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Quotients and weakly algebraic sets in pseudoeffect algebras

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Abstract In the paper, we show that the quotient $[E]_I$ of a lattice-ordered pseudoeffect algebra E with respect to a normal weak Riesz ideal I is linearly ordered if and only if I is a prime normal weak Riesz ideal, and $[E]_I$ is a representable pseudo MV-algebra if and only if I is an intersection of prime normal weak Riesz ideals. Moreover, we introduce the concept of weakly algebraic sets in pseudoeffect algebras, discuss the characterizations of weakly algebraic sets and show that weakly algebraic sets in pseudoeffect algebra E are in a one-to-one correspondence with normal weak Riesz ideals in pseudoeffect algebra E.

Keywords Pseudoeffect algebras · Pseudo MV-algebras · Normal weak Riesz ideals · Quotients · Weakly algebraic sets

1 Definitions and preliminaries

Effect algebra was introduced by Foulis and Bennett (1994) for modelling the set of self-adjoint operators on a Hilbert space lying between zero and identity. Independently, Kôpka and Chovanec introduced essentially equivalent structures called *D*-posets (see Kôpka 1992). Another equivalent structure called weak ortho-algebras was introduced by Giuntini and Greulingin (1989). For basic properties of effect algebras and D-posets, please refer to Dvurečenskij et al. (2000). In 2001, A. Dvurečenskij and T. Vetterlein defined another kind of structure closely connected to effect algebra, which is called pseudoeffect algebra (2001b, c), basically by dropping the axiom of commutativity. This idea is comparable to the way MV-algebra generalizes to pseudo MV-algebra (Dvurečenskij 2001; Georgescu and Iorgulescu 2001). Since 1994, there have been many papers studying the congruences, ideals and quotients in effect algebras (Avallone and Vitolo 2003; Chevalier and Pulmannová 2000; Jenča 2001; Jenča and Pulmannová 1999, Gudder and Pulmannová 1997) and in pseudoeffect algebras (Dvurečenskij and Vetterlein 2001a; Li and Li 2008), in which the order isomorphism relation between the Riesz congruences and Riesz ideals in effect algebra was established by Avallone and Vitolo (2003), the order isomorphism relation between the Riesz strong congruences and the normal weak Riesz ideals in pseudoeffect algebras was established by Li and Li (2008), and it was proved that the quotient $[E]_I$ of a lattice-ordered effect algebra E with respect to a Riesz ideal I is linearly ordered if and only if I is a prime Riesz ideal and $[E]_I$ is an MV-algebra if and only if I is an intersection of prime Riesz ideals by Jenča and Pulmannová (1999). Moreover, the congruences, quotients and weakly algebraic sets in partial abelian semigroup were studied by Pulmannová (1997), in which some conditions under which a congruence coincides with a perspectivity with respect to an appropriate set M were shown and, especially, conditions under which the corresponding quotient is a *D*-poset were found.

Inspired by Jenča and Pulmannová (1999) and Pulmannová (1997), in the paper, we study the quotient $[E]_I$ of a lattice-ordered pseudoeffect algebra E with respect to a normal weak Riesz ideal I in Sect. 2 and prove that the

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quotient $[E]_I$ is linearly ordered if and only if I is a prime normal weak Riesz ideal, and $[E]_I$ is a representable pseudo MV-algebra if and only if I is an intersection of prime normal weak Riesz ideals. The results extend those obtained in Jenča and Pulmannová (1999) for lattice-ordered effect algebras. In Sect. 3, we introduce the concept of weakly algebraic sets in pseudoeffect algebras, discuss the characterizations of weakly algebraic sets and show that weakly algebraic sets in pseudoeffect algebra E are in a one-to-one correspondence with normal weak Riesz ideals in pseudoeffect algebra E.

Definition 1.1 A partial algebra $(E, \oplus, 0, 1)$, where $\oplus$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra* (Dvurečenskij and Vetterlein 2001b, c) if, for all $a, b, c \in E$, the following hold:

(1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist if and only if $b \oplus c$ and $a \oplus (b \oplus c)$ exist, and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(2) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a \oplus d = e \oplus a = 1$;
(3) if $a \oplus b$ exists, there are elements $d, e \in E$ such that $a \oplus b = d \oplus a = b \oplus e$;
(4) if $1 \oplus a$ or $a \oplus 1$ exists, then $a = 0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a \oplus c = b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \le E$. It is possible to show that $a \le b$ if and only if $b = a \oplus c = d \oplus a$ for some $c, d \in E$. We write $c = b \backslash a$ and $d = b/a$. Then

$$(b/a) \oplus a = a \oplus (b \backslash a) = b$$

and we write $a^\sim = 1 \backslash a$, $a^- = 1/a$. We say that a pseudoeffect algebra $(E, \oplus, 0, 1)$ is a *lattice-ordered* pseudoeffect algebra if $a \wedge b$, and dually $a \vee b$ exist for any $a, b \in E$ with respect to the above partial order $\le$.

For basic properties of pseudoeffect algebras see Dvurečenskij and Vetterlein (2001b, c). We recall that if $\oplus$ is commutative, then$(E, \oplus, 0, 1)$ is said to be an *effect algebra*. For example, if $(G; u)$ is a unital (not necessary Abelian) po-group with a strong unit u (in fact it is sufficient to take a positive element u in G), and

$$\Gamma(G, u) = \{g \in G \mid 0 \le g \le u\},$$

then $(\Gamma(G, u), \oplus, 0, u)$ is a pseudoeffect algebra if we restrict the group addition $\oplus$ to $\Gamma(G, u)$.

In the following, we denote by E the pseudoeffect algebra $(E, \oplus, 0, 1)$.

