))-r_i^{-1}\bigvee_{j=1}^{n}\beta_{ij}g_j(r_jx_j(t-\tau_j(t)-\sigma)). \tag{6}$$

Theorem 1. *Assumption* (H) *holds. If there exists a set of* $r_i>0, i=1,2,\ldots,n$ *such that* $\mu+v<0$, *then system* (2) *will globally exponentially lag synchronize with system* (1), *where*

$$\begin{cases}\mu=\max\limits_{1\le j\le n}\left\{-1+(a_{jj}+w_{jj})c_j^{-1}L_j^f+\dfrac{r_j}{r_i}\sum\limits_{i=1,i\ne j}^{n}|a_{ij}+w_{ij}|c_j^{-1}L_j^f\right\},\\ v=\max\limits_{1\le j\le n}\left\{L_j^g r_j\sum\limits_{i=1}^{n}\dfrac{1}{r_i}(|\alpha_{ij}|+|\beta_{ij}|)\right\}>0.\end{cases} \tag{7}$$

Moreover, the exponential decay estimate is governed by

$$\|e(t)\|_1\le e^{-\theta t}\cdot\frac{\max_{1\le i\le n}r_i}{\min_{1\le i\le n}r_i}\cdot\sup_{-\tau_0\le s\le 0}\|e_\tau(s)\|_1,$$

where θ *is a bounded unique solution of equation*

$$\theta=-\rho\mu-\rho v e^{\theta\tau_0},$$

with $\rho=\min_{1\le i\le n}c_i$.

Proof. Define $FR,GR:\mathbb{R}^n\to\mathbb{R}^n$ as

$$FR(e(t))=F(Re(t))=[F_1(Re(t)),\ldots,F_n(Re(t))]^T,$$

$$GR(e_\tau(t))=G(Re_\tau(t))=[G_1(Re_\tau(t)),\ldots,G_n(Re_\tau(t))]^T,$$

where $e_\tau(t)=[e_1(t-\tau_1(t)),\ldots,e_n(t-\tau_n(t))]^T$ and

$$F_i(Re(t))=\sum_{j=1}^{n}(a_{ij}+w_{ij})(f_j(r_jy_j(t))-f_j(r_jx_j(t-\sigma))),$$

$$G_i(Re_\tau(t))=\bigwedge_{j=1}^{n}\alpha_{ij}g_j(r_jy_j(t-\tau_j(t)))-\bigwedge_{j=1}^{n}\alpha_{ij}g_j(r_jx_j(t-\tau_j(t)-\sigma))$$
$$+\bigvee_{j=1}^{n}\beta_{ij}g_j(r_jy_j(t-\tau_j(t)))-\bigvee_{j=1}^{n}\beta_{ij}g_j(r_jx_j(t-\tau_j(t)-\sigma)).$$

Then system (6) becomes

$$\frac{\mathrm{d}e(t)}{\mathrm{d}t}=-Ce(t)+R^{-1}FR(e(t))+R^{-1}GR(e_\tau(t)). \tag{8}$$

Take $Q = C^{-1}$ and by assumption (H), for any $e \in \mathbb{R}^n$ we compute that

$$\begin{aligned}
&\langle -CQe(t) + R^{-1}FR(Qe(t)), \mathrm{sign}(e(t)) \rangle \\
&\quad \le -\sum_{i=1}^{n} |e_i(t)| + \sum_{i=1}^{n} \frac{1}{r_i}(a_{ii} + w_{ii})(f_i(r_i c_i^{-1} y_i(t)) - f_i(r_i c_i^{-1} x_i(t-\sigma)))\,\mathrm{sign}(e_i(t)) \\
&\qquad + \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1, j\ne i}^{n} |a_{ij} + w_{ij}||(f_j(r_j c_j^{-1} y_j(t)) - f_j(r_j c_j^{-1} x_j(t-\sigma)))| \\
&\quad \le \sum_{i=1}^{n} (-1 + (a_{ii} + w_{ii}) c_i^{-1} L_i^f)|e_i(t)| + \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1, j\ne i}^{n} |a_{ij} + w_{ij}| c_j^{-1} L_j^f r_j |e_j(t)| \\
&\quad \le \mu \|e(t)\|_1,
\end{aligned}$$

where μ is defined by Eq. (7). Then by Definition 2, we have $m(-CQ + R^{-1}FRQ) \le \mu$ on $\mathbb{R}^n$.

