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Research Article

Multiple Positive Solutions for Nonlinear Semipositone Fractional Differential Equations

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We present some new multiplicity of positive solutions results for nonlinear semipositone fractional boundary value problem $D_{0+}^{\alpha}u(t) = p(t)f(t,u(t)) - q(t), 0 < t < 1, u(0) = u(1) = u'(1) = 0$, where $2 < \alpha \leq 3$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville differentiation. One example is also given to illustrate the main result.

1. Introduction

This paper is mainly concerned with the multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem (BVP for short)

$$\begin{gathered} D_{0^+}^{\alpha}u(t) = p(t)f(t,u(t)) - q(t), \quad 0 < t < 1, \\ u(0) = u(1) = u'(1) = 0, \end{gathered} \tag{1.1}$$

where $2 < \alpha \leq 3$ is a real number and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differentiation, and f, p, q is a given function satisfying some assumptions that will be specified later.

In the last few years, fractional differential equations (in short FDEs) have been studied extensively the motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For an extensive collection of such results, we refer the readers to the monographs by Kilbas et al. [1], Miller and Ross [2], Oldham and Spanier [3], Podlubny [4], and Samko et al. [5].

Some basic theory for the initial value problems of FDE involving the Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala [6–8], Babakhani and Daftardar-Gejji [9–11], and Bai [12], and others. Also, there are some papers that deal with the existence and multiplicity of solutions (or positive solution) for nonlinear FDE of BVPs by using techniques of nonlinear analysis (fixed point theorems, Leray-Schauders theory, topological degree theory, etc.), see [13–22] and the references therein.

Bai and Lü [15] studied the following two-point boundary value problem of FDEs

$$D_{0+}^{q}u(t) + f(t,u(t)) = 0, \quad u(0) = u(1) = 0,\ 0 < t < 1,\ 1 < q \le 2, \tag{1.2}$$

where D_{0+}^{q} is the standard *Riemann-Liouville* fractional derivative. They obtained the existence of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem.

Zhang [22] considered the existence and multiplicity of positive solutions for the nonlinear fractional boundary value problem

$$^{c}D_{0+}^{q}u(t) = f(t,u(t)), \quad 0 < t < 1,\ u(0) + u'(0) = 0,\ u(1) + u'(1) = 0, \tag{1.3}$$

where $1 < q \le 2$ is a real number, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$, and $^{c}D_{0+}^{q}$ is the standard Caputo's fractional derivative. The author obtained the existence and multiplicity results of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem.

From the above works, we can see the fact that although the fractional boundary value problems have been investigated by some authors to the best of our knowledge, there have been few papers that deal with the boundary value problem (1.1) for nonlinear fractional differential equation. Motivated by all the works above, in this paper we discuss the boundary value problem (1.1), using the Guo-Krasnosel'skii fixed point theorem, and we give some new existence of multiple positive solutions criteria for boundary value problem (1.1).

The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of the main results. In Section 3, we establish the existence of multiple positive solutions for boundary value problem (1.1) by the Guo-Krasnosel'skii fixed point theorem. In the end, we illustrate a simple use of the main result.

2. Preliminaries and Lemmas

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature such as [1, 4, 15].

Definition 2.1 (see [1, 4]). The Riemann-Liouville fractional integral of order $\alpha(\alpha > 0)$ of a function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \tag{2.1}$$

provided that the right side is pointwise defined on $(0,+\infty)$, where Γ is the gamma function.

Definition 2.2 (see [1, 4]). The Riemann-Liouville fractional derivative of order $\alpha(\alpha > 0)$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$\left(D_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-\alpha+1} f(s) ds, \tag{2.2}$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.3 (see [15]). *Let $\alpha > 0$. If one assumes $u \in C(0,1) \cap L(0,1)$, then fractional differential equation*

$$D^{\alpha} u(t) = 0 \tag{2.3}$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R},\ i = 1, 2, \ldots, N, \tag{2.4}$$

as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.4 (see [15]). *Assume that $h \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^{\alpha} D^{\alpha} h(t) = h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.5}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.5. *Let $h \in C[0,1]$ and $2 < \alpha \leq 3$, then the unique solution of*

$$\begin{gathered} D^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(1) = 0 \end{gathered} \tag{2.6}$$

is given by

$$u(t) = \int_{0}^{1} G(t,s) h(s) ds, \tag{2.7}$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if } 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-1} t^{\alpha-1}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.8}$$

The following properties of Green's function form the basis of our main work in this paper.

