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Riemann–Liouville abstract fractional Cauchy problem with damping☆

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Abstract

This paper is concerned with Riemann–Liouville abstract fractional Cauchy Problems with damping. The notion of Riemann–Liouville fractional (α, β, c) resolvent is developed, where $0 < \beta < \alpha \leq 1$. Some of its properties are obtained. By combining such properties with the properties of general Mittag-Leffler functions, existence and uniqueness results of the strong solution of Riemann–Liouville abstract fractional Cauchy Problems with damping are established. As an application, a fractional diffusion equation with damping is presented.

Keywords: Abstract fractional Cauchy problems; Riemann–Liouville fractional (α, β, c) resolvent; Strong solution

1. Introduction

In this paper, we are concerned with the following Riemann–Liouville abstract fractional Cauchy problem with damping

$$(\text{RLAFCP})_x \begin{cases} D_t^\alpha u(t) + cD_t^\beta u(t) = Au(t), & t > 0, \\ (g_{1-\alpha} * u)(0) = x, \end{cases}$$

where $0 < \beta < \alpha \leq 1, u(\cdot)$ is the state, $A : D(A) \subset X \to X$ is a closed linear operator, $(X, \|\cdot\|)$ is a Banach space, $D(A)$ is the domain of A endowed with the graph norm $\|\cdot\|_{D(A)} =$

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$\|\cdot\| + \|A\cdot\|$, D_t^α and D_t^β are respectively the α-order and β-order Riemann–Liouville fractional derivative operators, $g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, c is a real number. If the coefficient $c = 0$, we call $(\text{RLAFCP})_x$ to be Riemann–Liouville abstract fractional Cauchy problem. For fractional calculus involving fractional derivatives, we refer to [12].

Fractional derivatives possess memorizing properties, which makes fractional derivative more suitable than derivative of integer-order to describe the properties of various real materials. For example, fractional Cauchy problems have been used to model efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1,22] and the references therein), fractional random walk, etc. Berens and Westphal [4] first considered the abstract Cauchy problem involving the Riemann–Liouville fractional derivative operator. Hilfer studied in [15] the Riemann–Liouville fractional diffusion equation of order $0 < \alpha < 1$

$$D_t^\alpha f(r,t) = C_\alpha \Delta f(r,t)$$

where $f(r,t)$ denotes the unknown field and C_α denotes the fractional diffusion constant with dimensions $[cm/s^\alpha]$; he pointed out that the initial condition should be given in the form of $g_{1-\alpha}(t) * f(r,t)|_{t\to 0^+}$. This implies that the initial condition should have memorizing property. Heymans and Podlubny [14] demonstrated that in some material, the initial conditions for fractional differential equations with Riemann–Liouville derivatives have strong physical meaning, and that the corresponding quantities can be obtained from measurements.

Motivated by the fact that semigroup theory is closely related to the first-order abstract Cauchy problem, Da Prato and Iannelli [9] introduced the concept of resolvent families. Oka [25] introduced integrated solution families. Lizama [20] introduced (a,k)-regularized resolvents with a being continuous on $[0,\infty)$. Bazhlekova [3] carried out research on fractional evolution equations by using solution operators. Peng and Li [26] developed the notion of fractional semigroup to study fractional differential equations. Obviously, the above "families" have a common feature, that is, the families are strongly continuous at zero. However, Riemann–Liouville abstract fractional Cauchy problems with order $0 < \alpha < 1$ possess a singularity at zero. The singularity makes the above "families" unsuitable to study such Riemann–Liouville abstract fractional Cauchy problems.