By assumption (H) and Lemma 1, we obtain

$$\begin{aligned}
\|R^{-1}GR(Qe(t))\|_1 &\le \sum_{i=1}^{n} \frac{1}{r_i} \left| \bigwedge_{j=1}^{n} \alpha_{ij} g_j(r_j c_j^{-1} y_j(t)) - \bigwedge_{j=1}^{n} \alpha_{ij} g_j(r_j c_j^{-1} x_j(t-\sigma)) \right| \\
&\quad + \sum_{i=1}^{n} \frac{1}{r_i} \left| \bigwedge_{j=1}^{n} \alpha_{ij} g_j(r_j c_j^{-1} y_j(t)) - \bigwedge_{j=1}^{n} \alpha_{ij} g_j(r_j c_j^{-1} x_j(t-\sigma)) \right| \\
&\le \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1}^{n} |\alpha_{ij}||g_j(r_j c_j^{-1} y_j(t)) - g_j(r_j c_j^{-1} x_j(t-\sigma))| \\
&\quad + \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1}^{n} |\beta_{ij}||g_j(r_j c_j^{-1} y_j(t)) - g_j(r_j c_j^{-1} x_j(t-\sigma))| \\
&\le \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|) c_j^{-1} L_j^g r_j |e_j(t)| \le v \|e(t)\|_1,
\end{aligned}$$

where v is defined by Eq. (7). Then from Eq. (5), $0 < L(R^{-1}GRQ) \le v$ on $\mathbb{R}^n$. So, we obtain

$$m(-CQ + R^{-1}FRQ) + L(R^{-1}GRQ) \le \mu + v < 0 \tag{9}$$

on $\mathbb{R}^n$. By Eq. (4), for any $s > 0$,

$$\frac{\|e(t)\|_1 - \|e(t-s)\|_1}{s} \le \frac{1}{s} \langle e(t) - e(t-s), \mathrm{sign}(e(t)) \rangle .$$

Hence, from Eq. (6), the derivative of $\|e(t)\|_1$ satisfies almost everywhere in interval $(0,\infty)$ (see [31, Theorem 8.19]), and satisfies

$$\begin{aligned}
\frac{\mathrm{d}\|e(t)\|_1}{\mathrm{d}t} &\le \left\langle \frac{\mathrm{d}e(t)}{\mathrm{d}t}, \mathrm{sign}(e(t)) \right\rangle = \langle -Ce(t) + R^{-1}F(Re(t)) + R^{-1}G(Re_\tau(t)), \mathrm{sign}(e(t)) \rangle \\
&\le \langle -Ce(t) + R^{-1}F(Re(t)), \mathrm{sign}(Q^{-1}e(t)) \rangle + \langle R^{-1}G(Re_\tau(t)), \mathrm{sign}(e(t)) \rangle \\
&\le m(-CQ + R^{-1}FRQ)\|Q^{-1}e(t)\|_1 + L(R^{-1}GRQ)\|Q^{-1}e_\tau(t)\|_1 \\
&\le \{m(-CQ + R^{-1}FRQ)\|e(t)\|_1 + L(R^{-1}GRQ)\|e_t\|_1\}\rho \\
&= \rho\mu\|e(t)\|_1 + \rho v\|e_t\|_1,
\end{aligned}$$

where $\|e_t\|_1 = \sup_{t-\tau_0 \le s \le t} \|e_\tau(s)\|_1$, $\rho = \min_{1 \le i \le n} c_i$. Then by Eq. (9) and Halanay's inequality [32], system (8) is exponentially stable, which means that systems (1) and (2) are globally exponentially lag synchronized. And

$$\|e(t)\|_1 \le e^{-\theta t} \sup_{-\tau_0 \le s \le 0} \|e_\tau(s)\|_1.$$

Then we have

$$\|e(t)\|_1 \le e^{-\theta t} \cdot \frac{\max_{1\le i\le n} r_i}{\min_{1\le i\le n} r_i} \cdot \sup_{-\tau_0 \le s \le 0} \|e_\tau(s)\|_1,$$

where θ is a bounded exponential convergence rate, which is the unique solution of