It is easy to check the following proposition:

Proposition 1.2 *Let E be a pseudoeffect algebra. For all $a, b, c \in E$, we have*

(1) *If $a \le b$, then $b/a, b \backslash a \le b$ and $b \backslash (b/a) = b/(b \backslash a) = a$.*
(2) $a/a = a \backslash a = 0$.
(3) *If $a, b \le c$, and $c/a = c/b$ $(c \backslash a = c \backslash b)$, then $a = b$.*
(4) *If $a \le b \le c$, then $b/a \le c/a$, $b \backslash a \le c \backslash a$ and $(c/a)/(b/a) = c/b$, $(c \backslash a) \backslash (b \backslash a) = c \backslash b$.*
(4′) *If $a \le b \le c$, then $c/b \le c/a$, $c \backslash b \le c \backslash a$ and $(c/a) \backslash (c/b) = b/a$, $(c \backslash a)/(c \backslash b) = b \backslash a$.*
(5) *If $c \le a, b$, and $a/c = b/c$ (or $a \backslash c = b \backslash c$), then $a = b$.*
(6) *If $b \le c$ and $a \le c \backslash b$, then $b \le c/a$ and $(c/a) \backslash b = (c \backslash b)/a$.*
(6′) *If $b \le c$ and $a \le c/b$, then $b \le c \backslash a$ and $(c \backslash a)/b = (c/b) \backslash a$.*
(7) *$a \oplus b \le c$ if and only if $a \le c/b$ if and only if $b \le c \backslash a$.*
(8) *If $b \oplus a \le c$, then $(c \backslash b) \backslash a = c \backslash (b \oplus a)$ and $(c/a)/b = c/(b \oplus a)$.*
(9) *If $a \oplus b$ exists and $c \le b$, then $(a \oplus b)/c = a \oplus (b/c)$.*
(9′) *If $a \oplus b$ exists and $c \le a$, then $(a \oplus b) \backslash c = (a \backslash c) \oplus b$.*
(10) *If $a \le c, b \le d$ and $c \oplus d$ exists, then $(c \oplus d) \backslash (a \oplus b) = ((c \backslash a) \oplus d) \backslash b$.*

Moreover, if E is lattice-ordered, then

(11) $a/(a \wedge b) \vee b/(a \wedge b) = (a \vee b)/(a \wedge b); a \backslash (a \wedge b) \vee b \backslash (a \wedge b) = (a \vee b) \backslash (a \wedge b)$.

Definition 1.3 (*Georgescu and Iorgulescu* 2001) A *pseudo MV-algebra* is an algebra $(M, \oplus, -, \sim, 0, 1)$ of type (2, 1, 1, 0, 0) such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 1^- = 0$;
(A5) $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

We recall that a mapping $h : M_1 \longrightarrow M_2$ of two pseudo MV-algebras M_1 and M_2 is said to be a *homomorphism* if h preserves $\oplus, -, \sim, 0$ and 1. Let $\{(M_t, \oplus_t, -_t, \sim_t, 0_t, 1_t)\}_{t \in T}$ be a family of pseudo MV-algebras. The Cartesian product $M = \prod_{t \in T} M_t$, where $\oplus, -, \sim, 0$ and 1 are defined in a usual way by coordinates, is said to be a direct product of $\{(M_t, \oplus_t, -_t, \sim_t, 0_t, 1_t)\}_{t \in T}$. A pseudo

MV-algebra M is a *subdirect product* of a family of $\{(M_t, \oplus_t, -_t, \sim_t, 0_t, 1_t)\}_{t\in T}$ of pseudo MV-algebras iff there exists a one-to-one homomorphism $h: M \longrightarrow \prod_{t\in T} M_t$ of pseudo MV-algebras such that, for each $t \in T$, $\pi_t \circ h$ is a homomorphism from M onto M_t, where π_t is the t-th projection of $\prod_{t\in T} M_t$ onto M_t.

According to Georgescu and Iorgulescu (2001), we say that a pseudo MV-algebra M is *representable* if it can be represented as a subdirect product of linear pseudo MV-algebras.

If M is a pseudo MV-algebra, then the partial operation $a \oplus b$ is defined if and only if $a \leq b^-$, and then $(M, \oplus, 0, 1)$ is a pseudoeffect algebra. According to Dvurečenskij and Vetterlein (2001c), a lattice-ordered pseudoeffect algebra is a pseudo MV-algebra iff $a\backslash(a \wedge b) = (a \vee b)\backslash b$ equivalently, iff $a/(a \wedge b) = (a \vee b)/b$. By this conclusion, it is easy to prove that a linearly ordered pseudoeffect algebra and the direct product of a family of linearly ordered pseudoeffect algebras are all pseudo MV-algebras.

A binary relation $\sim$ on a pseudoeffect algebra E is a *congruence* if the following conditions are satisfied:

(C1) $\sim$ is an equivalence relation.

(C2) For any $a, a', b, b' \in E$ such that $a \sim a'$, $b \sim b'$ and $a \oplus b$, $a' \oplus b'$ both exist, we have $a \oplus b \sim a' \oplus b'$.

Let $[a]_\sim = \{a' \mid a' \sim a\}$ be the equivalence class of an $a \in E$, and let $[E]_\sim = \{[a]_\sim \mid a \in E\}$ be the set of all equivalence classes. Let $\oplus$ be the partial binary operation on $[E]_\sim$ such that, for any $a, b, c \in E$, $[a]_\sim \oplus [b]_\sim$ is defined and equals $[c]_\sim$ if and only if for some $a' \sim a$ and $b' \sim b$, $a' \oplus b'$ is defined and $a' \oplus b' \sim c$. Then the partial algebra $([E]_\sim, \oplus, [0]_\sim, [1]_\sim)$ is called the quotient of E induced by the congruence $\sim$. Owing to (C2), the partial binary operation $\oplus$ on $[E]_\sim$ is well defined, but $([E]_\sim, \oplus, [0]_\sim, [1]_\sim)$ need not be a pseudoeffect algebra (Dvurečenskij and Vetterlein 2001a, Example 3.2). If $([E]_\sim, \oplus, [0]_\sim, [1]_\sim)$ is a pseudoeffect algebra, then $\sim$ is called a *PE-algebra congruence* (see Dvurečenskij and Vetterlein 2001a).

We say that a congruence $\sim$ is a *strong congruence* (see Li and Li 2008) if it satisfies

(C3) If $a \oplus b$ exists, then for any $a' \sim a$, there is a $b' \sim b$ such that $a' \oplus b'$ exists and for any $b'' \sim b$, there is a $a'' \sim a$ such that $a'' \oplus b''$ exists.