In order to study the properties of Riemann–Liouville abstract fractional Cauchy problem, Li and Peng [19] recently developed a new notion, namely α-order fractional resolvent. Concretely, they considered a Riemann–Liouville abstract fractional Cauchy problem with

$$\lim_{t\to 0^+} \Gamma(\alpha) t^{1-\alpha} u(t) = x \tag{1}$$

instead of the initial value condition $(g_{1-\alpha} * u)(0) = x$ and defined the notion α-order fractional resolvent by

Definition 1 (*[19]*)**.** Let $0 < \alpha < 1$. A family $\{T(t)\}_{t>0}$ of bounded linear operators on Banach space X is called an α-order fractional resolvent if it satisfies the following assumptions:

(P1) for any $x \in X$, $T(\cdot)x \in C((0,\infty), X)$, and

$$\lim_{t\to 0+} \Gamma(\alpha) t^{1-\alpha} T(t)x = x \quad \text{for all } x \in X; \tag{2}$$

(P2) $T(s)T(t) = T(t)T(s)$ for all $t, s > 0$;

(P3) for all $t, s > 0$, there holds

$$T(t)J_s^\alpha T(s) - J_t^\alpha T(t)T(s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^\alpha T(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} J_t^\alpha T(t), \tag{3}$$

where J_t^α is α-order Riemann–Liouville fractional integral operator.

Obviously, it is easy to obtain by the dominated convergence theorem that if (1) holds then

$$(g_{1-\alpha} * u)(0) = \lim_{t\to 0^+} \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\sigma)^{-\alpha}\sigma^{\alpha-1}(t\sigma)^{1-\alpha}u(t\sigma)d\sigma = x.$$

In the special case $c = 0$, the system $(\text{RLAFCP})_x$ degrades into the system studied in [19]. Moreover, if $\alpha = \beta$ and $c \neq -1$, the system $(\text{RLAFCP})_x$ is converted to $D_t^\alpha u(t) = \frac{A}{1-c}u(t)$, which is also the system studied by Li and Peng [19] with $\frac{A}{1-c}$ replacing A.

However, in real material, the existence of damping is inevitable. Taking advantage of fractional calculus, fractional-order damping with a viscoelastic damping element provides a better model to describe a damping system [18]. In recent years, the dynamics and vibration analysis of fractional order damped systems have been of great interest to researchers [5,6, 8,11,16,28]. The above references indicate that the fractional damping may affect essentially the systems. Thus, the Riemann–Liouville abstract fractional differential equations discussed by Li and Peng [19] cannot exactly describe the practical dynamics. It is necessary to consider the fractional systems with the influence of fractional damping. The appropriate dynamical system should be described by $(\text{RLAFCP})_x$. Very recently, Ashyralyev [2] considered the well-posedness of differential equation with Riemann–Liouville fractional damping, which is just the special case of $(\text{RLAFCP})_x$ that $\alpha = 1, \beta = \frac{1}{2}, c = 1$ and $u(0) = 0$. For fractional differential equations with fractional damping, we have to mention the work [21] by Lizama. Concretely, [21] developed an operator theoretic approach to study systems described by

$$\begin{aligned} &D_t^\alpha u'(t) + \mu D_t^\alpha u(t) = Au(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}(u'(0) + \mu u(0)), \\ &t > 0,\ 0 < \alpha \le 1,\ \mu \ge 0. \end{aligned} \tag{4}$$

Denote ${}^C D_t^\alpha u(t) = D_t^\alpha u(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}u(0)$. Then ${}^C D_t^\alpha$ is the standard Caputo fractional derivative operator [12] and the system (4) is equivalent to ${}^C D_t^\alpha u'(t) + \mu {}^C D_t^\alpha u(t) = Au(t)$. Moreover, the (α, μ)-family used in [21] is strongly continuous on $[0, \infty)$, which is different from the fractional resolvent in [19] because zero is a singularity of fractional resolvent. However, the idea of [19] stems from Lizama [21] and Chen [7].

Motivated by the work in [19,21,7,2], the aim of this paper is to develop an operator theoretic approach to study the Riemann–Liouville abstract fractional Cauchy problem with damping $(\text{RLAFCP})_x$. The organization of this paper is as follows. In Section 2 some related definitions and preliminaries are presented. Section 3 is to introduce the notion of Riemann–Liouville fractional (α, β, c) resolvent, some properties are deduced. In Section 4 the strong solution of the system $(\text{RLAFCP})_x$ is considered. Moreover, a fractional diffusion equation with fractional damping is presented.