Lemma 2.6. *The function $G(t,s)$ defined by* (2.8) *possesses the following properties:*

(i) $G(t,s) = G(1-s, 1-t)$ *for* $t, s \in (0,1)$;

(ii) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \leq G(t,s)\Gamma(\alpha) \leq (\alpha-1)s(1-s)^{\alpha-1}$ *for* t, $s \in (0,1)$;

(iii) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \leq G(t,s)\Gamma(\alpha) \leq (\alpha-1)t^{\alpha-1}(1-t)$ *for* t, $s \in (0,1)$;

(iv) $G(t,s) > 0$ *for* $t, s \in (0,1)$.

The following Krasnosel'skii's fixed point theorem will play a major role in our next analysis.

Lemma 2.7 (see [23]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $\mathcal{A} : P \to P$ be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we establish some new existence results for the fractional differential equation (1.1). Given $a \in L^1(0,1)$, we write a $a \succ 0$, if $a \geq 0$ for $t \in [0,1]$, and it is positive in a set of positive measure.

Let us list the following assumptions:

(H1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $p, q \succ 0$;

(H2) there exists $\theta \in (0, 1/2)$, such that

$$\int_{\theta}^{1-\theta} p(s)s(1-s)^{\alpha-1}\, ds > 0. \tag{3.1}$$

In view of Lemmas 2.5 and 2.6, we obtain the following.

Lemma 3.1. *Let $q \in L^1[0,1]$ with $q > 0$ on (0,1), and $\gamma(t)$ is the unique solution of*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = q(t), \quad 0 < t < 1, \\ u(0) = u(1) = u'(1) = 0, \end{gathered} \tag{3.2}$$

Then

$$0 \leq \gamma(t) \leq \frac{\alpha-1}{\Gamma(\alpha)} \|q\|_1 t^{\alpha-1}(1-t) := Ct^{\alpha-1}(1-t), \ \ for\ t \in [0,1], \tag{3.3}$$

where $C = (\alpha - 1/\Gamma(\alpha))\|q\|_1$, $\|q\|_1 = \int_0^1 |q(t)|\, dt$.

Next, we consider

$$D_{0^+}^{\alpha}u(t) = p(t)g\big(t, u(t) - \gamma(t)\big), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(1) = 0, \tag{3.4}$$

where

$$g(t, u) = \begin{cases} f(t, u), & \text{if } u \geq 0 \\ f(t, 0), & \text{if } u < 0, \end{cases} \tag{3.5}$$

Then (3.4) is equivalent to the following integral equation:

$$u(t) = \int_0^1 G(t, s)p(s)g\big(s, u(s) - \gamma(s)\big)ds. \tag{3.6}$$

Lemma 3.2. *Let $u(t) \geq \gamma(t)$ for $t \in [0,1]$, and $u(t)$ is positive solution of the problem* (3.4). *Then $u(t) - \gamma(t)$ is positive solution of the problem* (1.1).

Proof. In fact, let $x(t) = u(t) - \gamma(t)$. Then $x(t) \geq 0$ and $u(t) = x(t) + \gamma(t)$. Since $u(t)$ is positive solution of the problem (3.4), we have

$$D_{0^+}^{\alpha}\big[x(t) + \gamma(t)\big] = p(t)g(t, x(t)), \quad 0 < t < 1,$$
$$\big(x + \gamma\big)(0) = \big(x + \gamma\big)(1) = \big(x + \gamma\big)'(1) = 0. \tag{3.7}$$

So

$$D_{0^+}^{\alpha}x(t) = p(t)g(t, x(t)) - q(t), \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(1) = 0. \tag{3.8}$$

For our constructions, we will consider the Banach space $E = C[0,1]$ equipped with standard norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|,\ u \in E$.