If $\sim$ is a strong congruence, then $([E]_\sim, \oplus, [0]_\sim, [1]_\sim)$ is a pseudoeffect algebra (Dvurečenskij and Vetterlein 2001a, Proposition 3.3(ii)), i.e., a strong congruence is a PE-algebra congruence. Moreover, condition (C3) is equivalent to the following two conditions (see Li and Li 2008):

(C4) If $a \sim b$, then $a^\sim \sim b^\sim$, $a^- \sim b^-$.

(C5) If $a \sim b \oplus c$, then there exist $a_1, a_2 \in E$ such that $a_1 \sim b$, $a_2 \sim c$ and $a = a_1 \oplus a_2$.

And, condition (C4) is equivalent to

(C6) If $a \sim b$ and $a \oplus a_1 \sim b \oplus b_1$ or $a_1 \oplus a \sim b_1 \oplus b$, then $a_1 \sim b_1$.

We will say that a strong congruence $\sim$ on E is a *Riesz strong congruence* (see Li and Li 2008) provided that the following holds for all $a, b \in E$:

(C7) If $a \sim b$, then there exist $c_1, c_2 \in E$ such that $c_1 \oplus a$, $c_1 \oplus b$, $a \oplus c_2$, $b \oplus c_2$ are defined and $c_1 \oplus a \sim 1 \sim c_1 \oplus b$, $a \oplus c_2 \sim 1 \sim b \oplus c_2$.

Definition 1.4 Let E be a pseudoeffect algebra.

(1) A subset I of E is called an *ideal* if (i) for any $a \in E$ and $r \in I$ with $a \leq r$ we have $a \in I$, and (ii) for any $r, s \in I$ such that $r \oplus s$ is defined we have $r \oplus s \in I$.

(2) An ideal I of E is called *normal* if for any $a, r, s \in E$ such that $r \oplus a$ and $a \oplus s$ exist and are equal, we have $r \in I$ if and only if $s \in I$.

(3) An ideal I of E is called *weak Riesz ideal* if for any $a, b \in E$ with $a \oplus b$ defined and for any $r \in I$ such that $r \leq a \oplus b$, there are $r_a, r_b \in I$ such that $r_a \leq a$, $r_b \leq b$ and $r \leq r_a \oplus r_b$.

If I is a normal ideal in E, we can define a binary relation $\sim_I$ on E by $a \sim_I b$ if there are $i, j \in E$, $i \leq a$, $j \leq b$ such that $a/i = b/j$. If $\sim_I$ is a congruence, then the quotient of E induced by $\sim_I$ is called the quotient of E induced by normal ideal I, and denoted by $[E]_I$ or E/I. Furthermore, the following lemmas hold.

Lemma 1.5 (Li and Li 2008) *Let E be a pseudoeffect algebra, $a, b \in E$, and I a normal ideal. If $\sim_I$ is defined as above, then the following statements are equivalent:.*

(1) $a \sim_I b$.

(2) *There are $i, j \in I$ such that $a\backslash i = b\backslash j$.*

(3) *There is $k \in E$ such that a/k, $b/k \in I$.*

(4) *There is $k \in E$ such that $a\backslash k$, $b\backslash k \in I$.*

Lemma 1.6 (Li and Li 2008) *Let I be a normal ideal in E. Then $\sim_I$ is a strong congruence if and only if I is a normal weak Riesz ideal.*

Lemma 1.7 (Li and Li 2008)

(1) *If I is a normal weak Riesz ideal, then $\sim_I$ is a Riesz strong congruence and $[0]_{\sim_I} = I$.*

(2) *If $\sim$ is a Riesz strong congruence, then $I = [0]_\sim$ is a normal weak Riesz ideal and $\sim_I = \sim$.*

2 Quotients in lattice-ordered pseudoeffect algebras

In this section, we study the quotient in lattice-ordered pseudoeffect algebra. Let E be a pseudoeffect algebra, I a normal weak Riesz ideal. Then by Lemma 1.6 $\sim_I$ is a strong congruence and hence the quotient $[E]_I$ (or E/I) is a pseudoeffect algebra.

Let E, F be pseudoeffect algebras. A mapping h: $E \longrightarrow F$ is a *morphism* if $h(1) = 1$, $h(a \oplus b) = h(a) \oplus h(b)$, and a morphism h: $E \longrightarrow F$ is a *homomorphism* if $h(a \wedge b) = h(a) \wedge h(b)$ whenever $a \wedge b$ exists in E.

Recall that an ideal I of a lattice L is *prime* if $a \wedge b \in I \Longrightarrow a \in I$ or $b \in I$.

Lemma 2.1 *Let I be a normal weak Riesz ideal in E. The mapping ϕ: $E \longrightarrow [E]_I$ is defined by $\phi(a) = [a]_I (\forall a \in E)$. We call it the canonical projection. Then ϕ is homomorphism.*

Proof It is easy to check that ϕ is a morphism by Lemma 1.6. Clearly, $[a \wedge b]_I \leq [a]_I,\ [b]_I$. Assume $[x]_I \leq [a]_I,\ [b]_I$ for some $x \in E$. Then there are $x_1 \sim_I x,\ a_1 \sim_I a$ with $x_1 \leq a_1$, which implies that $a_1^{\sim} \sim_I a^{\sim}$ and $x_1 \oplus a_1^{\sim}$ is defined. By (C3), there is $y_1 \sim_I x_1$ with $y_1 \oplus a^{\sim}$ is defined, i.e., $y_1 \leq a$. Similarly, there is $y_2 \sim_I x$ with $y_2 \leq b$. Now $y_1 \sim_I y_2$ means that there are $i,\ j \in I$ such that $y_1/i = y_2/j$. Put $y_1/i = y_2/j = z$, then $z \leq a,\ b$. Therefore, $z \leq a \wedge b$, and $[z]_I = [x]_I \leq [a \wedge b]_I$. It follows that $[a \wedge b]_I$ is the greatest lower bound of $[a]_I,\ [b]_I$ in $[E]_I$. Then ϕ is homomorphism.