$$\theta = -\rho\mu - \rho v e^{-\theta\tau_0}.$$

This completes the proof. □

Remark 1. In Theorem 1, one needs to determine the controller gain matrix $W=(w_{ij})_{n\times n}$. By Eq. (7), given a set of $r_i>0$, v is a fixed constant and the values $|w_{ij}+a_{ij}|(i\ne j)$ will impact the choice of w_{ii}. So, $w_{ij}(i\ne j)$ should be chosen as close to $-a_{ij}$ as possible to avoid a strong requirement of w_{ii}. In fact, we can take $w_{ij}=-a_{ij}$, $i\ne j$, then

$$\mu = \max_{1\le j\le n}\{-1+(a_{jj}+w_{jj})c_j^{-1}L_j^f\},$$

and we only need to determine w_{ii} in the controller.

By the discussion of Remark 1, we have:

Corollary 1. *Assume* (H) *holds, and choose* $w_{ij}=-a_{ij}, i\ne j$. *Then if there exists a set of* $r_i>0$, $i=1,2,\ldots,n$ *such that* $\mu+v<0$, *system* (2) *will globally exponentially lag synchronize with system* (1), *where*

$$\begin{cases} \mu = \max\limits_{1\le j\le n}\{-1+(a_{jj}+w_{jj})c_j^{-1}L_j^f\}, \\ v = \max\limits_{1\le j\le n}\left\{L_j^g r_j \sum\limits_{i=1}^{n}\dfrac{1}{r_i}(|\alpha_{ij}|+|\beta_{ij}|)\right\} > 0. \end{cases} \quad (10)$$

And the exponential decay estimate is governed by

$$\|e(t)\|_1 \le e^{-\theta t} \cdot \frac{\max_{1\le i\le n} r_i}{\min_{1\le i\le n} r_i} \cdot \sup_{-\tau_0\le s\le 0} \|e_\tau(s)\|_1,$$

where θ *is a bounded unique solution of equation* $\theta=-\rho\mu-\rho v e^{\theta\tau_0}$ *with* $\rho=\min_{1\le i\le n} c_i$.

When $r_i=1$, $i=1,2,\ldots,n$ in Theorem 1 and Corollary 1, one can easily obtain the following corollary.

Corollary 2. *Assumption* (H) *holds. If* $\mu+v<0$, *then system* (2) *will globally exponentially lag synchronize with system* (1), *where*

$$\begin{cases} \mu = \max\limits_{1\le j\le n}\left\{-1+(a_{jj}+w_{jj})c_j^{-1}L_j^f + \sum\limits_{i=1, i\ne j}^{n}|a_{ij}+w_{ij}|c_j^{-1}L_j^f\right\}, \\ v = \max\limits_{1\le j\le n}\left\{L_j^g \sum\limits_{i=1}^{n}(|\alpha_{ij}|+|\beta_{ij}|)\right\} > 0. \end{cases} \quad (11)$$

Furthermore, the exponential decay estimate is governed by

$$\|e(t)\|_1 \le e^{-\theta t} \cdot \sup_{-\tau_0 \le s \le 0} \|e_\tau(s)\|_1,$$

where θ is a bounded unique solution of equation $\theta = -\rho\mu - \rho v e^{\theta\tau_0}$ with $\rho = \min_{1 \le i \le n} c_i$.

Corollary 3. *Assume* (H) *holds, and choose $w_{ij} = -a_{ij}, i \ne j$. Then if $\mu + v < 0$, system* (2) *will globally exponentially lag synchronize with system* (1), *where*

$$\begin{cases} \mu = \max_{1 \le j \le n} \{-1 + (a_{jj} + w_{jj}) c_j^{-1} L_j^f\}, \\ v = \max_{1 \le j \le n} \left\{ L_j^g \sum_{i=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|) \right\} > 0. \end{cases} \tag{12}$$

And the exponential decay estimate is governed by

$$\|e(t)\|_1 \le e^{-\theta t} \cdot \sup_{-\tau_0 \le s \le 0} \|e_\tau(s)\|_1,$$

where θ is a bounded unique solution of equation $\theta = -\rho\mu - \rho v e^{\theta\tau_0}$ with $\rho = \min_{1 \le i \le n} c_i$.