Define a cone K by

$$K = \left\{u \in E : u(t) \geq \frac{t^{\alpha-1}(1-t)}{\alpha - 1}\|u\|,\ \forall t \in [0,1], \alpha \in (2,3]\right\}. \tag{3.9}$$

Let the operator $\mathcal{A} : K \to E$ be defined by the formula

$$(\mathcal{A}x)(t) := \int_0^1 G(t, s)p(s)g\big(s, x(s) - \gamma(s)\big)ds, \quad 0 \leq t \leq 1,\ u \in K. \tag{3.10}$$

□

Lemma 3.3. *Assume that (H1) holds. Then $\mathcal{A}(K) \subset K$.*

Proof. Notice from (3.10) and Lemma 2.6 that, for $x \in K$, $\mathcal{A}x(t) \geq 0$ on $[0,1]$ and

$$\|\mathcal{A}x\| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha-1)s(1-s)^{\alpha-1} p(s) g(s, x(s) - \gamma(s)) ds. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}\mathcal{A}x(t) &\geq \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} p(s) g(s, x(s) - \gamma(s)) ds \\ &\geq \frac{t^{\alpha-1}(1-t)}{(\alpha-1)\Gamma(\alpha)} \|\mathcal{A}x\|.\end{aligned} \tag{3.12}$$

Thus we have $\mathcal{A}(K) \subset K$. The proof is finished. □

It is standard that $\mathcal{A} : K \to K$ is continuous and completely continuous.

For convenience, we introduce the following notations: $\widetilde{N} = ((\alpha-1)\|p\|_1/\Gamma(\alpha)) \max_{0\leq s\leq 1} s(1-s)^{\alpha-1}$, $\overline{N} = \sigma \int_\theta^{1-\theta} (s(1-s)^{\alpha-1}/\Gamma(\alpha)) p(s) ds$, $\sigma = \min_{\theta\leq t\leq 1-\theta}(1-t)t^{\alpha-1}$.

Theorem 3.4. *Assume that (H1) and (H2) are satisfied. Also suppose the following conditions are satisfied:*

(A1) *there exists a constant* $R_1 > (\alpha-1)\mathcal{C}$ *such that* $\widetilde{N} f(t,u) \leq R_1$ *for all* $(t,u) \in [0,1] \times [0, R_1]$;

(A2) *there exists a constant* $R_2 > 2R_1$ *such that* $\overline{N} f(t,u) > R_2$ *for all* $(t,u) \in [0,1] \times [\sigma R_2, R_2]$;

(A3) $\lim_{u\to+\infty} \max_{0\leq t\leq 1}(f(t,u)/u) = 0$.

Then the problem (1.1) *has at least two positive solutions.*

Proof. To show that (1.1) has at least two positive solutions, we will assume the problem (3.4) has at least two positive solutions x_1 and x_2 with $R_1 \leq \|x_1\| < R_2 < \|x_2\| \leq R_3$.

We now show

$$\|\mathcal{A}x\| \leq \|x\|, \quad \text{for } x \in K \cap \partial\Omega_1, \tag{3.13}$$

To see this, let $\Omega_1 = \{x \in K \mid \|x\| < R_1\}$, then for $x \in K \cap \partial\Omega_1$, $t \in [0,1]$, by Lemma 3.1 and (A1), we have

$$\begin{gathered} x(t) - \gamma(t) \leq x(t) \leq \|x\| = R_1, \\ x(t) - \gamma(t) \geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} R_1 - \mathcal{C} t^{\alpha-1}(1-t) \geq \left(\frac{R_1}{\alpha-1} - \mathcal{C}\right) t^{\alpha-1}(1-t) \geq 0. \end{gathered} \tag{3.14}$$

Thus, we see, from Lemma 2.6 and (A1), that

$$\begin{aligned}
\|\mathcal{A}x\| &= \max_{0\le t\le 1}(\mathcal{A}u)(t) = \max_{0\le t\le 1}\int_0^1 G(t,s)p(s)g\big(s,x(s)-\gamma(s)\big)ds \\
&\le \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}p(s)f\big(s,x(s)-\gamma(s)\big)ds \\
&\le \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 \frac{R_1}{\overline{\overline{N}}}s(1-s)^{\alpha-1}p(s)ds \\
&\le R_1,
\end{aligned} \tag{3.15}$$

from which we see that $\|\mathcal{A}x\| \le \|x\|$, for $x \in K \cap \partial\Omega_1$.

