Neural Comput & Applic (2012) 21:289–297
DOI 10.1007/s00521-010-0426-0

ORIGINAL ARTICLE

Pinning a stochastic neural network to the synchronous state

Tao He · Jigen Peng · Jikai Lei

Received: 6 January 2010 / Accepted: 28 June 2010 / Published online: 15 July 2010

Abstract In this paper, the asymptotic stability of the pinning synchronous solution of stochastic neural networks with and without time-delays is analyzed. The delays are time-varying, and the uncertainties are norm-bounded that enter into all the parameters of network and control. The aim of this paper is not only to establish easily verifiable conditions under which the pinning synchronous solution of stochastic neural network is globally asymptotically stable but also to give a feasible way to offset the limitation of network itself in order to reach synchronization. In addition, a specific neurobiological network is also introduced, and some numerical examples are provided to illustrate the applicability of the proposed criteria.

Keywords Pinning mechanism · Synchronization · Stochastic neural network

1 Introduction

Synchronization control ensures two or more systems to share a common dynamical behavior by coupling or external forcing. In recent years, it has received much attention because of its extensive application in engineering, such as neurobiology, secure communication, ship control, navigation, and so on [1–4]. In Kyrkjeb et al. [1], a leader–follower synchronization output feedback control scheme is presented for the ship replenishment problem. In Chea et al. [2], the synchronization control of two coupled chaotic FitzHugh-Nagumo (FHN) neurons under external electric stimulation is investigated. The FHN model is a two-dimensional simplification of the widely known Hodgkin–Huxley model describing the signal transmission across axons in neurobiology. In Chena et al. [3], the evolutionary programming (EP)-based proportional-integral-derivative (PID) control design is presented for synchronization of chaotic systems with application in secure communication. And a robust fuzzy sliding mode control (FSMC) scheme for the synchronization of two chaotic nonlinear gyros subject to uncertainties and external disturbances is presented in Yau [4].

Since neural networks can exhibit complex dynamics, the synchronization control of neural networks has been applied in various areas, including signal processing, image processing, and pattern recognition, and there are a lot of synchronization schemes which are designed for neural networks. For example, synchronization of an array of linearly coupled neural networks with and without time-delay have been investigated in Lu and Chen [5], and a mild condition for coupled delayed neural networks to reach synchronization and its applications to chaotic cellular neural networks are given in Chen et al. [6]. In particular, adaptive synchronization in networks has been a focal subject for research, and the adaptive techniques have been proved to an effective method to help a complex network reach synchronization. Pinning mechanism, as a special adaptive control, has been introduced recently in view of impracticality of controlling all the nodes in a large-scale network. Related results show that it is possible

This work was supported by the NCET and in part by the NSFC under the contact 10531030.

T. He (✉) · J. Peng
Department of Mathematics, Xi'an Jiaotong University, 710049 Xi'an, China
e-mail: ht1012@gmail.com

J. Lei
Department of Computer Science and Technology, Xi'an Jiaotong University, 710049 Xi'an, China

to help all the nodes reach synchronization by controlling only a small part of nodes in the network [7–14].

However, all of these models previously did not consider the existence of noise. Actually, in real nervous networks, the synaptic transmission is a noise process brought on by random fluctuation from the release of neurotransmitters and other stochastic inputs. Also, in the man-made communication networks, signal transmission often involves frame losses, bit errors, environment disturbance, and so on. Therefore, the synchronization analysis of a stochastic neural network becomes increasingly important and we try to answer the following three fundamental questions in stochastic neural networks: (1) which nodes should be controlled to help the whole network to reach synchronization? (2) how to set the coupling strength so that a general stochastic neural network with pinning control can realize synchronization? (3) how to add noise to a network in order to offset its limitation in topology and in turn to realize synchronization.

In this paper, the globally asymptotic stability of the pinning synchronous solution of stochastic neural networks with and without time-delays is analyzed. The delays are time-invariant, and the uncertainties are norm-bounded that enter into all the parameters of network and control. Using Lyapunov function methods and some well-known inequalities, some topology-based sufficient conditions are derived for pinning synchronization. A concrete neurobiological network with a specific pinning control is introduced, and some numerical results are denoted to show the effectiveness and applicability of the proposed criteria.

The paper is organized as follows. A general stochastic neural network model under pinning control is introduced in Sect. 2. In Sect. 3, globally adaptive pinning synchronization criteria for stochastic neural networks with and without time-delay are investigated. Especially, a specific neurobiological network model is discussed in Sect. 3, and numerical results are showed in Sect. 4. Conclusions are finally drawn in Sect. 5.

2 Model formulation

As the most important component of neural network, the complicated interconnections between neurons are responsible for implementing every function of human's body. For every neuron in the network, its dynamics is not only deprived by its own property, but also by the information from their neighbors. Besides, it is important and of prime importance to consider stochastic effects to the synchronization of real networks. The detail description of the stochastic neural network is as follows.

2.1 Notation

First of all, there are some necessary notations which will be used throughout the paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices, I_l is the identity matrix of l dimensions. T denotes the transpose of a matrix or a vector. $|\cdot|$ stands for the 2-norm of a vector, i.e., $\|x\| = \sqrt{x^T x}$. $\lambda_{\max}(\cdot)$ represents the maximum eigenvalue of a square matrix. The notation $X > 0$ (respectively, $X < 0$), for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite). $\mathbb{E}(\cdot)$ denotes the expectation. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathbb{P}$ is the probability measure on $\mathcal{F}$. $C([-\tau, 0], \mathbb{R})$ denotes the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}$. Matrix Φ_l represents the sub-matrix of Φ by removing all the $i_k (1 \le i_k \le N, 1 \le k \le l, 1 \le l \le N)$ row–column pairs of Φ.

