MATHEMATICA CODES FOR KLOOSTERMAN V.S. HECKE – EIGENVALUES

PING XI / JAN. 2018

Determination of ρ : Analysis for H_ 1 (X) and H_ 2 (X)

```
TableMeasures[N0_, x_] :=
 Module[{delta = 1 / N0, helpme = Table[Sqrt[1 - (i / N0) ^ 2], {i, 1, N0}],
   lValues = Table[0, {x}], uValues = Table[0, {x}]},
  lValues[[1]] = Table[N[2 / Pi (i delta * helpme[[i]] + ArcSin[i delta])],
    {i, 1, N0}];
  uValues[[1]] = lValues[[1]];
  For[k = 2, k ≤ x, k++,
   lValues[[k]] =
    4 / Pi * Table[Sum[N[helpme[[j]] * lValues[[k - 1]][[Floor[N0 i / j]]]],
        {j, i + 1, N0}], {i, 1, N0}] / N0 + lValues[[1]];
   uValues[[k]] = 4 / Pi * Table[N[Sum[N[helpme[[j]] * uValues[[k - 1]][[Ceiling[
            N0 i / j]]]], {j, i, N0 - 1}] / N0], {i, 1, N0}] + uValues[[1]];
  ];
  {lValues, uValues}
 ]
N0 = 2000
```

```
2000
```

```
t = TableMeasures[N0, 8]; // Timing
```

```
{339.543, Null}
```

```
Pos[L_, x_] := Module[{down = 1, up = Length[L]},
  Which[x < L[[down]], down = 0,
   x > L[[up]], down = Length[L], L[[down]] ≤ x ≤ L[[up]],
   While[up - down > 1,
     If[x <= L[[Floor[(down + up) / 2]]],
       up = Floor[(down + up) / 2], down = Floor[(down + up) / 2]];
    ];
  ];
  down]
Cj[j_] :=
 4^j * 4 / Pi * Sum[(i / N0) ^2 * (Pos[t[[2]][[j - 1]], 1 - t[[2]][[1]][[i]]] / N0) ^2 *
    N[Sqrt[1 - (i / N0) ^2]] / N0, {i, 1, N0 - 1}]
Cjs = Table[Cj[j], {j, 2, 9}]
```

```
{0.233839, 0.0997795, 0.0474737, 0.0235236,
 0.0116855, 0.00556737, 0.00242014, 0.000894193}
```

```
f[L0_] := Module[{L = L0, res = 1},
   L = Append[L, 1 - Total[L]];
   1 / Apply[Times, L]];
BasicInt[f_, a_, b_, delta0_] :=
  Module[{Nd = 1, res = 0, delta = delta0, x = 0},
```

```
    If[a > b, 0,
     Nd = Ceiling[(b - a) / delta];
     delta = (b - a) / Nd;
     res = Sum[f[a + delta / 2 + i delta], {i, 0, Nd - 1}];
     res * delta
    ]
   ];
A2[u_, deltaout_] := BasicInt[(1 / (# (1 - #))) &, 3 / 7, 1 / 2, deltaout]
A3[u_, deltain_, deltaout_] :=
 BasicInt[Function[x, BasicInt[(1 / (x # (1 - x - #))) &,
    Max[x, (1 - x) / 3], (1 - x) / 2, deltain]], 1 / 12, 1 / 3, deltaout]
A4[u_, deltain_, deltamid_, deltaout_] :=
 BasicInt[Function[x, BasicInt[
    Function[y, BasicInt[(1 / (x y # (1 - x - y - #))) &, Max[y, (1 - x) / 3 - y],
      (1 - x - y) / 2, deltain]], x, (1 - x) / 3, deltamid]], 1 / 12, 1 / 4, deltaout]
A5[u_, deltain_, deltamidi_, deltamido_, deltaout_] :=
  BasicInt[Function[x, BasicInt[
     Function[y, BasicInt[Function[z, BasicInt[(1 / (x y z # (1 - x - y - z - #))) &,
         Max[z, (1 - x) / 3 - y - z, (x + y + z) / 2], (1 - x - y - z) / 2, deltain]],
       y, (1 - x - y) / 3, deltamidi]], x, (1 - x) / 4,
     deltamido]], 1 / 12, 1 / 5, deltaout];
A6[u_, deltain_, deltamidi_, deltamido_, deltamidoo_, deltaout_] :=
  BasicInt[
   Function[w, BasicInt[Function[x, BasicInt[Function[y, BasicInt[Function[
          z, BasicInt[(1 / (x y z w # (1 - x - y - z - w - #))) &, Max[z,
            (1 - w) / 3 - x - y - z, (x + y + z + w) / 2], (1 - x - y - z - w) / 2, deltain]],
         y, (1 - x - y - w) / 3, deltamidi]], x, (1 - x - w) / 4, deltamido]],
     w, (1 - w) / 5, deltamidoo]], 1 / 12, 1 / 6, deltaout];
A7[u_, deltain_, deltamidi_, deltamido_, deltamidoo_,
   deltamidooo_, deltaout_] :=
  BasicInt[Function[t, BasicInt[
     Function[w, BasicInt[Function[x, BasicInt[Function[y,
          BasicInt[Function[z, BasicInt[(1 / (x y z w t # (1 - x - y - z - w - t - #))) &,
            Max[z, (1 - t) / 3 - x - y - z - w, (x + y + z + w + t) / 2], (1 - x - y - z -
              w - t) / 2, deltain]], y, (1 - x - y - w - t) / 3, deltamidi]], x,
         (1 - x - w - t) / 4, deltamido]], w, (1 - w - t) / 5, deltamidoo]],
     t, (1 - t) / 6, deltamidooo]], 1 / 12, 1 / 7, deltaout];
A8[u_, deltain_, deltamidi_, deltamido_, deltamidoo_,
   deltamidooo_, deltamidoooo_, deltaout_] :=
  BasicInt[Function[s, BasicInt[Function[t,
      BasicInt[Function[w, BasicInt[Function[x, BasicInt[Function[y, BasicInt[
            Function[z, BasicInt[(1 / (x y z w t s # (1 - x - y - z - w - t - s - #))) &,
              Max[z, (1 - s) / 3 - x - y - z - w - t, (x + y + z + w + t + s) / 2],
              (1 - x - y - z - w - t - s) / 2, deltain]], y, (1 - x - y - w - t - s) / 3,
            deltamidi]], x, (1 - x - w - t - s) / 4, deltamido]], w,
         (1 - w - t - s) / 5, deltamidoo]], t, (1 - t - s) / 6, deltamidooo]],
     s, (1 - s) / 7, deltamidoooo]], 1 / 12, 1 / 8, deltaout];
A9[u_, deltain_, deltamidi_, deltamido_, deltamidoo_, deltamidooo_,
   deltamidoooo_, deltamidooooo_, deltaout_] :=
  BasicInt[Function[r, BasicInt[
     Function[s, BasicInt[Function[t, BasicInt[Function[w,
          BasicInt[Function[x, BasicInt[Function[y, BasicInt[Function[z,
                BasicInt[(1 / (x y z w t s r # (1 - x - y - z - w - t - s - r - #))) &,
                 Max[z, (1 - r) / 3 - x - y - z - w - t - s, (x + y + z + w + t + s + r) / 2],
                 (1 - x - y - z - w - t - s - r) / 2, deltain]], y, (1 - x - y - w - t - s - r) /
                3, deltamidi]], x, (1 - x - w - t - s - r) / 4, deltamido]],
           w, (1 - w - t - s - r) / 5, deltamidoo]], t, (1 - t - s - r) / 6,
         deltamidooo]], s, (1 - s - r) / 7, deltamidoooo]],
     r, (1 - r) / 8, deltamidooooo]], 1 / 12, 1 /
```

```
    9, deltaout];
N[
 A2[
  4,
  1 /
   30 000]]
```