2. Preliminaries

In this section we shall introduce some preliminaries. Let $\gamma > 0$, $m = [\gamma]$ denotes the smallest integer greater than or equal to γ. Denote by $\mathbb{C}$ the set consisting of all complex numbers. For

$z \in \mathbb{C}$, $\operatorname{Re} z$ denotes the real part of z. Let $(X, \|\cdot\|)$ be a Banach space and A linear operator on X. We denote the resolvent operator of A by $R(\lambda, A) = (\lambda - A)^{-1}$ with λ being in the resolvent set $\rho(A)$. For $T \geq 0$, $L^1((0,T); X)$ denotes the space of X-valued Bochner integrable functions $u : (0,T) \to X$ with the norm $\|u\|_{L^1((0,T);X)} = (\int_0^T \|u(t)\|dt)$. The Banach space of continuous functions $u : [0,T] \to X$ with the norm $\|u\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|u(t)\|$ is denoted by $C([0,T], X)$. Obviously, $L^1((0,T); X)$ is a Banach space. We denote the convolution of two functions by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geq 0.$$

The Laplace transform of a function $u \in L^1_{\text{loc}}(R^+, X)$ is defined by

$$\hat{u}(\lambda) := \int_0^\infty e^{-\lambda t} u(t)dt$$

for suitable λ such that the integral $\int_0^\infty e^{-\lambda t} u(t)dt$ is convergent on X.

For $\beta \geq 0$, let

$$g_\beta(t) = \begin{cases} \dfrac{t^{\beta-1}}{\Gamma(\beta)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. For any $u \in L^1((0,T); X)$, the α-order Riemann–Liouville fractional integral of u is defined by

$$J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)d\tau. \tag{6}$$

Denote $J_t^0 u(t) = u(t)$. The fractional integral operators $\{J_t^\alpha\}_{\alpha \geq 0}$ satisfy the semigroup property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \quad \alpha, \beta \geq 0. \tag{7}$$

Definition 3. Let $0 < \alpha \leq 1$. The α-order Riemann–Liouville fractional derivative is defined by

$$D_t^\alpha u(t) = \frac{d}{dt}(g_{1-\alpha} * u)(t) = \frac{d}{dt} J_t^{1-\alpha} u(t), \tag{8}$$

where $u \in L^1((0,T); X)$, $g_{1-\alpha} * u \in W^{1,1}((0,T); X)$.

Definition 4. The general Mittag-Leffler function is defined by

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^\infty \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (z, \beta, \gamma \in \mathbb{C}; \operatorname{Re}(\alpha) > 0),$$

where

$$(\gamma)_n := \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1, & n = 0 \\ \gamma(\gamma+1)\cdots(\gamma+n-1), & n \neq 0. \end{cases}$$

Remark 1. If $\gamma = 1$ the general Mittag-Leffler function is equal to the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$. Furthermore, if, in addition, $\beta = 1$, the general Mittag-Leffler function is equal to the one-parameter Mittag-Leffler function $E_\alpha(\cdot)$.

Remark 2. The one-parameter Mittag-Leffler function was introduced by Mittag-Leffler [24,23]; Wiman [30,31] defined two-parameter Mittag-Leffler function; while Prabhakar [27] introduced the general Mittag-Leffler function. For more details of Mittag-Leffler function, we refer to [10,13,17,29].

We denote the two-parameter Mittag-Leffler integral operator by

$$\mathbb{E}_t^{\alpha,\beta,a} f(t) = \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(a(t-s)^\alpha) f(s)ds, \quad t > 0. \tag{9}$$

The Mittag-Leffler function $E^\gamma_{\alpha,\beta}$ is related to the Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E^\gamma_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\gamma\alpha-\beta}}{(\lambda^\alpha - \omega)^\gamma}, \quad \operatorname{Re}\lambda > |\omega|^{1/\alpha}. \tag{10}$$

Moreover, the Mittag-Leffler function is related to the following theorem.