2.2 Stochastic neural network without time-delay under pinning control

Generally, we assume the stochastic neural network to be described by the following set equations:

$$dx_i(t) = \left[f(x_i(t), t) + \sum_{j=1}^{N} a_{ij} g(x_j(t)) + u_i(t) \right] dt + \sigma_i(x_i(t) - x_0(t)) d\omega(t), \tag{1}$$

where $1 \le i \le N$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state vector of the ith node, $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a smooth nonlinear vector function, individual neuron dynamics is $\dot{x}(t) = f(x(t), t)$, $\sigma_i \in R^n$ is called the noise intensity function vector, $\omega(t)$ is a 1-dimension Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is out-coupling weight configuration matrix, $g : R^n \to R^n$ called inner-coupling vector function, which describes the input information from other nodes. If there is a link between node i and node j $(i \ne j)$, then $a_{ij} = a_{ji} > 0$, otherwise $a_{ij} = 0$. Besides, $a_{ii} = -\sum_{j=1, j\ne i} a_{ij}$. u_i are adaptive controllers designed as follows:

$$\begin{aligned} u_{i_k} &= -\alpha_{i_k}(x_{i_k}(t) - x_0(t)) && 1 \le k \le l \\ \dot{\alpha}_{i_k} &= \beta_{i_k} \|x_{i_k}(t) - x_0(t)\|^2 && 1 \le k \le l \\ &= \beta_{i_k}(x_{i_k}(t) - x_0(t))^T (x_{i_k}(t) - x_0(t)) && \\ u_{i_k} &= 0 && \text{otherwise} \end{aligned} \tag{2}$$

where $x_0(t)$, which is a trajectory of the uncoupled system, stands for a synchronous state, i.e, $\dot{x}_0(t) = f(x_0(t), t)$. The adaptive scheme above assumes that i_1th, i_2th,… ,i_lth nodes are controlled and it means that only a small fraction of neurons are involved in the direct control.

2.3 Stochastic neural network with time-delay under pinning control

In neural network model, data or axon signal transmission is always accompanied by a non-zero time interval between the initial and the delivery time of a message of signals. For example, the finite switching speed of amplifier circuits in neural networks can cause time-delay [15]. Also, to deal with motion-related problems, such as moving image processing [16], the factor of time-delay should be considered. The existence of time-delay may result in instability or other poor performance of neural networks. Similarly, a neural network under pinning control may not reach synchronization if there exist time-delays in the network. Therefore, consider the following stochastic neural network with time-varying delay under pinning control described by

$$\begin{aligned} dx_i(t) = &\left[f(x_i(t), t) + \sum_{j=1}^{N} a_{ij} g(x_j(t)) \right. \\ &\left. + \sum_{j=1}^{N} b_{ij} h(x_j(t-\tau(t))) + u_i(t) \right] dt \\ &+ \sigma_i'(e_i(t), e_i(t-\tau(t))) d\omega(t) \end{aligned} \quad (3)$$

where $\tau(t)$ is the time-delay in coupling, $e_i(t) = \|x_i(t) - x_0(t)\|, e_i(t-\tau(t)) = \|x_i(t-\tau(t)) - x_0(t-\tau(t))\|, h: R^n \to R^n$ is delayed inner-coupling vector function, which describes the input information from other neurons under delayed condition. Let $\tau_m = \sup_{t \in R^+} \tau(t)$, and $x_{i0} = (x_{i1}^0, x_{i2}^0, \ldots, x_{in}^0) \in C([-\tau_{m,0}], \mathbb{R}^n)$ denotes the initial condition of ith neuron. The adaptive controller is as same as (2). Besides, we assumes that under delayed condition, noise intensity function σ'_i is involved with $e_i(t)$ and $e_i(t - \tau(t))$.

3 Main results

This section further investigates the globally adaptive pinning synchronization of a stochastic neural network. Several network synchronization criteria are drawn.

3.1 Preliminaries

To proceed with our analysis, the following assumptions and lemmas are needed.

Assumption 1 The time-delay $\tau(t)$ is a differentiable function satisfying

$$-\theta \le \dot{\tau}(t) \le \theta \quad (4)$$

where $0 \le \theta < 1$ is a constant.

Remark 1 Since $\tau(t)$ often change slowly in real engineering system, this assumption is reasonable and practical.

Assumption 2 Suppose that there exists a positive constant μ satisfying

$$(\xi_1 - \xi_2)^T (f(\xi_1, t) - f(\xi_2, t)) \le \mu (\xi_1 - \xi_2)^T (\xi_1 - \xi_2) \quad (5)$$

for any two vectors $\xi_1, \xi_2 \in \mathbb{R}^n$.

Assumption 3 Assume that there exists two positive constants v_1, v_2 satisfying

$$\begin{aligned} v_1 \|\xi_1 - \xi_2\|^2 &\le (\xi_1 - \xi_2)^T (g(\xi_1, t) - g(\xi_2, t)) \\ &\le \|\xi_1 - \xi_2\| \|g(\xi_1, t) - g(\xi_2, t)\| \\ &\le v_2 \|\xi_1 - \xi_2\|^2 \end{aligned} \quad (6)$$

for any two vectors $\xi_1, \xi_2 \in \mathbb{R}^n$.

Assumption 4 Suppose that the delayed inner-coupling function $h(\xi)$, where $\xi \in \mathbb{R}^n$, satisfies

$$\begin{aligned} \omega_1 \|\xi_1 - \xi_2\|^2 &\le (\xi_1 - \xi_2)^T (h(\xi_1, t) - h(\xi_2, t)) \\ &\le \|\xi_1 - \xi_2\| \|h(\xi_1, t) - h(\xi_2, t)\| \\ &\le \omega_2 \|\xi_1 - \xi_2\|^2 \end{aligned} \quad (7)$$

for any two vectors $\xi_1, \xi_2 \in \mathbb{R}^n$, where ω_1, ω_2 are two positive constants.

Assumption 5 Suppose that the noise intensity vector $\sigma: \mathbb{R}^n \to \mathbb{R}^n$, which is locally Lipschitz continuous, satisfies

$$\text{trace}\left[\sigma_i^T(e_i(t)) \sigma_i(e_i(t))\right] \le e_i(t)^T H_i e_i(t) \quad (8)$$

where $i = 1, \ldots, N, H_i$ is positive semi-definite matrix.