Remark 2. Parameters $r_i (i = 1, \ldots, n)$ are introduced to adjust the value of v and term $(r_j/r_i) \sum_{i=1, i \ne j}^{n} |a_{ij} + w_{ij}| c_j^{-1} L_j^f$ in μ such that the globally exponentially lag synchronization can be achieved in our result. By Remark 1, we should choose $w_{ij} = -a_{ij} (i \ne j)$ in the controllers. Then, we should first determine v then w_{ii} by Eq. (10). A proper selected set of $r_i > 0$ can make the value of v smaller than that in Corollary 2. Thus by Eq. (10) and $\mu + v < 0$, a relatively weaker requirement upon w_{ii} is needed, and one can determine a suitable value of w_{ii}.

Remark 3. For systems (1) and (2), the differentiability of time delays is often required using Lyapunov direct method, see [24–28]. However, time delays can occur in an irregular fashion, and sometimes the time-varying delays are not differentiable. Our results removed the requirement of differentiability of time delays based on nonlinear measure method. Moreover, one can easily see that the results in this paper have no relation to lag delay σ. So our results are still hold for $\sigma = 0$.

4. Numerical simulations

In this section, we give some examples and numerical simulations to illustrate the effectiveness of the controlling laws presented in Section 3. First, we consider a time delayed system in which the time-varying delays are not differentiable.

Example 1. Consider two coupled identical two-order FCNNs with time-varying delay. The drive systems are described by the following equations:

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t)) + \bigwedge_{j=1}^{2} \alpha_{ij} f_j(x_j(t-\tau(t))) + \bigvee_{j=1}^{2} \beta_{ij} f_j(x_j(t-\tau(t))) + I_i, \quad i = 1, 2, \tag{13}$$

where $A = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & 2.0 \end{pmatrix}$, $\alpha = \beta = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -1.6 \end{pmatrix}$, $I = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}$ and $f(x) = \tanh(x)$, $\tau(t) = 1.38|\sin(t)|$. For the parameters chosen above, the system exhibits chaotic behavior. Fig. 1 shows

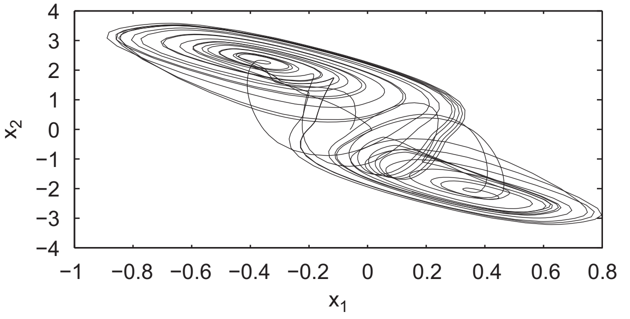

Fig. 1. The chaotic attractor of system (13).

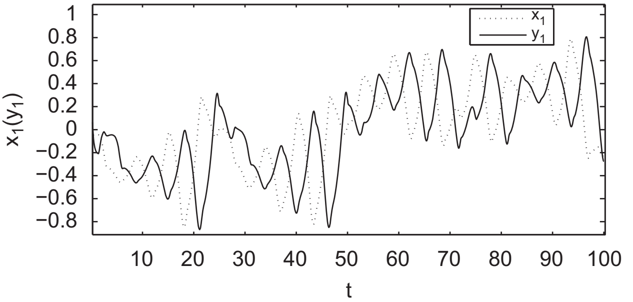

Fig. 2. Lag synchronization of x_1 and y_1 of systems (13) and (14) for $\sigma=3$.

a chaotic attractor of this system with the initial condition $x_1(s)=-0.06$, $x_2(s)=-0.01$, $-1.38\le s\le 0$.

The controlled response system is described by

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t}=-y_i(t)+\sum_{j=1}^{2}a_{ij}f_j(y_j(t))+\bigwedge_{j=1}^{2}\alpha_{ij}f_j(y_j(t-\tau_j(t)))$$
$$+\bigvee_{j=1}^{2}\beta_{ij}f_j(y_j(t-\tau_j(t)))+I+u_i(t),\quad i=1,2, \tag{14}$$