Theorem 1 (*[13]*)**.** *For $\alpha, \delta, \xi, \gamma, \mu > 0, \beta > \tau > 0$, there hold the following three equalities:*

$$J_t^\xi (t^{\beta-1} E^\gamma_{\alpha,\beta}(wt^\alpha)) = t^{\beta+\xi-1} E^\gamma_{\alpha,\beta+\xi}(wt^\alpha), \tag{11}$$

$$\int_0^t (t-s)^{\beta-1} s^{\mu-1} E^\gamma_{\alpha,\beta}(w(t-s)^\alpha) E^\delta_{\alpha,\mu}(ws^\alpha)ds = t^{\beta+\mu-1} E^{\gamma+\delta}_{\alpha,\beta+\mu}(wt^\alpha), \tag{12}$$

and

$$D_t^\tau t^{\beta-1} E^\gamma_{\alpha,\beta}(wt^\alpha) = t^{\beta-\tau-1} E^\gamma_{\alpha,\beta-\tau}(wt^\alpha). \tag{13}$$

3. Riemann–Liouville fractional (α, β, c) resolvent

In this section we introduce the notion Riemann–Liouville fractional (α, β, c) resolvent and obtain some of its properties. The notion and properties will be used in the rest of this paper. In this section we assume $0 < \beta < \alpha \le 1$.

Definition 5. A family $\{T(t)\}_{t>0}$ of bounded linear operators is called a Riemann–Liouville fractional (α, β, c) resolvent on Banach Space X, if it satisfies the following assumptions:

(a) For any $x \in X$, $T(\cdot)x \in C((0,\infty), X)$, and

$$\lim_{t\to 0^+} \Gamma(\alpha) t^{1-\alpha} T(t)x = x \quad \text{for all } x \in X; \tag{14}$$

(b) $T(s)T(t) = T(t)T(s)$ for all $t, s > 0$;

(c) for all $t, s > 0$, there holds

$$\begin{aligned} &T(s)\mathbb{E}_t^{\alpha-\beta,\alpha,-c} T(t) - \mathbb{E}_s^{\alpha-\beta,\alpha,-c} T(s)T(t) \\ &\quad = s^{\alpha-1} E_{\alpha-\beta,\alpha}(-s^\alpha) \mathbb{E}_t^{\alpha-\beta,\alpha,-c} T(t) - t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^\alpha) \mathbb{E}_s^{\alpha-\beta,\alpha,-c} T(s). \end{aligned} \tag{15}$$

Remark 3. The integrals in (15) are understood strongly in the sense of Bochner.

Remark 4. Definition 5 is a natural extension of the known definition in the case that there is no damping ($c = 0$), see [7,19]. We have to mention that in [7], the "α-resolvent operator function" is defined on $[0, \infty)$. The reason why we do not consider in Definition 5 the value of $T(\cdot)$ at 0 is that in (15) the function $t \mapsto t^{\alpha-1} E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})$ possesses a singularity at zero.

Remark 5. Define an operator $(PT)(\cdot)$ as follows: for any $x \in X$,

$$(PT)(t)x \triangleq \begin{cases} \Gamma(\alpha)t^{1-\alpha}T(t)x, & t > 0, \\ x, & t = 0. \end{cases}$$

Then, (a) of Definition 5 implies that $(PT)(\cdot)$ is strongly continuous on $[0, \infty)$. By the uniform boundedness theorem, it follows that the function $t \mapsto \Gamma(\alpha)t^{1-\alpha}T(t)$ is bounded over the interval $(0, b]$ with $b > 0$.