Lemma 1 *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} A(x) & B(x) \\ B^T(x) & C(x) \end{pmatrix} < 0,$$

where $A^T(x) = A(x)$, $C^T(x) = C(x)$, *is equivalent to one of the following conditions* [17]:

(i) $A(x) < 0$ *and* $C(x) - B^T(x) A(x)^{-1} B(x) < 0$;
(ii) $C(x) < 0$ *and* $A(x) - B^T(x) C(x)^{-1} B(x) < 0$.

Lemma 2 *If M is a diagonal matrix whose i_kth* ($1 \le i_k \le N$, $1 \le k \le l$, $1 \le l \le N$) *diagonal elements are m, which is a positive constant, and others are* 0, *then for a symmetric matrix G which has the same dimension as matrix M the following two conditions are equivalent when m is large enough* [14]:

(a) $G - M < 0$;
(b) $G_l < 0$

3.2 Synchronization analysis of general model

Theorem 1 *Suppose that Assumptions* 2, 3, 5 *hold. Then, the synchronous solution of the controlled stochastic neural network* (1) *is globally asymptotically stable with adaptive pinning controller* (2) *if provided:*

$$\lambda_{\max}(\bar{A})_l < -(\mu+\gamma) \tag{9}$$

where $\gamma = \frac{1}{2}\sum_{i=1}^{N}\lambda_{\max}(H_i)$, $\bar{A}$ *is a modified matrix of* A, *and its diagonal elements* $\bar{a}_{ii} = v_1 a_{ii}$ *and others* $\bar{a}_{ij} = v_2 a_{ij}$.

Proof Construct the following Lyapunov candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}(x_i(t)-x_0(t))^T(x_i(t)-x_0(t)) + \sum_{k=1}^{l}\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t)-\alpha)^2$$

where α is a positive constant which is large enough. By *Itô*'s differential formula [18], the stochastic derivative of V along the solution of (1) can be obtained as

$$\begin{aligned} dV = & \sum_{i=1}^{N}(x_i(t)-x_0(t))^T\Bigg[f(x_i(t),t)-f(x_0(t),t) \\ & + \sum_{j=1}^{N}a_{ij}g(x_j(t)) + u_i(t)\Bigg]dt \\ & + \sum_{i=1}^{N}(x_i(t)-x_0(t))^T\sigma_i(x_i(t)-x_0(t))d\omega \\ & + \sum_{k=1}^{l}(\alpha_{i_k}(t)-\alpha)\|x_{i_k}(t)-x_0(t)\|^2 dt \\ & + \frac{1}{2}\sum_{i=1}^{N}\mathrm{trace}[\sigma^T{}_i(x_i(t)-x_0(t))\sigma_i(x_i(t)-x_0(t))]dt \\ = & \ LVdt + \sum_{i=1}^{N}(x_i(t)-x_0(t))^T\sigma_i(x_i(t)-x_0(t))d\omega, \end{aligned}$$

where L is diffusion operator. Then, taking Assumption 2, 3, 5 into account, we have

$$LV \le \Delta^T(t)\left[\left(\mu + \frac{1}{2}\sum_{i=1}^{N}\lambda_{\max}(H_i)\right)I_N + \bar{A} - \Lambda\right]\Delta(t)$$

where $\Delta(t) = (\|x_1(t)-x_0(t)\|, \|x_2(t)-x_0(t)\|, \ldots, \|x_N(t)-x_0(t)\|)^T$, $\Lambda \in R^{N\times N}$ is a diagonal matrix, whose i_kth $(1 \le k \le l)$ elements are α while others are 0. Since (9) is satisfied, we have $\bar{A} + (\mu+\gamma)I_N - \Lambda < 0$ according to Lemma 2 if α is large enough. Then, it follows $E[dV] = E[LVdt] < 0$. From the above analysis, it is easy to show the synchronous solution of stochastic neural network (1) with pinning controller (2) is globally asymptotically stable. This completes the proof. □

From this conclusion above, we know that if l nodes can be found satisfying the mild topology-based condition (9), synchronization of this type of neural network (1) with adaptive pinning control can be achieved. Also, it should be noted that the condition is sufficient but not necessary. In other words, it means that sometimes even if the condition is not satisfied, the stochastic neural network (1) may also reach synchronization.

Next we will consider the ability of synchronous solution of the delayed stochastic neural network (3) with pinning controller (2). Since the noise intensity vector function $\sigma_i' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ enters into two factors, $e_i(t)$ and $e_i(t-\tau(t))$, a new assumption about σ'_i is given as follows:

Assumption 6 Suppose that the noise vector function σ'_i in delayed model (3) is locally Lipschitz continuous and satisfies

$$\begin{aligned} & \mathrm{trace}\left[\sigma_i'^T(e_i(t), e_i(t-\tau(t)))\sigma_i'(e_i(t), e_i(t-\tau(t)))\right] \\ & \quad \le e_i(t)^T H_i^1 e_i(t) + e_i(t-\tau(t))^T H_i^2 e_i(t-\tau(t)) \end{aligned} \tag{10}$$

where $i = 1, \ldots, N$, H_i^1 and H_i^2 are positive semi-definite. Then we have the following results.

Theorem 2 *Suppose that Assumptions* 1–4, 6 *hold. Then, the synchronous solution of the delayed stochastic neural network* (3) *is globally asymptotically stable with adaptive pinning controller* (2) *if provided:*

$$\lambda_{\max}\left(\bar{A} + \frac{\bar{B}^2}{4[-\rho(1-\theta)+\lambda_2]}\right)_l < -(\mu+\rho+\lambda_1) \tag{11}$$

and

$$-\rho(1-\theta) + \lambda_2 < 0 \tag{12}$$

where $\lambda_1 = \frac{1}{2}\sum_{i=1}^{N}\lambda_{\max}(H_i^1)$, $\lambda_2 = \frac{1}{2}\sum_{i=1}^{N}\lambda_{\max}(H_i^2)$, ρ *can be any positive constant,* $\bar{B}$ *is a modified matrix of* B, and its diagonal elements $\bar{b}_{ii} = \omega_1 b_{ii}$ *and others* $\bar{b}_{ij} = \omega_2 b_{ij}$.