Proposition 2.2 *Let I be a normal weak Riesz ideal in a lattice-ordered pseudoeffect algebra E. Then I is a lattice ideal and $\sim_I$ is a lattice congruence. Moreover, for all $a,\ b \in E$, $a \sim_I b$ if and only if $(a/(a \wedge b)) \vee (b/(a \wedge b)) \in I$ (equivalently, $(a \backslash (a \wedge b)) \vee (b \backslash (a \wedge b)) \in I$), i.e., $(a \vee b)/(a \wedge b) \in I$ (equivalently, $(a \vee b) \backslash (a \wedge b) \in I$).*

Proof We have $I = \{a \mid \phi(a) = 0\}$. By Lemma 2.1 ϕ is a lattice homomorphism and hence I is a lattice ideal and $\sim_I$ is a lattice congruence. By Lemma 1.5, $a \sim_I b$ if and only if there exists $k \in E$ such that $k \leq a,\ b$ and $a/k,\ b/k \in I$. Then $k \leq a \wedge b$ and $a/(a \wedge b) \leq a/k,\ b/(a \wedge b) \leq b/k$, and since I is a lattice ideal, we get $(a/(a \wedge b)) \vee (b/(a \wedge b)) = (a \vee b)/(a \wedge b) \in I$. Conversely, take $k = a \wedge b$, then $a/(a \wedge b),\ b/(a \wedge b) \in I$ and hence $a \sim_I b$.

Theorem 2.3 *A quotient $[E]_I$ of a lattice-ordered pseudoeffect algebra with respect to a normal weak Riesz ideal I is a linearly ordered pseudoeffect algebra if and only if I is a prime normal weak Riesz ideal.*

Proof Let I be a prime normal weak Riesz ideal. By Lemma 1.6, the quotient $[E]_I$ is a pseudoeffect algebra. Let $a,\ b \in E$, then $(a/(a \wedge b)) \wedge (b/(a \wedge b)) = 0$ implies that $a/(a \wedge b) \in I$ or $b/(a \wedge b) \in I$, which entails $a \sim_I a \wedge b$ or $b \sim_I a \wedge b$. Hence $[a]_I \leq [b]_I$ or $[b]_I \leq [a]_I$, i.e., $[E]_I$ is a linearly ordered pseudoeffect algebra.

Conversely, let the quotient $[E]_I$ be linearly ordered. Assume $a \wedge b \in I,\ a,\ b \in E$. Then $[a]_I \leq [b]_I$ implies $[a]_I = [a \wedge b]_I$, i.e., $a \sim_I a \wedge b$. Similarly, $[b]_I \leq [a]_I$ implies $b \sim_I a \wedge b$. It follows $a \in I$ or $b \in I$; hence I is a prime normal weak Riesz ideal.

Lemma 2.4 *Let E, F be lattice-ordered pseudoeffect algebras, and h: $E \longrightarrow F$ a homomorphism. Then $I_h = \{a \in E \mid h(a) = 0\}$, the kernel of h, is a normal weak Riesz ideal, and $a \sim_{I_h} b$ if and only if $h(a) = h(b)$. Moreover, if F is linearly ordered, then I_h is a prime normal weak Riesz ideal.*

Proof Clearly, I_h is an ideal of lattice-ordered pseudoeffect algebra E. Let $x,\ y,\ z \in E$ and $x \oplus y = y \oplus z$. If $x \in I_h$, then $h(y) = h(x) \oplus h(y) = h(y) \oplus h(z)$ and hence $h(z) = 0$, which implies $z \in I_h$. Similarly, if $z \in I_h$, then $x \in I_h$. So I_h is a normal ideal.

To prove I_h is a normal weak Riesz ideal, it suffices to show that $\sim_{I_h}$ is a strong congruence by Lemma 1.6. First, we show that $a \sim_{I_h} b$ if and only if $h(a) = h(b)$. Indeed, let $a \sim_{I_h} b$. Then there are $i,\ j \in I_h$ with $a/i = b/j$. It follows $h(a) = h((a/i) \oplus i) = h(a/i) = h(b/j) = h(b)$. Conversely, assume that $h(a) = h(b)$. Then $h(a/(a \wedge b)) \oplus h(a \wedge b) = h(a) = h(a) \wedge h(b) = h(a \wedge b)$ by h is a homomorphism, which implies $h(a/(a \wedge b)) = 0$. Thus $a/(a \wedge b) \in I_h$. Similarly $b/(a \wedge b) \in I_h$. Therefore, $a \sim_{I_h} b$ by Lemma 1.5. Second, we show that $\sim_{I_h}$ is a strong congruence. The proof of (C1) and (C2) is straightforward. It suffices to verify condition (C3). Assume $a \sim_{I_h} b$ and $b \oplus c$ is defined. Put $d = c \wedge a^{\sim}$. Then $d^{-} = (c \wedge a^{\sim})^{-} \geq a^{\sim -} = a$, and hence $a \oplus d$ is defined. And $h(d) = h(c) \wedge h(a^{\sim}) = h(c) \wedge h(b^{\sim}) = h(c)$, which implies $d \sim_{I_h} c$. Assume $a \sim_{I_h} b$ and $c \oplus b$ is defined. Put $d = c \wedge a^{-}$. Then $d^{\sim} = (c \wedge a^{-})^{\sim} \geq a^{- \sim} = a$ and hence $d \oplus a$ is defined. And $h(d) = h(c) \wedge h(a^{-}) = h(c) \wedge h(b^{-}) = h(c)$, which implies $d \sim_{I_h} c$. Therefore, $\sim_{I_h}$ is a strong congruence.

Let F be linearly ordered and $a \wedge b \in I_h$. Then $h(a \wedge b) = h(a) \wedge h(b) = h(a) (\text{or}\, h(b)) = 0$. It implies that $a \in I_h$ (or $b \in I_h$), and hence I_h is a prime normal weak Riesz ideal.