Next we now show

$$\|\mathcal{A}x\| \ge \|x\|, \quad \text{for } x \in K \cap \partial\Omega_2. \tag{3.16}$$

To see this, let $\Omega_2 = \{x \in K \mid \|x\| < R_2\}$; then, for $x \in K \cap \partial\Omega_2$, $t \in [0,1]$, by $R_2 > 2R_1$, we have

$$x(t)-\gamma(t) \ge \frac{t^{\alpha-1}(1-t)}{\alpha-1}R_2 - Ct^{\alpha-1}(1-t) \ge \frac{t^{\alpha-1}(1-t)}{2(\alpha-1)}R_2. \tag{3.17}$$

For $x \in \partial\Omega_2$; $t \in [\theta, 1-\theta]\sigma$, then, it follows from (3.17) that

$$R_2 \le \frac{t^{\alpha-1}(1-t)}{2(\alpha-1)}R_2 \le x(t)-\gamma(t) \le R_2. \tag{3.18}$$

In view of (A2), (3.17) and Lemma 2.6, we have that for all $x \in \partial\Omega_2$, $t \in [\theta, 1-\theta]\sigma$

$$\begin{aligned}
\|\mathcal{A}u\| &\ge \int_0^1 G(t,s)p(s)g\big(s,x(s)-\gamma(s)\big)ds \\
&\ge t^{\alpha-1}(1-t)\int_0^1 \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f\big(s,x(s)-\gamma(s)\big)ds \\
&> t^{\alpha-1}(1-t)\int_\theta^{1-\theta} \frac{s(1-s)^{\alpha-1}R_2}{\Gamma(\alpha)\overline{N}}p(s)ds \\
&\ge \sigma\int_\theta^{1-\theta} \frac{s(1-s)^{\alpha-1}R_2}{\Gamma(\alpha)\overline{N}}p(s)ds \\
&= R_2,
\end{aligned} \tag{3.19}$$

from which we see that $\|\mathcal{A}x\| > \|x\|$, for $x \in K \cap \partial\Omega_2$.

On the other hand, let $\varepsilon > 0$, where

$$\frac{\varepsilon(\alpha-1)}{\Gamma(\alpha)}\max_{0\le t\le 1} t(1-t)^{\alpha-1}\|p\|_1 \le 1. \tag{3.20}$$

Supposing that (A3) holds, one can find $N > R_2 > 0$, so that

$$f(t,u) \le \varepsilon u, \quad \forall t \in [0,1],\ u \ge N. \tag{3.21}$$

Setting

$$R_3 = \frac{(\alpha-1)\max_{0\le t\le 1} t(1-t)^{\alpha-1}\ \|p\|_1 \max_{(t,u)\in[0,1]\times[0,N]} f(t,u)}{\left(\Gamma(\alpha)-\varepsilon(\alpha-1)\max_{0\le t\le 1} t(1-t)^{\alpha-1}\|p\|_1\right)} + N, \tag{3.22}$$

then $R_3 > N > R_2$, and so

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)p(s)g\big(s,x(s)-\gamma(s)\big)ds \\ &\le \int_0^1 \frac{(\alpha-1)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\max_{(s,u)\in[0,1]\times[0,N]} f(s,u)ds \\ &\quad + \int_0^1 \frac{(\alpha-1)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\varepsilon\big(x(s)-\gamma(s)\big)ds \\ &\le R_3, \end{aligned} \tag{3.23}$$

from which we see that $\|\mathcal{A}x\| \le \|x\|$, for $x \in K \cap \partial\Omega_3$.