Proof Consider a Lyapunov candidate as

$$\begin{aligned} V(t) = & \frac{1}{2}\sum_{i=1}^{N}(x_i(t)-x_0(t))^T(x_i(t)-x_0(t)) \\ & + \sum_{k=1}^{l}\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t)-\alpha)^2 \\ & + \rho\sum_{i=1}^{N}\int_{t-\tau(t)}^{t}(x_i(s)-x_0(s))^T(x_i(s)-x_0(s))ds \end{aligned}$$

where ρ can be any positive constant, α is a positive constant large enough. Similarly, by *Itô*'s differential formula, the stochastic derivative of V along the solution of (3) can be obtained as:

$$
\begin{aligned}
dV &= \sum_{i=1}^{N} e_i^T(t)\Bigg[f(x_i(t),t) - f(x_0(t),t) \\
&\quad + \sum_{j=1}^{N} a_{ij} g(x_j(t)) + u_i(t)\Bigg]dt \\
&\quad + \sum_{i=1}^{N} e_i^T(t)\sigma_i'(e_i(t), e_i(t-\tau(t)))d\omega \\
&\quad + \sum_{k=1}^{l} (\alpha_{i_k}(t) - \alpha)\|e_{i_k}(t)\|^2 dt \\
&\quad + \sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} b_{ij} h(x_j(t-\tau(t)))dt \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\|\sigma_i'(e_i(t), e_i(t-\tau(t)))\|^2 dt \\
&\quad + \rho\sum_{i=1}^{N}[\|e_i(t)\|^2 - (1-\dot{\tau}(t))\|e_i(t-\tau(t))\|^2]dt \\
&= LVdt + \sum_{i=1}^{N}(x_i(t) - x_0(t))^T \sigma_i'(e_i(t), e_i(t-\tau(t)))d\omega
\end{aligned}
$$

Then taking Assumption 1–4, 6 into account, we have

$$
LV \le \begin{pmatrix} \Delta(t) \\ \Delta(t-\tau(t)) \end{pmatrix}^T M \begin{pmatrix} \Delta(t) \\ \Delta(t-\tau(t)) \end{pmatrix},
$$

where

$$
M = \begin{pmatrix} (\mu+\rho+\lambda_1)I_N + \bar{A} - \Lambda & \frac{1}{2}\bar{B} \\ \frac{1}{2}\bar{B} & [-\rho(1-\theta)+\lambda_2]I_N \end{pmatrix}.
$$

According to Lemmas 1 and 2, provided with condition (11) and condition (12), we have $E[dV] = E[LVdt] < 0$. Hence, the stochastic neural network (3) has a globally asymptotically stable synchronous solution $x_0(t)$ under pinning control (2). This completes the proof.

Also, the condition is only a sufficient criteria. In this theorem, we give an possible approach to help the neural network to realize synchronization by selecting proper intensity function of noise without changing the structure of network. It is especially useful when the network structure itself has “limitation”. □

3.3 Synchronization analysis of a specific neurobiological network

In this section, Hindmarsh-Rose (HR) model, a concrete stochastic neural network under adaptive control will be discussed. It is a mathematical model abstracted from neurobiological network and reflects the electrochemical properties of sodium and potassium ion flow. Besides, it has been widely applied to proceed research about dynamical behaviors in nonlinear neuroscience since it has a simple form. For a individual HR neuron, its dynamics can be described as:

$$
\dot{x}(t) = f(x(t)),
$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$,

$$
f(x(t)) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} = \begin{pmatrix} x_2(t) - ax_1^3(t) + bx_1^2(t) + I_{\text{ext}} - x_3(t) \\ c - dx_1^2(t) - x_2(t) \\ r(S(x_1(t) - e) - x_3(t)) \end{pmatrix}.
$$

In this system, $x_1(t)$ represents the membrane action potential, $x_2(t)$ is a recovery variable associated with fast current, $x_3(t)$ is a slowly changing adaptation current. a, b, c, d, e, r, S are constants, and it should be noted that a, b, c, d, r, S are often chosen as positive constants, while e is negative one [19–22]. I_{ext} is the external direct current. Here, we consider the coupled input from other neurons and added adaptive control as the I_{ext}. From the description above, we can find that neurons are coupled with the membrane potential in H-R neuron model. In addition, only the membrane potentials can be observed by in practical experiments. Therefore, we can construct H-R stochastic neuron networks as (1) and (3), where out-coupling vector function $g(x_j(t))$, delayed inner-coupling vector function $h(x_j(t))$ and noise intensity vector functions $\sigma_i(x_i(t) - x_0(t)), \sigma_i'(e_i(t), e_i(t-\tau(t)))$, can be designed as

$$
g(x_j(t)) = \begin{pmatrix} g_1(x_{j1}(t)) \\ 0 \\ 0 \end{pmatrix},
$$

$$
h(x_j(t-\tau(t))) = \begin{pmatrix} h_1(x_{j1}(t-\tau(t))) \\ 0 \\ 0 \end{pmatrix},
$$

$$
\sigma_i(x_i(t) - x_0(t)) = \begin{pmatrix} \sigma_{i1}(x_{i1}(t) - x_{01}(t)) \\ 0 \\ 0 \end{pmatrix},
$$

$$
\sigma_i'(e_i(t), e_i(t-\tau(t))) = \begin{pmatrix} \sigma_{i1}'(e_{i1}(t), e_{i1}(t-\tau(t))) \\ 0 \\ 0 \end{pmatrix}.
$$

To realize synchronization in this kind of H-R stochastic neural network, construct controller as follows:

$$
\begin{aligned}
u_{i_k} &= \begin{pmatrix} -\alpha_{i_k}(x_{i_k1}(t) - x_{01}(t)) \\ 0 \\ 0 \end{pmatrix} \qquad 1 \le k \le l \\
\dot{\alpha}_{i_k} &= \beta_{i_k}\|x_{i_k1}(t) - x_{01}(t)\|^2 \\
&= \beta_{i_k}(x_{i_k1}(t) - x_{01}(t))^T(x_{i_k1}(t) - x_{01}(t)) \\
u_{i_k} &= 0 \qquad \text{otherwise.}
\end{aligned} \tag{13}
$$

Next we introduce three concrete assumptions to discuss the ability of synchronous solution of H-R stochastic neural network without time-delay under adaptive pinning control (13). It should be noted that these assumptions are similar as the assumptions before, and the reason of outlining these ones here is to attained pinning synchronization theorems the specific H-R neural network since various functions in and controller are more specific than before.