Lemma 2.5 *Let I be a normal weak Riesz ideal in a lattice-ordered pseudoeffect algebra E such that $I = \bigcap_{\alpha \in A} I_\alpha$, where I_α is a prime normal weak Riesz ideal for every $\alpha \in A$. Then $I_\alpha/I = \{[a]_I \mid a \in I_\alpha\}$ is a prime normal weak Riesz ideal in E/I for every $\alpha \in A$.*

Proof Applying Lemma 1.6, $\sim_I$ and $\sim_{I_\alpha}$ are all strong congruence and hence $E/I, E/I_\alpha$ are all pseudoeffect algebras, and by Theorem 2.3 E/I_α is linear. Since $I \subset I_\alpha$ for every $\alpha \in A$, we have that if $a \sim_I a_1$, then $a \sim_{I_\alpha} a_1$. Define the mapping h: $E/I \longrightarrow E/I_\alpha$ by $h([a]_I) = [a]_{I_\alpha}$. It is easy to check that h: $E/I \longrightarrow E/I_\alpha$ is a homomorphism.

Then, by Lemma 2.4, $I_h = \{[a]_I \in E/I \mid h([a]_I) = [0]_{I_\alpha}\}$ is a normal weak Riesz ideal. Moreover, we have $I_h = I_\alpha/I$. Indeed, assume that $[a]_I \in I_h$. Then $h([a]_I) = [a]_{I_\alpha} = [0]_{I_\alpha}$, which implies $a \in I_\alpha$ and hence $[a]_I \in I_\alpha/I$. Conversely, if $[a]_I \in I_\alpha/I$, then there exists $a^* \in I_\alpha$ such that $a \sim_I a^*$. Therefore, $h([a]_I) = h([a^*]_I) = [a^*]_{I_\alpha} = [0]_{I_\alpha}$, i.e., $[a]_I \in I_h$. By Theorem 2.3 and Lemma 2.4, we have I_α/I is a prime normal weak Riesz ideal.

Theorem 2.6 *Let I be a normal weak Riesz ideal in a lattice-ordered pseudoeffect algebra E. The quotient E/I is a representable pseudo MV-algebra iff $I = \bigcap_{\alpha\in A} I_\alpha$, where I_α is a prime normal weak Riesz ideal for every $\alpha \in A$.*

Proof Assume $I = \bigcap_{\alpha\in A} I_\alpha$, where I_α is a prime normal weak Riesz ideal for every $\alpha \in A$. We have, by Lemma 2.5, I_α/I is a prime normal weak Riesz ideal in E/I and $E/I_\alpha = (E/I)/(I_\alpha/I)$. Moreover, by Theorem 2.3, for every $\alpha \in A, E/I_\alpha$ is a linear pseudoeffect algebra. Consider $M = \prod_{\alpha\in A}(E/I)/(I_\alpha/I)$. As a direct product of linearly ordered pseudoeffect algebras, M is a pseudo MV-algebra. Define the mapping g: $E/I \longrightarrow M$ by $g([a]_I) = ([a]_I/(I_\alpha/I))_\alpha = ([a]_{I_\alpha})_\alpha$. Then g is a homomorphism. Owing to $I = \bigcap_{\alpha\in A} I_\alpha, g$ is injective. Indeed, $g([a]_I) = g([b]_I)$ implies $[a]_I/(I_\alpha/I) = [b]_I/(I_\alpha/I)$ for all α, and hence $[a]_{I_\alpha} = [b]_{I_\alpha}$ for all α. That is $a \sim_{I_\alpha} b$ for all α, and hence $a \sim_I b$. So $[a]_I = [b]_I$. Moreover, if π_α: $M \longrightarrow (E/I)/(I_\alpha/I)$ is the α-th projection, then $\pi_\alpha \circ g$ is surjective. Therefore E/I is a subdirect product of M, and hence E/I is a representable pseudo MV-algebra.

Conversely, assume that I is a normal weak Riesz ideal of E such that E/I is a representable pseudo MV-algebra. That is E/I is a subdirect product of linearly ordered pseudo MV-algebras M_α. Let ϕ: $E/I \longrightarrow \prod_\alpha M_\alpha$ be the corresponding homomorphism. Then for every α, $\pi_\alpha \circ \phi$ is a homomorphism as well. By Lemma 2.4, the set $I_\alpha = \{[a]_I \in E/I \mid \pi_\alpha \circ \phi([a]_I) = 0_\alpha\}$ is a prime normal weak Riesz ideal of E/I. Since ϕ is injective, we have $a \sim_I b$ if and only if $\phi([a]_I) = \phi([b]_I)$ if and only if $\pi_\alpha \circ \phi([a]_I) = \pi_\alpha \circ \phi([b]_I)$ for every α, i.e., if and only if $a \sim_{I_\alpha} b$ for every α (Lemma 2.4). Therefore $I = \bigcap_{\alpha\in A} I_\alpha$.

3 Congruences and weakly algebraic sets

In this section, inspired by the work of Pulmannová (1997) on congruences, quotients and weakly algebraic sets in partial abelian semigroup, we study congruences and weakly algebraic sets in pseudoeffect algebras and show that weakly algebraic sets in pseudoeffect algebra are in a one-to-one correspondence with normal weak Riesz ideals in pseudoeffect algebra.

Let E be a pseudoeffect algebra and let M be any subset of E. We will say that a is a *left complement* of c and c is a *right complement* of a with respect to M if $a \oplus c \in M$. We will say that $a,\ b \in E$ are *left perspective* with respect to M if they share a common left complement, that is, there is a $c \in E$ such that $c \oplus a \in M$, $c \oplus b \in M$. We will write $a \sim_L^M b(c)$ or $a \sim_L^M b$, $a \sim_L b(c)$, $a \sim_L b$ if c or M or both are not emphasized, respectively. Similarly, we call $a, b \in E$ are *right perspective* with respect to M if they share a common right complement, that is, there is a $d \in E$ such that $a \oplus d \in M$, $b \oplus d \in M$, and we denote by $a \sim_R^M b(d)$ or $a \sim_R^M b, a \sim_R b(d)$, $a \sim_R b$ if d or M or both are not emphasized, respectively. We will say that $a, b \in E$ are *perspective* with respect to M if they are both right and left perspective.

A subset M of E is *left* [*right*] *dominating* if for every $a \in E$ there is a $c[d]$ such that $c \oplus a \in M[a \oplus d \in M]$. In other words, every $a \in E$ has at least one left [right] complement with respect to M. If M is not only left dominating but also right dominating, then we say that M is *dominating*.