In view of Lemma 2.7, the problem (3.4) has at least two positive solutions x_1 and x_2 with $R_1 \le \|x_1\| < R_2 < \|x_2\| \le R_3$. Since $R_2 > R_1 > (\alpha-1)\mathcal{C}$, we have

$$\begin{aligned} x_1(t)-\gamma(t) &\ge \frac{t^{\alpha-1}(1-t)}{\alpha-1}R_1 - \mathcal{C}t^{\alpha-1}(1-t) \ge \left(\frac{R_1}{\alpha-1}-\mathcal{C}\right)t^{\alpha-1}(1-t) \ge 0, \\ x_2(t)-\gamma(t) &\ge \frac{t^{\alpha-1}(1-t)}{\alpha-1}R_2 - \mathcal{C}t^{\alpha-1}(1-t) \ge \left(\frac{R_2}{\alpha-1}-\mathcal{C}\right)t^{\alpha-1}(1-t) \ge 0. \end{aligned} \tag{3.24}$$

Therefore x_1, x_2 are solutions of the problem (1.1). This completes the proof. □

Theorem 3.5. *Suppose that (H1), (H2) are satisfied. Furthermore assume that*

(A4) *there exists a constant $R_1 > 2(\alpha-1)\mathcal{C}$ such that $\overline{N}f(t,u) \ge R_1$ for all $(t,u) \in [0,1] \times [\sigma R_1, R_1]$;*

(A5) *there exists a constant $R_2 > max\{R_1, (R_1/\overline{N})\widetilde{N}\}$ such that $\widetilde{N}f(t,u) < R_2$ for all $(t,u) \in [0,1] \times [0, R_2]$;*

(A6) $\lim_{u\to+\infty}\min_{\theta\le t\le 1-\theta}(f(t,u)/u) = +\infty$.

Then the problem (1.1) *has at least two positive solutions.*

4. An Example

As an application of the main results, we consider

$$D^{5/2}y(t) = f(y) - \frac{1}{\sqrt{t}}, \quad 0 < t < 1,$$
$$y(0) = y'(0) = y(1) = 0, \tag{4.1}$$

Set

$$f(y) = \begin{cases} -2(y-7)^2 + 1100, & \text{if} \quad 0 \le y \le 7, \\ -2(y-7) + 1100, & \text{if} \quad 7 \le y \le 450, \\ (y-450)^2 + 214, & \text{if} \quad y \ge 450, \end{cases} \tag{4.2}$$

Then we have $\mathcal{C} = (\alpha-1)/\Gamma(\alpha)\|q\|_1 \approx 2.25676, \widetilde{N} = ((\alpha-1)\|q\|_1/\Gamma(\alpha))\max_{0\le s\le 1} s(1-s)^{\alpha-1} \approx 0.4$, letting, $\theta = 1/4$, then $\sigma = \min_{\theta\le t\le 1-\theta}(1-t)t^{\alpha-1} \approx 0.09375$, $\overline{N} = \sigma\int_{1/4}^{3/4}(s(1-s)^{\alpha-1}/\Gamma(\alpha))ds \approx 0.008$, choosing $R_1 = 7$, $R_2 = 450$, then $R_1 > 2(\alpha-1)\mathcal{C} = 6.77$, $R_2 > \max\{R_1, (R_1/\overline{N})\widetilde{N}\} = \max\{7, 350\} = 350$; therefore, we have $\overline{N} f(u) = 0.008[-2(y-7)^2 + 1100] \ge 8.156 > R_1$, $(t,u) \to [1/4, 3/4] \times [0.65625, 7]$, $\widetilde{N} f(u) = 0.4[-2(y-7)^2 + 1100] \le 440 < R_2$, $(t,u) \to [0,1] \times [0,7]$, $\widetilde{N} f(u) = 0.4[-2(y-7) + 1100] \le 440 < R_2$, $(t,u) \to [0,1] \times [7,450]$, and $\lim_{y\to+\infty}(f(y)/y) = \lim_{y\to+\infty}(((y-450)^2 + 214)/y) = +\infty$.

It is clear that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Since all the conditions of Theorem 3.5 are satisfied, the problem (4.1) has at least two positive solutions.

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