Assumption 3* Assume that $g_1(y)$, where $y \in \mathbb{R}$, satisfies

$$v_1' \le \frac{g_1(y_1) - g_1(y_2)}{y_1 - y_2} \le v_2'$$

for any two scalars $y_1, y_2 \in \mathbb{R}$, where v'_1, v'_2 are two positive constants.

Assumption 4* Assume that $h_1(y)$, where $y \in \mathbb{R}$, satisfies

$$\omega_1' \le \frac{h_1(y_1) - h_1(y_2)}{y_1 - y_2} \le \omega_2'$$

for any two scalars $y_1, y_2 \in \mathbb{R}$, where ω'_1, ω'_2 are two positive constants.

Assumption 5* Assume that $\sigma_{i1}(y)$, where $y \in \mathbb{R}$, satisfies

$$\sigma_{i1}^2(y) \le \chi_i y^2$$

for any scalar $y \in \mathbb{R}$, where χ is a positive constant, $i = 1, 2, \ldots, N$.

Assumption 6* Assume that $\sigma'_{i1}(y, z)$, where $y, z \in \mathbb{R}$, satisfies

$$\sigma_{i1}'^2(y, z) \le \chi_{i1} y^2 + \chi_{i2} z^2$$

for any two scalars $y, z \in \mathbb{R}$, where χ_1, χ_2 are two positive constants, $i = 1, 2, \ldots, N$.

Similarly, denote $\bar{A}'$, in which the diagonal elements a_{ii} are replaced by $v'_1 a_{ii}$ and other a_{ij} are replaced by $v'_2 a_{ij}$, as a modified matrix of A. Also denote $\bar{B}'$, in which the diagonal elements b_{ii} are replaced by $\omega'_1 b_{ii}$ and other b_{ij} are replaced by $\omega'_2 b_{ij}$, as a modified matrix of B. Then, we can attain the globally adaptive pinning synchronization theorems for H-R stochastic neural networks with and without time-delay, respectively.

Theorem 3 *Assume that Assumption 3*, 5* hold. Then, the synchronous solution of the controlled H-R stochastic neural network without time-delay under pinning control (13), which is described above, is globally asymptotically stable if provided:*

$$\lambda_{\max}(\bar{A}')_l < -\left(2bM + \phi + \left(dM + \frac{1}{2}\right)^2 + \frac{(rs+1)^2}{4r}\right) \tag{14}$$

where $\phi = \frac{1}{2}\sum_{i=1}^{N} \chi_i$, and M is a positive constant and represents the boundary of the membrane potential in H-R neurodynamics.

Proof As two vectors, $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))$ and $x_0(t) = (x_{01}(t), x_{02}(t), x_{03}(t))$ in the H-R equation satisfy the following inequations.

$$\begin{aligned}
&(x_{i1}(t) - x_{01}(t))(f_1(x_i(t) - x_0(t)))\\
&\quad = (x_{i1}(t) - x_{01}(t))(x_{i2}(t) - ax_{i1}^3(t) + bx_{i1}^2(t) - x_{i3}(t)\\
&\qquad - x_{02}(t) + ax_{01}^3(t) - bx_{01}^2(t) + x_{03}(t))\\
&\quad = (x_{i1}(t) - x_{01}(t))[(x_{i2}(t) - x_{02}(t)) - a(x_{i1}(t) - x_{01}(t))\\
&\qquad \times (x_{i1}^2(t) + x_{i1}(t)x_{01}(t) + x_{01}^2(t)) + b(x_{i1}(t) - x_{01}(t))\\
&\qquad \times (x_{i1}(t) + x_{01}(t)) - (x_{i3}(t) - x_{03}(t))]\\
&\quad \le 2bM|x_{i1}(t) - x_{01}(t)| + |x_{i1}(t) - x_{01}(t)||x_{i2}(t) - x_{02}(t)|\\
&\qquad + |x_{i1}(t) - x_{01}(t)||x_{i3}(t) - x_{03}(t)|
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
&(x_{i2}(t) - x_{02}(t))(f_2(x_i(t) - x_0(t))) \le 2dM|x_{i2}(t)\\
&\quad - x_{02}(t)||x_{i1}(t) - x_{01}(t)| - |x_{i2}(t) - x_{02}(t)|^2
\end{aligned}$$

and

$$\begin{aligned}
&(x_{i3}(t) - x_{03}(t))(f_3(x_i(t) - x_0(t))) \le rS|x_{i3}(t)\\
&\quad - x_{03}(t)||x_{i1}(t) - x_{01}(t)| - r|x_{i3}(t) - x_{01}(t)|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&\sum_{i=1}^{N}(x_i - x_0)(f(x_i) - f(x_0))\\
&\quad \le \sum_{i=1}^{N}\begin{pmatrix}|x_{i1} - x_{01}|\\ |x_{i2} - x_{02}|\\ |x_{i3} - x_{03}|\end{pmatrix}^T K \begin{pmatrix}|x_{i1} - x_{01}|\\ |x_{i2} - x_{02}|\\ |x_{i3} - x_{03}|\end{pmatrix}
\end{aligned}$$

where

$$K = \begin{pmatrix} 2bM & dM + \frac{1}{2} & \frac{rS+1}{2}\\ dM + \frac{1}{2} & -1 & 0\\ \frac{rS+1}{2} & 0 & -r \end{pmatrix}.$$