We will say that $M \subset E$ is *left* [*right*] *weakly algebraic* if the left [right] perspectivity $\sim_L^M$ [$\sim_R^M$] is a strong congruence, and we will say that $M \subset E$ is *weakly algebraic* if it is both left and right algebraic. By the following proposition, we have $\sim_R^M \equiv \sim_L^M$ if M is weakly algebraic. Thus we denote them by $\sim^M$.

Proposition 3.1 *If $\sim_L^M$ and $\sim_R^M$ are all strong congruences, then $\sim_L^M \equiv \sim_R^M$.*

Proof Let $a \sim_L^M b$, then there is an $x \in E$ such that $x \oplus a \in M$ and $x \oplus b \in M$. Hence $x \oplus a \sim_R^M x \oplus b$ and, by $\sim_R^M$ is a strong congruence, we have $a \sim_R b$. Similarly, we can prove the converse conclusion.

We will say that a subset M of a pseudoeffect algebra E is *left* [*right*] *closed* if $x \in M, y \sim_L^M x$ imply $y \in M[x \in M, y \sim_R^M x$ imply $y \in M]$. For any subset $M \subset E$, the set $\overline{M}^L = \{x \in E \mid \exists y \in M,\ x \sim_L^M y\}[\overline{M}^R = \{x \in E \mid \exists y \in M,\ x \sim_R^M y\}]$ is obviously the smallest left [right] closed subset which contains M. We call it the *left closure* [*right closure*] of M. Clearly, if M is a weakly algebraic subset of E, then $\overline{M}^L = \overline{M}^R$. We call it the *closure* of M and denote by $\overline{M}$.

Theorem 3.2 *Let E be a pseudoeffect algebra and M a left [right] weakly algebraic subset of E. Then*

(1) *M is left [right] dominating.*
(2) *$\overline{M}^L = \{x \in E \mid \exists r \in E$ such that $r \oplus x \in M$ and $r \sim_L^M 0\}[\overline{M}^R = \{x \in E \mid \exists s \in E$ such that $x \oplus s \in M$ and $s \sim_R^M 0\}]$.*

(3) $\overline{M}^L = \{x \in E \mid x \sim_L^M z \oplus y,\ y \in M \text{ and } z \in E\}$
$[\overline{M}^R = \{x \in E \mid x \sim_R^M y \oplus z,\ y \in M \text{ and } z \in E\}]$.

(4) $\overline{M}^L$ *is left weakly algebraic* $[\overline{M}^R$ *is right weakly algebraic*], *and* $\sim_L^{\overline{M}^L} \equiv \sim_L^M [\sim_R^{\overline{M}^R} \equiv \sim_R^M]$.

Proof

(1) For every $a \in E$, since the perspectivity $\sim_L^M$ is an equivalence, we have $a \sim_L^M a$, and hence there is a $b \in E$ with $b \oplus a \in M$. So M is left dominating.

(2) Suppose that $x \in \overline{M}^L$, then there is a $y \in M$ with $x \sim_L^M y$. By (1), M is left dominating and hence there is an $r \in E$ such that $r \oplus x \in M$. Then $r \oplus x \sim_L^M y$ and $r \sim_L^M 0$ by $x \sim_L^M y$. Conversely, if there is an $r \in E$ such that $r \oplus x \in M$ and $r \sim_L^M 0$, then $r \oplus x \sim_L^M x$ which implies $x \in \overline{M}^L$.

(3) Let $x \sim_L^M z \oplus y$, $y \in M$ and $z \in E$. Then there is a $c \in E$ such that $c \oplus x \in M$ and $c \oplus z \oplus y \in M$. So $c \oplus z \oplus y \sim_L^M y$ and hence $c \oplus z \sim_L^M 0$ which implies that $z \sim_L^M 0$ and $x \sim_L^M z \oplus y \sim_L^M y$. Hence $x \in \overline{M}^L$ by (2). The converse is obvious.

(4) It suffices to prove that $\sim_L^{\overline{M}^L}$ implies $\sim_L^M$. Let $a \sim_L^{\overline{M}^L} b$. Then there is an $x \in E$ such that $x \oplus a \in \overline{M}^L$ and $x \oplus b \in \overline{M}^L$. By (2), there are $r,\ s \in E$ such that $r \sim_L^M 0$, $s \sim_L^M 0$, $r \oplus x \oplus a \in M$ and $s \oplus x \oplus b \in M$ and thus $r \oplus x \oplus a \sim_L^M s \oplus x \oplus b$ which implies that $a \sim_L^M b$.

Corollary 3.3 *Let E be a pseudoeffect algebra and M a weakly algebraic subset of E. Then*

(1) *M is dominating.*

(2) $\overline{M} = \overline{M}^L = \{x \in E \mid \exists r \in E$ *such that* $r \oplus x \in M$ *and* $r \sim_L^M 0\} = \overline{M}^R = \{x \in E \mid \exists s \in E$ *such that* $x \oplus s \in M$ *and* $s \sim_R^M 0\}$.

(3) $\overline{M} = \overline{M}^L = \{x \in E \mid x \sim_L^M z \oplus y,\ y \in M \text{ and } z \in E\} = \overline{M}^R = \{x \in E \mid x \sim_R^M y \oplus z,\ y \in M \text{ and } z \in E\}$.

(4) $\overline{M}$ is weakly algebraic, and $\sim_L^{\overline{M}} \equiv \sim^M$.

Theorem 3.4 *A left closed subset M of a pseudoeffect algebra E is left weakly algebraic if and only if*

(1) *M is left dominating.*

(2) $a \oplus x \in M, b \oplus y \in M, x \sim_L^M y \Longrightarrow a \sim_L^M b$ *and* $a \sim_L^M b \Longrightarrow x \sim_L^M y$.

(3) $a \sim_L^M b$, *and* $a \oplus x$ *is defined* $\Longrightarrow \exists y \in E$ *such that* $y \sim_L^M x$ *and* $b \oplus y$ *is defined and* $x \oplus a$ *is defined* $\Longrightarrow \exists y \in E$ *such that* $y \sim_L^M x$ *and* $y \oplus b$ *is defined.*

Proof If M is left weakly algebraic, properties (1), (2) and (3) follow from the fact that $\sim_L^M$ is a strong congruence.