0.287682

```
N[A3[4, 1 / 2000, 1 / 2000]]
```

1.04781

```
N[A4[4, 1 / 1000, 1 / 1000, 1 / 1000]]
```

0.850196

```
N[A5[4, 1 / 800, 1 / 800, 1 / 800, 1 / 800]]
```

0.148932

```
N[A6[4, 1 / 700, 1 / 700, 1 / 700, 1 / 700, 1 / 700]]
```

0.00424227

```
N[A7[4, 1 / 600, 1 / 600, 1 / 600, 1 / 600, 1 / 600, 1 / 600]]
```

7.25032×10^{-6}

```
BB[2] := 0.233838;
BB[3] := 0.099779;
BB[4] := 0.047473;
BB[5] := 0.023523;
BB[6] := 0.011685;
BB[7] := 0.005567;
l[i_, y_] := Max[0, 1 - 4 y * (8 / 3 / Pi) ^ (i - 1) * (11 / 12) ^ i + BB[i] * y^2];
u[i_, y_] :=
  16 y^4 + 32 y^3 * (11 / 12) ^ (i - 1) + 24 y^2 + (8 * Sqrt[5]) ^ (i - 1) * y + 2^ (i - 1);
CC[i_, y_] := Sqrt[l[i, y] ^3 / u[i, y]];
A1[y_] :=
   CC[2, y] * 0.28768 +
   CC[3, y] * 1.04781 +
   CC[4, y] * 0.85019 +
   CC[5, y] * 0.14893 +
   CC[6, y] * 0.004242 +
   CC[7, y] * 7.25032 / 1 000 000;
A1[1]
```