Remark 6. For any $x \in X$, we have $T(\cdot)x \in L^1_{\text{loc}}([0, \infty), X)$. In fact, (14) implies that there exists $\delta > 0$, such that

$$\|T(t)x\| \leq \frac{2t^{\alpha-1}}{\Gamma(\alpha)}\|x\|, \quad \forall 0 < t < \delta. \tag{16}$$

Thus, for any $\tau_0 > 0$, we have that

$$\int_0^{\tau_0} \|T(t)x\|dt \leq \int_0^{\tau_0} \frac{2t^{\alpha-1}}{\Gamma(\alpha)}dt\|x\| = \frac{2\tau_0^{\alpha}}{\Gamma(\alpha+1)}\|x\|, \quad \text{if } \tau_0 \leq \delta,$$

and

$$\begin{aligned} \int_0^{\tau_0} \|T(t)x\|dt &= \int_0^{\delta} \|T(t)x\|dt + \int_{\delta}^{\tau_0} \|T(t)x\|dt \\ &\leq \int_0^{\delta} \frac{2t^{\alpha-1}}{\Gamma(\alpha)}dt\|x\| + \int_{\delta}^{\tau_0} \|T(t)x\|dt \\ &\leq \frac{2\delta^{\alpha}}{\Gamma(\alpha+1)}\|x\| + \int_{\delta}^{\tau_0} \|T(t)x\|dt, \quad \text{if } \tau_0 > \delta. \end{aligned}$$

Therefore, $T(\cdot)x \in L^1_{\text{loc}}([0, \infty), X)$.

Remark 6 guarantees that many integrals, such as $\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)$, $\int_0^t T(t-\sigma)f(\sigma)d\sigma$ with $f \in L^1_{\text{loc}}([0, \infty), X)$, make sense in the rest of this paper.

Now we define the generator of Riemann–Liouville fractional (α, β, c) resolvent as follows.

Definition 6. The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t \to 0+} \frac{t^{1-\alpha}T(t)x - E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x}{t^{\alpha}} \text{ exists} \right\}$$

and

$$Ax = \Gamma(2\alpha) \lim_{t \to 0+} \frac{t^{1-\alpha}T(t)x - E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x}{t^{\alpha}} \quad \text{for } x \in D(A)$$

is the generator of the Riemann–Liouville fractional (α, β, c) resolvent $\{T(t)\}_{t>0}$, where $D(A)$ is the domain of A.

Proposition 1. *Let $\{T(t)\}_{t>0}$ be a Riemann–Liouville fractional (α, β, c) resolvent on Banach space X, and A its generator.*

Then

(a) $T(t)D(A) \subset D(A)$ *and* $AT(t)x = T(t)Ax$ *for each* $x \in D(A)$.

(b) *For each* $x \in X, t > 0$,

$$T(t)x = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x + A\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x. \tag{17}$$

(c) *For each* $x \in D(A), t > 0$,

$$T(t)x = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x + \mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)Ax. \tag{18}$$

(d) *A is closed and densely defined.*

Proof. (a) Let $x \in D(A)$. Combining (b) of Definition 5 and the boundedness of $T(t)$ $(t > 0)$, we derive

$$\begin{aligned}
&\Gamma(2\alpha)\lim_{s\to 0+}\frac{s^{1-\alpha}T(s)T(t)x - E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})T(t)x}{t^{\alpha}}\\
&= T(t)\Gamma(2\alpha)\lim_{s\to 0+}\frac{s^{1-\alpha}T(s)x - E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})x}{s^{\alpha}}\\
&= T(t)Ax, \quad t > 0.
\end{aligned}$$

This means that for any $t > 0$, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$.

(b) Obviously, $s \mapsto E_{\alpha,\alpha-\beta}(-s^{\alpha})$ is uniformly continuous on [0, 1] and