Construct a Lyapunov candidate

$$\begin{aligned}
V(t) &= \frac{1}{2}\sum_{i=1}^{N}(x_i(t) - x_0(t))^T(x_i(t) - x_0(t))\\
&\quad + \sum_{k=1}^{l}\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t) - \alpha)^2
\end{aligned}$$

where α is a positive constant which is large enough. Similarly, we can obtain the expectation of stochastic derivative of V along the trajectory (1), (13) and (13) as follows:

$$E[dV] = E[LV] \le \xi_1^T G_1 \xi_1$$

where $\xi_1=(|x_{11}(t)-x_{01}(t)|,\ldots,|x_{N1}(t)-x_{01}(t)|,|x_{12}(t)-x_{02}(t)|,\ldots,|x_{N2}(t)-x_{02}(t)|,|x_{13}(t)-x_{03}(t)|,\ldots,|x_{N3}(t)-x_{03}(t)|)^T$,

$$G_1=\begin{pmatrix} \bar{A}'+2bMI_N-\Lambda+\phi & (dM+\frac{1}{2})I_N & \frac{rS+1}{2}I_N \\ (dM+\frac{1}{2})I_N & -I_N & 0 \\ \frac{rS+1}{2}I_N & 0 & -rI_N \end{pmatrix},$$

and Λ is a diagonal matrix whose i_kth elements are α and the others are 0. □

Provided with (14) in Theorem 3, we have

$$\bar{A}'+\left(2bM+\phi+\left(dM+\frac{1}{2}\right)^2+\frac{(rs+1)^2}{4r}\right)I_N-\Lambda<0$$

when α is large enough according to Lemma 2. Then, it follows $G_1<0$ according to Lemma 1. Hence, the synchronous solution of the stochastic neural network without time-delay under pinning control (13) is globally asymptotically stable. The proof is completed.

Theorem 4 *Assume that Assumptions 1, 3*, 4*, 6* hold. Then, the synchronous solution of the controlled H-R stochastic neural network with time-delay under pinning control (13), which is described above, is globally asymptotically stable if provided:*

$$\lambda_{\max}\left(\bar{A}'+\frac{\bar{B}'^2}{4[-\rho(1-\theta)+\lambda_2']}\right)_l<-\left(\rho+\lambda_1'+\Omega\right) \tag{15}$$

$$-\rho(1-\theta)+\lambda_2'<0 \tag{16}$$

where $\lambda_1'=\frac{1}{2}\sum_{i=1}^N\chi_{i1}$, $\lambda_2'=\frac{1}{2}\sum_{i=1}^N\chi_{i2}$, $\Omega=2bM+(dM+\frac{1}{2})^2+(rs+1)^2/4r$ *and* ρ *is a positive constant.*

Proof Consider a Lyapunov candidate as

$$\begin{aligned} V(t)=&\frac{1}{2}\sum_{i=1}^N(x_i(t)-x_0(t))^T(x_i(t)-x_0(t))\\ &+\sum_{k=1}^l\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t)-\alpha)^2\\ &+\rho\sum_{i=1}^N\int_{t-\tau(t)}^t(x_{i1}(s)-x_{01}(s))^2\mathrm{d}s \end{aligned}$$

Following the same method as the proof of Theorem 3, we can obtain the result described in Theorem 4. In addition, it should be noted that the conditions in the two theorems are just sufficient. □

4 Numerical simulation

In this section, we will give two examples to justify theorems obtained above. Although four theorems are obtained in this paper, the last two are the special examples of the first two if Assumption 2 is satisfied. For simplicity, we consider stochastic neural network consisting of three nodes.

First consider a H-R network (1), (13) without time-delay under pinning control (13). Set parameters in a H-R equation as follows: $a=1$, $b=3$, $c=1$, $d=3$, $r=0.6$, $S=1.0$, $e=-1.6$. Assume that the inner-coupling matrix is

$$A=\begin{pmatrix} -5 & 0 & 5 \\ 0 & -6 & 6 \\ 5 & 6 & -11 \end{pmatrix}.$$

Besides, $g_1(x_{i1})=x_{i1}(1\le i\le 3)$ is assumed to be the inner-coupling function, and $\sigma_{i1}=(x_{i1}(t)-x_{01}(t))^2$ $(1\le i\le 3)$ is the noise intensity function. It is easy to verify that *Assumption* 3*, 5* hold and $v'_1=v'_2=1$, $\chi_i=1(1\le i\le 3)$. Choose initial values as $x_i=(0.1+0.1i,\ 0.2+0.1i,\ 0.3+0.1i)(0\le i\le 3)$ and set $\alpha_{i_k}(0)=\beta_{i_k}=1$ for $1\le k\le 2$ in the simulation. In this case, the bound M of the first variable in H-R equation is 0.4. By simple calculation, we can get $-(2bM+\phi+(dM+\frac{1}{2})^2+\frac{(rs+1)^2}{4r})=-7.8567$. Since $\lambda_{\max}(\bar{A})'_2=-11<-7.8567$ when the first two nodes are pinned, the stochastic network can reach synchronization under pinning control (13) according to *Theorem* 3. The synchronization errors $x_{ij}(t)-x_{0j}(t)(1\le j\le 3,\ 1\le i\le 3,)$ of H-R stochastic neural network without time-delay under pinning control are displayed in Figs. 1, 2, and 3, respectively.

Remark 1 As mentioned above, the *Theorem* 3 is the special situation of *Theorem* 1. Therefore, the example showed above can also verify the *Theorem* 1, since it satisfies *Assumption* 2. For $\lambda_{\max}(K)=3.2469>0$, there exists a positive constant μ satisfying $(\xi_1-\xi_2)^T(f(\xi_1,t)-f(\xi_2,t))\le\mu(\xi_1-\xi_2)^T(\xi_1-\xi_2)$. Actually, μ can be selected as $\lambda_{\max}(K)$. Actually, $-(\mu+\gamma)=-4.7496$,

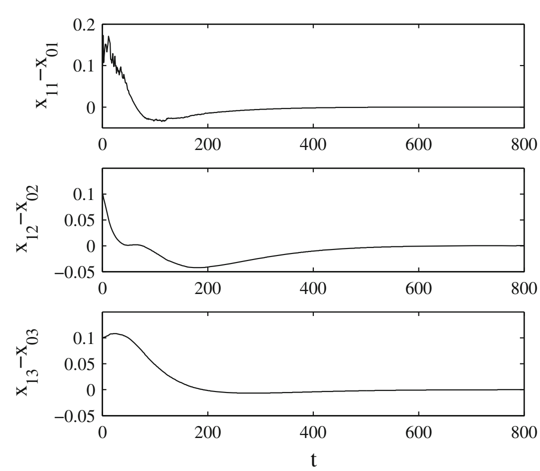

Fig. 1 Synchronization errors of the first neuron in the stochastic H-R network without time-delay for pinning the first two neurons