1.06464×10^{-10}

```
ff[s_] := 2 /; s > 0 && s ≤ 2;
ff[s_] := 4 Log[s / 2] + 2 /; s > 2 && s ≤ 4;
ff[s_] := 4 Log[s]^2 - 8 Log[2] * Log[s] + 4 Log[2] + 2 /; s > 4 && s ≤ 6;
ff[s_] := 0 /; s > 6;
ff[s_] := 0 /; s ≤ 0;
sigma02[s_] := s^2 / (8 Exp[2 * 0.57721566]);
sigma24[s_] := s^2 (4 + Log[4] - 2 Log[s] - 8 / s + 4 / s^2) / (8 Exp[2 * 0.57721566]);
Dsigma02[s_] := 2 s / (8 Exp[2 * 0.57721566]);
Dsigma24[s_] :=
  ((-8/s^3 + 8/s^2 - 2/s) s^2 + 2 s (4 + 4/s^2 - 8/s + Log[4] - 2 Log[s])) / (8 Exp[2 * 0.57721566]);
F[u_] := NIntegrate[ff[6 u - 2 v] * (2 ff[6 u] - ff[6 u - 2 v]), {v, 0, 3 u}];
CC1[x_] := NIntegrate[2 * ff[6 u]^2 / (6 - 6 u)^2, {u, 0, 1 / 3}] +
   NIntegrate[Dsigma24[6 - 6 u] * ff[6 u]^2, {u, 1 / 3, 2 / 3}] +
   NIntegrate[Dsigma02[6 - 6 u] * ff[6 u]^2, {u, 2 / 3, x}];
CC2[x_] := NIntegrate[2 * F[u] / (6 - 6 u)^2, {u, 0, 1 / 3}] +
   NIntegrate[Dsigma24[6 - 6 u] * F[u], {u, 1 / 3, 2 / 3}] +
   NIntegrate[Dsigma02[6 - 6 u] * F[u], {u, 2 / 3, x}];
A2[y_, x_] := (2 + y) * 64 * 8 * Exp[2 * 0.57721566] (2 CC1[x] / 3 + CC2[x] / 9);
A2[y, 1]
```

```
2460.9 (2 + y)
```

Analysis for the different η

```
Reduce[CC[2, y] > 0 && y > 0]
```

```
0 < y < 0.361202 || y > 11.8396
```

```
Reduce[CC[3, y] > 0 && y > 0]
```

```
0 < y < 0.459982 || y > 21.7881
```

```
Reduce[CC[4, y] > 0 && y > 0]
```

```
0 < y < 0.588463 || y > 35.796
```

```
Reduce[CC[5, y] > 0 && y > 0]
```

```
0 < y < 0.75401 || y > 56.3807
```

```
Reduce[CC[6, y] > 0 && y > 0]
```

```
0 < y < 0.966705 || y > 88.5274
```

```
Reduce[CC[7, y] > 0 && y > 0]
```

```
0 < y < 1.2395 || y > 144.922
```

```
Rho[y_] := 2461 (2 + y) / A1[y];
Rho[2018]
```

```
133590.
```

```
Rho[1 / 2018]
```

```
4963.94
```

```
RR[s_] := (1 / 2) ^ s * Sum[(3 Pi / 4) ^ j * Binomial[s, j], {j, 0, s / 2}];
Table[N[RR[j]], {j, 77, 82}]
```

$$\{2.06059 \times 10^{13},\ 4.93597 \times 10^{13},\ 4.80033 \times 10^{13},\ 1.14946 \times 10^{14},\ 1.11857 \times 10^{14},\ 2.67752 \times 10^{14}\}$$

```
Table[N[RR[j]], {j, 22, 36}]
```

$$\{3260.73,\ 3009.75,\ 7410.13,\ 6875.54,\ 16\,883.4,\ 15\,736.8,\ 38\,553.2,\ 36\,079.2,\ 88\,207.4,\ 82\,841.3,\ 202\,158.,\ 190\,465.,\ 464\,019.,\ 438\,435.,\ 1.06652 \times 10^{6}\}$$