$$E_{\alpha-\beta,\alpha}(0) = \frac{1}{\Gamma(\alpha)}. \tag{19}$$

For any $x \in X$, we have

$$\begin{aligned}
&\left\|\Gamma(2\alpha)\frac{\mathbb{E}_s^{\alpha-\beta,\alpha,-c}T(s)x}{s^{2\alpha-1}} - x\right\|\\
&= \left\|\Gamma(2\alpha)\int_0^s (s-\sigma)^{\alpha-1}s^{1-2\alpha}E_{\alpha-\beta,\alpha}(-c(s-\sigma)^{\alpha-\beta})T(\sigma)xd\sigma - x\right\|\\
&= \left\|\Gamma(2\alpha)\int_0^1 (1-\sigma)^{\alpha-1}s^{1-\alpha}E_{\alpha-\beta,\alpha}(-c(s-s\sigma)^{\alpha-\beta})T(s\sigma)xd\sigma - x\right\|\\
&= \left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(s-s\sigma)^{\alpha-\beta})\right.\\
&\qquad\left.\times\,\Gamma(\alpha)(s\sigma)^{1-\alpha}T(s\sigma)xd\sigma - x\right\|\\
&= \left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-1}\Gamma(\alpha)E_{\alpha-\beta,\alpha}(-c(s-s\sigma)^{\alpha-\beta})\right.\\
&\qquad\left.\cdot\,\Gamma(\alpha)(s\sigma)^{1-\alpha}T(s\sigma)xd\sigma - \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-1}xd\sigma\right\|
\end{aligned}$$

$$\leq \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-1} d\sigma$$
$$\times \sup_{\sigma\in(0,1]} \|\Gamma(\alpha)E_{\alpha-\beta,\alpha}(-c(s-s\sigma)^{\alpha-\beta})\Gamma(\alpha)(s\sigma)^{1-\alpha}T(s\sigma)x - x\|$$
$$= \sup_{\sigma\in(0,1]} \|\Gamma(\alpha)E_{\alpha-\beta,\alpha}(-c(s-s\sigma)^{\alpha-\beta})\Gamma(\alpha)(s\sigma)^{1-\alpha}T(s\sigma)x - x\|. \tag{20}$$

The combination of (19), (20), (a) of Definition 5, and Remark 5 implies that

$$\lim_{s\to 0^+} \Gamma(2\alpha)\frac{\mathbb{E}_s^{\alpha-\beta,\alpha,-c}T(s)x}{s^{2\alpha-1}} = x. \tag{21}$$

By Definition 5 and (21), we obtain

$$\begin{aligned}
&A\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x \\
&= \Gamma(2\alpha) \lim_{s\to 0+} \frac{s^{1-\alpha}T(s)\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x - E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x}{s^{\alpha}} \\
&= \Gamma(2\alpha) \lim_{s\to 0+} \frac{\mathbb{E}_s^{\alpha-\beta,\alpha,-c}T(s)\Big(T(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x\Big)}{s^{2\alpha-1}} \\
&= T(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x,
\end{aligned}$$

which implies that (b) holds.

(c) Let $x \in D(A)$. By Definition 6, the limit

$$\lim_{t\to 0+} \frac{t^{1-\alpha}T(t)x - E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x}{t^{\alpha}}$$

exists, which implies that the function

$$g(s) = \frac{t^{1-\alpha}T(t)x - E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x}{t^{\alpha}}$$

is bounded for sufficiently small $s > 0$. For $t > 0$, by the dominated convergence theorem, we derive

$$\begin{aligned}
T(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x &= A\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x \\
&= \Gamma(2\alpha) \lim_{s\to 0+} \frac{s^{1-\alpha}T(s) - E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})}{s^{\alpha}} \\
&\quad \cdot \int_0^t (t-\sigma)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-\sigma)^{\alpha-\beta})T(\sigma)x d\sigma \\
&= \Gamma(2\alpha) \lim_{s\to 0+} \int_0^t (t-\sigma)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-\sigma)^{\alpha-\beta}) \\
&\quad \cdot T(\sigma)\frac{s^{1-\alpha}T(s) - E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})}{s^{\alpha}} x d\sigma \\
&= \mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)Ax.
\end{aligned}$$

(d) Assume that $t > 0$. Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. It follows from part (c) that

$$\begin{aligned} T(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x &= \lim_{n\to\infty}\Big(T(t)x_n - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x_n\Big) \\ &= \lim_{n\to\infty}\int_0^t (t-\sigma)^{\alpha-1}E_{\alpha-\beta,\alpha} \\ &\quad \times (-c(t-\sigma)^{\alpha-\beta})T(\sigma)Ax_n d\tau \\ &= \int_0^t (t-\sigma)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-\sigma)^{\alpha-\beta})T(\sigma)y d\tau \\ &= \mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)y. \end{aligned} \tag{22}$$

Using (21), we have

$$\begin{aligned} Ax &= \Gamma(2\alpha)\lim_{t\to 0+}\frac{t^{1-\alpha}T(t)x - E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x}{t^{\alpha}} \\ &= \Gamma(2\alpha)\lim_{t\to 0+}\frac{\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)y}{t^{2\alpha-1}} \\ &= y. \end{aligned} \tag{23}$$

The closedness of A is proved.