where $u_i(t)$ is defined as in Eq. (3). Obviously, assumption (H) holds with $L_i^f=1$, $0\le\tau(t)\le 1.38$, $t\ge 0$ and $\tau(t)$ is not differentiable. So the conditions of [24–28] are not satisfied and the corresponding methods cannot be used for this example. According to Corollary 3, let $w_{12}=0.1$, $w_{21}=5$ and it is easy to get that $\mu=\max\{1+w_{11},1+w_{22}\}$, $v=3.4$. Inequality $\mu+v<0$ implies that $w_{ii}<-4.4$, $i=1,2$. Take $w_{ii}=-5(i=1,2)$, then the lag synchronization can be achieved with an exponential convergence rate 0.0995. Let $\sigma=3$ and initial values $(x_1(s),\ x_2(s))=(-0.06,-0.01)$, $(y_1(s),y_2(s))=(0.4,-1.8)$ for $-1.38\le s\le 0$, the time response of lag synchronization between systems (13) and (14) is presented in Figs. 2 and 3. Correspondingly, Fig. 4 shows the lag synchronization errors of systems (13) and (14).

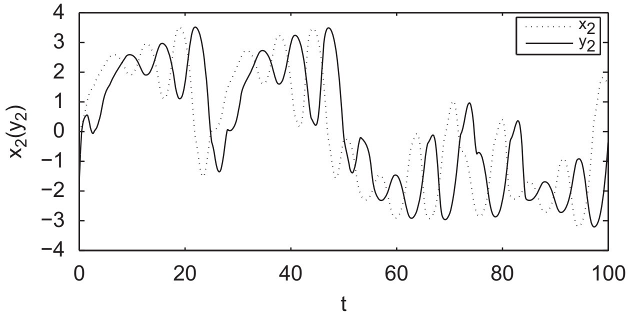

Fig. 3. Lag synchronization of x_2 and y_2 of systems (13) and (14) for $\sigma=3$.

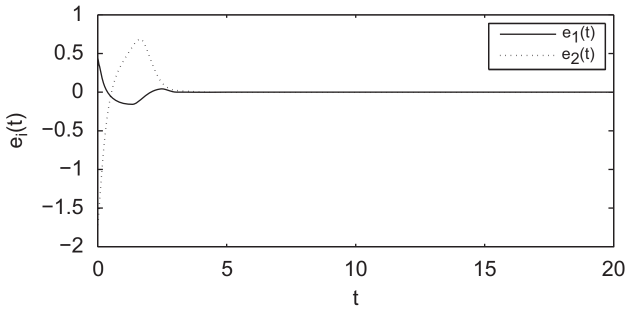

Fig. 4. Lag synchronization error $e_i(t)$ $(i=1,2)$ of systems (13) and (14) with $\sigma=3$.

Another example is given to illustrate the role of r_i, $i=1,\ldots,n$ for designing the controllers. Lag delay σ is not the key quantity we consider here, and as a special case of lag synchronization, we take $\sigma=0$.

Example 2. Consider that the drive systems are described by the following FCNNs with time-varying delays:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}=-x_i(t)+\sum_{j=1}^{2}a_{ij}f_j(x_j(t))+\bigwedge_{j=1}^{2}\alpha_{ij}f_j(x_j(t-\tau(t)))$$
$$+\bigvee_{j=1}^{2}\beta_{ij}f_j(x_j(t-\tau(t)))+I_i,\quad i=1,2, \tag{15}$$

where $A=\begin{pmatrix}\frac{2.0}{1.21^3} & \frac{-0.1}{1.21^3}\\ \frac{-4.0}{1.21^3} & \frac{3.0}{1.21^3}\end{pmatrix}$, $\alpha=\beta=\begin{pmatrix}\frac{-1.5}{1.21^3} & \frac{-1.2}{1.21^3}\\ \frac{-0.1}{1.21^3} & \frac{-2.2}{1.21^3}\end{pmatrix}$, $I=\binom{0}{0.1}$ and $f(x)=\tanh(1.21^3x)$, $\tau(t)=1+1.08e^{-t}$. The system exhibits chaotic behavior with the parameters chosen above. Fig. 5 shows a chaotic attractor of this system with the initial condition $x_1(s)=-0.01$, $x_2(s)=0.03$, $-2.08\le s\le 0$.

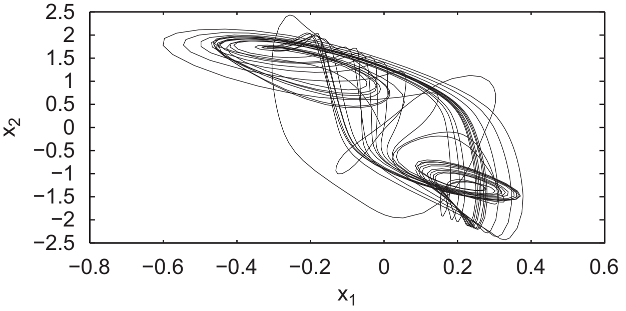

Fig. 5. The chaotic attractor of system (15).