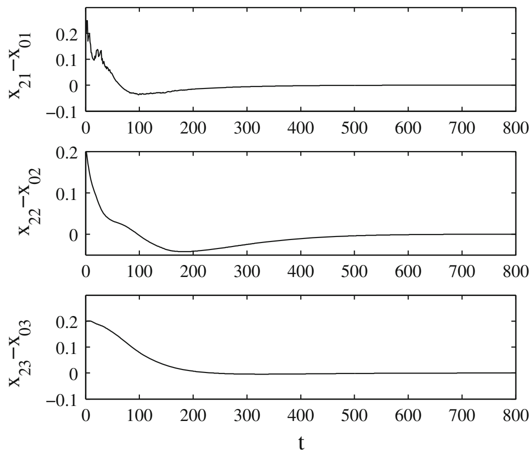

Fig. 2 Synchronization errors of the second neuron in the stochastic H-R network without time-delay for pinning the first two neurons

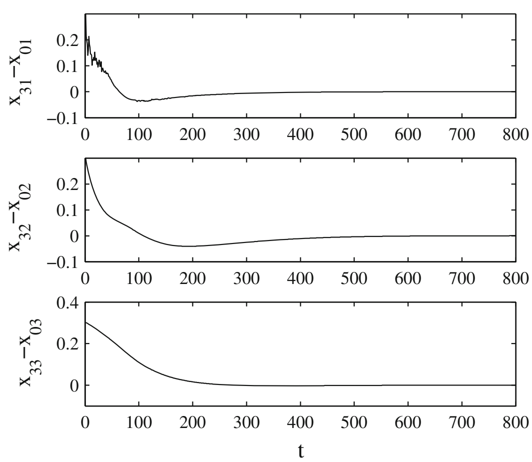

Fig. 3 Synchronization errors of the third neuron in the stochastic H-R network without time-delay for pinning the first two neurons

and $\lambda_{\max}(\bar{A})_2 = -11 < -4.7496$. Therefore, the condition in (9) is satisfied.

Next, consider the H-R network (3), (13) with time-delay under pinning control (13). Also, set parameters in a H-R equation as before, i.e. $a = 1, b = 3, c = 1, d = 3, r = 0.6, S = 1.0, e = -1.6$. Assume that the instantaneous and delayed coupling matrices are

$$A = \begin{pmatrix} -5 & 0 & 5 \\ 0 & -6 & 6 \\ 5 & 6 & -11 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Besides, $g_1(x_{i1}) = h_1(x_{i1}) = x_{i1}(1 \le i \le 3)$ are assumed to be the instantaneous and delayed inner-coupling functions, and $\sigma'_{i1} = (x_{i1}(t) - x_{01}(t))^2 + (x_{i1}(t-\tau(t)) - x_{01}(t-\tau(t)))^2(1 \le i \le 3)$ is the noise intensity function. Suppose the time-delay $\tau(t) = 1$. It is easy to verify that *Assumptions* 1, 3^*, 4^*, 6^* hold and $v'_1 = v'_2 = 1, \omega'_1 =$

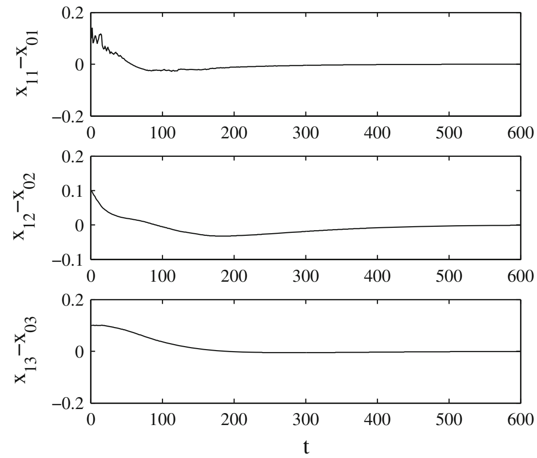

Fig. 4 Synchronization errors of the first neuron in the stochastic H-R network with time-delay for pinning the first two neurons

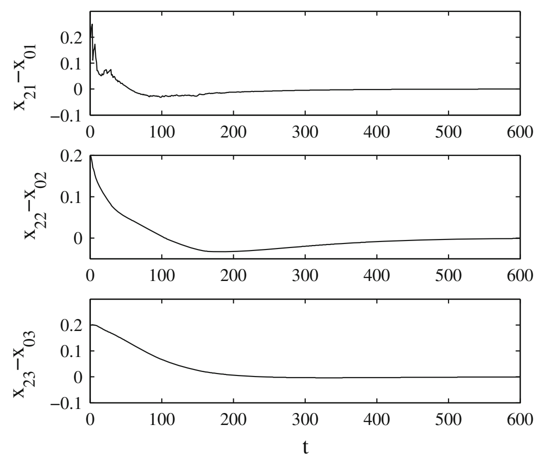

Fig. 5 Synchronization errors of the second neuron in the stochastic H-R network with time-delay for pinning the first two neurons

$\omega'_2 = 1, \chi_{i1} = \chi_{i2} = 1(1 \le i \le 3)$. Similarly, choose initial values as $x_i = (0.1 + 0.1i, 0.2 + 0.1i, 0.3 + 0.1i)$ $(0 \le i \le 3)$ and set $\alpha_{i_k}(0) = \beta_{i_k} = 1$ for $1 \le k \le 2$ in the simulation. Selecting $\rho = 1.6$, we can get $-(\rho + \lambda_1{}' + \Omega) = -12.6667$. Since $\lambda_{\max}(\bar{A}' + \frac{\bar{B}'^2}{4[-\rho(1-\theta)+\lambda_2']})_2 = -16 < -12.6667$ when the first two nodes are pinned, the stochastic network can reach synchronization under pinning control (13) according to *Theorem* 4. The synchronization errors $x_{ij}(t) - x_{0j}(t)(1 \le j \le 3, 1 \le i \le 3,)$ of H-R stochastic neural network without time-delay under pinning control are displayed in Figs. 4, 5, and 6, respectively.