Conversely, let (1), (2) and (3) be satisfied for a subset M of E. Since M is left dominating, for any $a \in E$ there is an $x \in E$ such that $x \oplus a \in M$ and hence $a \sim_L^M a$. Symmetry of $\sim_L^M$ is obvious. To prove transitivity, we first show that (4) holds:

(4) $a \sim_L^M b,\ x \oplus a \in M \Longrightarrow \exists y \sim_L^M x$ such that $y \oplus b \in M$.

Indeed, by (3) there is a $y \in E$ such that $y \sim_L^M x$ and $y \oplus b$ is defined. Since M is left dominating, there is a $z \in E$ such that $z \oplus y \oplus b \in M$. By (2) and $x \oplus a \in M$, $a \sim_L^M b$, we have $x \sim_L^M z \oplus y$. Hence there is a $u \in E$ such that $u \oplus x \in M$ and $u \oplus z \oplus y \in M$. By (2) and $y \sim_L^M x$, we have $u \sim_L^M u \oplus z$. So there is a $v \in E$ such that $v \oplus u \in M$ and $v \oplus u \oplus z \in M$. Again by (2) and $v \oplus u \sim_L^M v \oplus u$, we have $z \sim_L^M 0$. Hence $y \oplus b \in M$ by (2) and M is a closed subset.

Assume that $a \sim_L^M b$, $b \sim_L^M c$. Then there is a $u \in E$ such that $u \oplus b \in M$ and $u \oplus c \in M$. By (4), $a \sim_L^M b$, $u \oplus b \in M$ imply that there exists a $v \in E$ such that $v \sim_L^M u$, $v \oplus a \in M$. By (2), $c \sim_L^M a$ and hence $\sim_L^M$ is an equivalence.

Now let $a_1 \sim_L^M b_1$, $a_2 \sim_L^M b_2, a_1 \oplus a_2$ and $b_1 \oplus b_2$ be defined. Since M is left dominating, there are $u,\ v \in E$ such that $u \oplus a_1 \oplus a_2 \in M$, $v \oplus b_1 \oplus b_2 \in M$. By (2), we have $y_1 \sim_L^M y_2$. By (2) and $a_2 \sim_L^M b_2$, we have $u \oplus a_1 \sim_L^M v \oplus b_1$. Hence there is a $w \in E$ such that $w \oplus u \oplus a_1 \in E$ and $w \oplus v \oplus b_1 \in E$. By (2) and $a_1 \sim_L^M b_1$, we have $w \oplus u \sim_L^M w \oplus v$. Then there is a $t \in E$ such that $t \oplus w \oplus u \in M$ and $t \oplus w \oplus v \in E$. Again by (2) and $t \oplus w \sim_L^M t \oplus w$, we have $u \sim_L^M v$. Therefore $a_1 \oplus a_2 \sim_L^M b_1 \oplus b_2$. Taking (3) into account, we prove that $\sim_L^M$ is a strong congruence.

For right weakly algebraic or weakly algebraic, we have the similar conclusion.

For $M \subset E$, we define $M^- = \{a^- \mid a \in M\}, M^\sim = \{a^\sim \mid a \in M\}$.

Lemma 3.5 *Let E be a pseudoeffect algebra.*

(1) *For any $M \subset E, M^-$ is a normal weak Riesz ideal iff $M^\sim$ is a normal weak Riesz ideal.*

(2) *If $M \subset E$ is left (or right) weakly algebraic, then $M^- = M^\sim$.*

(3) *If $I \subset E$ is a normal weak Riesz ideal, then $I^\sim$ is left weakly algebraic, I^- is right weakly algebraic and* $\sim_I \equiv \sim_L^{I^\sim} \equiv \sim_R^{I^-}$.

Proof

(1) Assume that M^- is a normal weak Riesz ideal. First, we prove $M^\sim$ is an ideal. If $a \in M^\sim$ and $b \leq a$, then $a^= \in M^-$ and $b^= \leq a^=$. Since M^- is an ideal, $b^= \in$

M^- and thus $b \in M^\sim$. Suppose that $a, b \in M^\sim$ and $a \oplus b$ is defined. Then $a^=, b^= \in M^-$ and $a^= \oplus b^=$ is defined. So $a^= \oplus b^= \in M^-$ and hence $(a \oplus b)^- = (a^= \oplus b^=)^\sim \in M$, which implies that $a \oplus b \in M^\sim$. Hence $M^\sim$ is an ideal.

For any $a, b, c \in E$ with $a \oplus b = b \oplus c$, we have $(a^= \oplus b^=)^\sim = (a \oplus b)^- = (b \oplus c)^- = (b^= \oplus c^=)^\sim$ and hence $a^= \oplus b^= = b^= \oplus c^=$. If $a \in M^\sim$, then $a^= \in M^-$ and hence $c^= \in M^-$ by M^- is a normal ideal, which entails $c \in M^\sim$. Similarly, if $c \in M^\sim$, then $a \in M^\sim$. Therefore, $M^\sim$ is a normal ideal.

Finally, we prove $M^\sim$ is a weak Riesz ideal. Let $a \in M^\sim$ and $a \leq b \oplus c$. Then $a^= \in M^-$ and $a^= \leq b^= \oplus c^=$. Since M^- is a Riesz ideal, there exist $x, y \in M^-$ such that $a^= \leq x \oplus y$, $x \leq b^=$, $y \leq c^=$, and thus $a \leq x^\approx \oplus y^\approx, x^\approx \leq b$, $y^\approx \leq c$. Hence $M^\sim$ is a weak Riesz ideal while $x^\approx, y^\approx \in M^\sim$.

The converse conclusion is proved similarly.