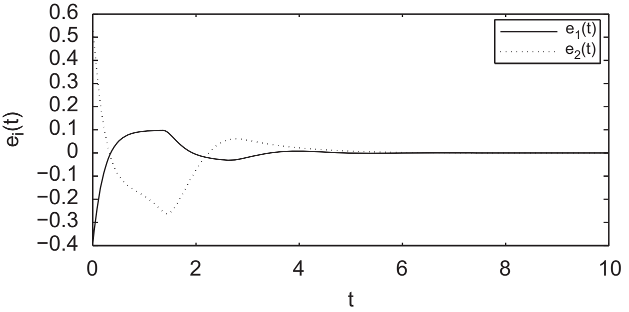

Fig. 6. Synchronization error $e_i(t)$ $(i=1,2)$ of systems (15) and (16).

The controlled response system is described by

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -y_i(t) + \sum_{j=1}^{2} a_{ij}f_j(y_j(t)) + \bigwedge_{j=1}^{2} \alpha_{ij}f_j(y_j(t-\tau_j(t)))$$

$$+ \bigvee_{j=1}^{2} \beta_{ij}f_j(y_j(t-\tau_j(t))) + I + u_i(t), \quad i=1,2, \tag{16}$$

where $u_i(t)$ is defined as in Eq. (3). Obviously, assumption (H) holds with $L_i^f = 1.21^3$, $0<\tau(t)\leq 2.08$, $t\geq 0$. $\tau(t)$ is differentiable but do not satisfy $0<\dot{\tau}(t)<1$, $t\geq 0$. Again, the conditions of [24–28] are not satisfied and the corresponding methods cannot be used here. According to Corollary 1, let $w_{12} = \frac{0.1}{1.21^3}$, $w_{21} = \frac{4}{1.21^3}$ and it is easy to get that $\mu = \max\{1+1.21^3 w_{11}, 2+1.21^3 w_{22}\}$, $v = \max\{3+2.4r_1/r_2, 0.2r_2/r_1+4.4\}$. From Remark 2, we need first to determine the value of v, then w_{ii}. From the fact that $v=4.5563$ when $r_1/r_2=0.64$ while $v=5.4$ when $r_1/r_2=1$, we see that the choice $r_1/r_2=0.64$ is more suitable. Then, inequality $\mu+v<0$ implies that $w_{11}<\frac{-5.5563}{1.21^3}$, $w_{22}<\frac{-6.5563}{1.21^3}$. Take $w_{11}=\frac{-5.7}{1.21^3}$, $w_{22}=\frac{-6.8}{1.21^3}$, the lag synchronization of networks (15) and (16) can be achieved with an exponential convergence rate 0.0135. Fig. 6 shows the synchronization errors of systems (15) and (16) with initial values $(x_1(s), x_2(s)) = (-0.01, 0.03)$, $(y_1(s), y_2(s)) = (-0.4, 0.6)$ for $-2.08 \le s \le 0$.

5. Conclusion

There have been many good results concerned about the synchronization of FCNNs. Most of them consider the fuzzy networks with constant delay. Compared with the systems with constant delay, time-varying delay systems represent a more natural framework for mathematical modeling of many real world phenomena. However, for FCNNs with time-varying delay, many authors often require that the time-varying delay is differentiable. In this paper, we obtained some sufficient conditions for the exponential lag synchronization of FCNNs using the nonlinear measure method. The requirement of derivability of the time-varying delay is removed and the exponential decay rate of synchronization error is estimated. Moreover, we also show how to determine the controller gain matrix for synchronization under this method. Finally, simulation examples are given to illustrate the effectiveness of our results.

In our results as well as works mentioned in this paper, controllers are put to all the components of response system, and sometimes some controllers (but not all) may not work in implementation. What should we do to avoid this situation? If one has a method to deal with this situation, a new interesting problem in synchronization model will appear, that is, it is not needed to put the controllers to all the components of response system. Then, which components should be controlled to guarantee the synchronization? It is an interesting problem and will become our future investigate direction.

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