Remark 2 This numerical example also verify the *Theorem* 2. Actually, $-(\mu + \rho + \lambda_1) = -6.3496$, and $\lambda_{\max}(\bar{A} + \frac{\bar{B}^2}{4[-\rho(1-\theta)+\lambda_2]})_2 = -16 < -6.3496$. Therefore, the conditions in (11) and (12) are satisfied.

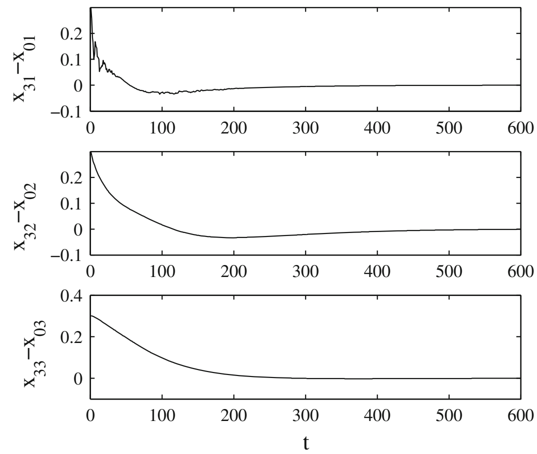

Fig. 6 Synchronization errors of the third neuron in the stochastic H-R network with time-delay for pinning the first two neurons

5 Conclusion

In this paper, we considered pinning synchronization control of stochastic neural networks with and without time-delay. We use Lyapunov functional method and matrix inequality technique to solve this problem. Several sufficient conditions have been derived to ensure the global asymptotical stability for the synchronization error, and thus all the nodes in the network can realize synchronization after a period of time under adaptive pinning control. Besides, a specific stochastic neural network consisting of H-R neurons with pinning controller is discussed and related results are exhibited. Finally, numerical examples are given to verify the theorems proposed above. The criteria obtained in this paper can not only help us verify whether a stochastic network with pinning controller could reach synchronization or not but also give a feasible way to help a network system reach synchronization by adding noises. More importantly, the criteria also provide a method to select the pinning nodes. Since every coupling function in networks is nonlinear and the stochastic elements are involved with the state of networks, a lot of existing conclusions could be included in the framework of this paper.

References

1. Kyrkjeb E, Pettersen KY, Wondergem M, Nijmeijer H (2007) Output synchronization control of ship replenishment operations: theory and experiments. Control Eng Pract 15(6):741–755
2. Chea Y-Q, Wang J, Zhou S-S, Deng B (2009) Robust synchronization control of coupled chaotic neurons under external electrical stimulation. Chaos Solutions Fractals 40(3):1333–1342
3. Chena H-C, Changb J-F, Yanb J-J, Liao T-L (2008) EP-based PID control design for chaotic synchronization with application in secure communication. Expert Syst Appl 34(2):1169–1177
4. Yau H-T (2008) Chaos synchronization of two uncertain chaotic nonlinear gyros using fuzzy sliding mode control. Mech Syst Signal Process 22(2):408–418
5. Lu W, Chen T (2004) Synchronization of coupled connected neural networks with delays. Circuits Syst I Regul Pap IEEE Trans 51(12):2491–2503
6. Chen G, Zhou J, Liu Z (2004) Global synchronization of coupled delayed neural networks and applications to chaotic CNN model. Int J Bifur Chao 14(7):2229–2240
7. Li X, Wang X, Chen G (2004) Pinning a complex dynamical network to its equilibrium. Circuits Syst I Regul Pap IEEE Trans 51(10):2074–2086
8. Guo W, Austin F, Chen S, Sun W (2009) Pinning synchronization of the complex networks with non-delayed and delayed coupling. Phys Lett A 373:1563–1572
9. Wang X, Chen G (2002) Pinning control of scale-free dynamical networks. Physica A 310:521–531
10. Yu W, Chen G, Lü J (2009) On pinning synchronization of complex dynamical networks. Automatica 45:429–435
11. Sorrentino F, Bernardo M, Garofalo F, Chen G (2007) Controllability of complex networks via pinning. Phys Rev E 75(4): 046103-1–046103-6
12. Lu W (2007) Adaptive dynamical networks via neighborhood information: synchronization and pinning control. Chaos 17(2): 023122-1–023122-18
13. Chen T, Liu X, Lu W (2007) Pinning complex networks by a single controller. Circuits Syst I Regul Pap IEEE Trans 54(6): 1317–1326
14. Zhou J, Wu X, Yu W, Small M, Lu J (2008) Pinning synchronization of delayed neural networks. Chaos 18(4):043111-1–043111-9
15. Marcus C, Westervelt R (1989) Stability of analog neural networks with delays. Phys Rev A 39:347–359
16. Roska T, Wu C, Balsi M, Chua L (1992) Stability and dynamics of delay-type cellular neural networks. Circuits Syst I Regul Pap IEEE Trans 39(6):487–490
17. Boyd S, Ghaoui L, Feron E, Balakrishnan V (1994) Linear matrix inequalities in system and control theory. SIAM, Philadelphia
18. Arnold L (1974) Stochastic different equations: theroy and applications. Wiley, London
19. Sabbagh H (2000) Control of chaotic solutions of the Hindmarsh-Rose equations. Chaos Solutions Fractals 11(8):1213–1218
20. Shi X, Lu QS (2005) Coherence resonance and synchronization of Hindmarsh-Rose neurons with noise. Chin Phys 14(6): 1088–1094
21. Shi X, Lu QS (2004) Complete synchronization of coupled Hindmarsh-Rose neurons with ring structure. Chin Phys Lett 21(9):1695–1698
22. Baltanas J, Casado J (2002) Noise-induced resonances in the Hindmarsh-Rose neuronal model. Phys Rev E 65(041915): 1317–1326