(2) Assume that $M \subset E$ is left weakly algebraic. Then M is left dominating by Theorem 3.2 and hence for every $a \in E$ there is a $c \in E$ such that $c \oplus a \in M$, which implies that $1 \in M$. Hence $a \in M$ if and only if $a \sim_L^M 1$. Let $I = \{a \in E \mid a \sim_L^M 0\}$. Then we have $I = M^-$. Indeed, if $a \sim_L^M 0$, then $a^\sim \sim_L^M 1$ by (C4). So $a^\sim \in M$ which implies that $a \in M^-$. Conversely, if $a \in M^-$, then $a^\sim \in M$, which implies that $a^\sim \sim_L^M 1$ and thus $a \sim_L^M 0$, that is $a \in I$. Similarly, $I = M^\sim$ and hence $M^- = M^\sim$.

(3) Assume that $I \subset E$ is a normal weak Riesz ideal. Then $\sim_I$ is a strong congruence by Lemma 1.6. We show that $\sim_I \equiv \sim_R^{I^-}$, which entails that I^- is right weakly algebraic. If $a \sim_I b$, then $a^\sim \sim_I b^\sim$ by $\sim_I$ is a strong congruence. Hence there exist $i, j \in I$ such that $a^\sim / i = b^\sim / j$, and thus $i^- \setminus a = j^- \setminus b$. As $i^- = a \oplus (i^- \setminus a) = a \oplus (j^- \setminus b) \in I^-$ and $j^- = b \oplus (j^- \setminus b) \in I^-$, we have $a \sim_R^{I^-} b$. Conversely, if $a \sim_R^{I^-} b$, then there exists a $c \in E$ such that $a \oplus c = i^-$, $b \oplus c = j^-$ where $i, j \in I$. Then $c = i^- \setminus a = j^- \setminus b$ and hence $a^\sim / i = b^\sim / j$ which implies that $a \sim_I b$. Similarly, we can show that $\sim_I \equiv \sim_L^{I^\sim}$.

Theorem 3.6 *Let E be a lattice-ordered pseudoeffect algebra. Then $M \subset E$ is weakly algebraic if and only if M^- (or $M^\sim$) is a normal weak Riesz ideal. This yields an order isomorphism between weakly algebraic sets and normal weak Riesz ideals.*

Proof If M^- is a normal weak Riesz ideal, then by Lemma 3.5 (1) $M^\sim$ is a normal weak Riesz ideal. So $\sim_{M^-}$ and $\sim_{M^\sim}$ are all strong congruences. It follows from Lemma 3.5(3), we have $\sim_L^M$ and $\sim_R^M$ are all strong congruences and hence $\sim^M$ is a strong congruence.

Conversely, assume that $M \subset E$ is weakly algebraic. It is sufficient to show that the mapping $\phi: E \longrightarrow [E]_{\sim^M}$ defined by $\phi(a) = [a]_{\sim^M}$ $(\forall a \in E)$ is homomorphism by Lemma 2.4. In fact, it is easy to check that ϕ is a morphism. Clearly, $[a \wedge b]_{\sim^M} \leq [a]_{\sim^M}, [b]_{\sim^M}$. Assume $[x]_{\sim^M} \leq [a]_{\sim^M}, [b]_{\sim^M}$ for some $x \in E$. Then there are $x_1 \sim^M x$, $a_1 \sim^M a$ with $x_1 \leq a_1$, which implies that $a_1^\sim \sim^M a^\sim$ and $x_1 \oplus a_1^\sim$ is defined. By (C3), there is $y_1 \sim^M x_1$ with $y_1 \oplus a^\sim$ defined, i.e., $y_1 \leq a$. Similarly, there is $y_2 \sim^M x$ with $y_2 \leq b$, which entails $y_1 \sim^M y_2$ and hence $y_1^- \sim^M y_2^-$. Then there is a $c \in E$ with $y_1^- \oplus c \in M$ and $y_2^- \oplus c \in M$, which implies that $c \leq y_1, y_2$. Since $a \in M$ if and only if $a \sim^M 1$, we have $y_1^- \oplus c \sim^M 1 \sim^M y_2^- \oplus c$. Thus $y_1 \backslash c = (y_1^- \oplus c)^\sim \sim^M 0$ and $y_2 \backslash c = (y_2^- \oplus c)^\sim \sim^M 0$, so that by (C2) $y_1 \sim^M y_2 \sim^M c$. Therefore, $c \leq a \wedge b$, and $[c]_{\sim^M} = [x]_{\sim^M} \leq [a \wedge b]_{\sim^M}$. It follows that $[a \wedge b]_{\sim^M}$ is the greatest lower bound of $[a]_{\sim^M}$ and $[b]_{\sim^M}$ in $[E]_{\sim^M}$. Then ϕ is homomorphism.

Theorem 3.7 *Let E be a lattice-ordered pseudoeffect algebra, $\sim$ be a Riesz strong congruence on E, and ϕ: $E \longrightarrow [E]_\sim$ be the canonical mapping. Then $\sim \equiv \sim^M$, where $M = \phi^{-1}([1]_\sim)$.*

Proof Let $I_\phi := \{a \in E \mid \phi(a) = [a]_\sim = [0]_\sim\}$. Then I_ϕ is a normal weak Riesz ideal and $\sim \equiv \sim_{I_\phi}$ by Lemma 1.7. By Lemma 3.5(3), we have $\sim_{I_\phi} \equiv \sim_L^{I_\phi^\sim} \equiv \sim_R^{I_\phi^-}$. To prove the conclusion, it suffices to show that $I_\phi{}^\sim = I_\phi{}^- = \phi^{-1}([1]_\sim)$. Indeed, $b \in I_\phi{}^\sim \Longleftrightarrow b^- \in I_\phi \Leftarrow \Rightarrow [b^-]_\sim = [0]_\sim \Longleftrightarrow [b]_\sim = [1]_\sim \Longleftrightarrow b \in \phi^{-1}([1]_\sim)$. Similarly, $I_\phi{}^- = \phi^{-1}([1]_\sim)$.

Corollary 3.8 *Let E be a lattice-ordered pseudoeffect algebra, $\sim$ be a Riesz strong congruence on E, and ϕ: $E \longrightarrow [E]_\sim$ be the canonical mapping. Then $\sim \equiv \sim^M \equiv \sim_I$, where $M = \phi^{-1}([1]_\sim)$ and $I = \{a \in E \mid \phi(a) = [a]_\sim = [0]_\sim\}$.*

Proof It follows from Lemma 1.7 and Theorem 3.7.

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