For every $x \in X$, set $x_t = \mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t)x$, from part (b) it follows that $x_t \in D(A)$, and by (21) we have $\Gamma(2\alpha)t^{1-2\alpha}x_t \to x$ as $t \to 0^+$. Thus $\overline{D(A)} = X$. □

Remark 7. Proposition 1 is a generalization of [7, Proposition 3.3] and the proof is inspired by the proof of [19, Theorem 3.1].

Theorem 2. *Let $\{T(t)\}_{t>0}$ and $\{S(t)\}_{t>0}$ be the Riemann–Liouville fractional (α,β,c) resolvents with generators A and B respectively. Then $T(t) = S(t)$ for $t > 0$, provided A is equal to B.*

Proof. It follows from (c) and (a) of Proposition 1 that

$$\begin{aligned} t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta}) * S(t)x &= (T(t) - \mathbb{E}_t^{\alpha-\beta,\alpha,-c}AT(t)) * S(t)x \\ &= T(t) * S(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta}) * AT(t) * S(t)x \\ &= T(t) * S(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta}) * T(t) * AS(t)x \\ &= T(t) * (S(t)x - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta}) * AS(t)x) \\ &= T(t) * t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x \\ &= t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta}) * T(t)x, \quad x \in D(A). \end{aligned}$$

Therefore, by Titchmarsh's theorem, we obtain

$$T(t)x = S(t)x, \quad x \in D(A),\ t > 0.$$

Since $D(A)$ is dense in X, we derive $T(t) = S(t),\ t > 0$. The proof is completed. □

Remark 8. Theorem 2 tells us that every closed and densely defined operator A generates at most one Riemann–Liouville fractional (α,β,c) resolvent.

Definition 7. The Riemann–Liouville fractional (α, β, c) resolvent $\{T(t)\}_{t>0}$ is called exponentially bounded if there exist constants $M \geq 1, \omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad t > 0. \tag{24}$$

An operator A is said to belong $\mathcal{C}^{\alpha,\beta}(M, \omega)$ if A generates a Riemann–Liouville fractional (α, β, c) resolvent satisfying (24). Denote $\mathcal{C}^{\alpha,\beta}(\omega) = \bigcup\{\mathcal{C}^{\alpha,\beta}(M, \omega); M \geq 1\}$.

In the rest of this section we will introduce a necessary and sufficient condition such that a given closed linear operator generates an exponentially bounded Riemann–Liouville fractional (α, β, c) resolvent, which is a generalization of Proposition 3.5 of [7].

Theorem 3. *$A \in \mathcal{C}^{\alpha,\beta}(M, \omega)$ if and only if $(\omega^\alpha + c\omega^\beta, \infty) \subset \rho(A)$ and there is a family $\{T(t)\}_{t>0}$ of bounded linear operators satisfying*

(1) *for any $x \in X$, $T_\alpha(\cdot)x \in C((0, \infty), X)$, and*

$$\lim_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}T(t)x = x \quad \text{for all } x \in X; \tag{25}$$

(2) $T(t)T(s) = T(s)T(t),\ t, s > 0$;
(3) $\|T(t)\| \leq Me^{\omega t}, M \geq 1,\ t > 0$;
(4) *there holds*

$$R(\lambda^\alpha + c\lambda^\beta, A)x = \int_0^\infty e^{-\lambda t}T(t)x dt, \quad \lambda > \omega,\ x \in X. \tag{26}$$

If this is the case, $\{T(t)\}_{t>0}$ is the Riemann–Liouville fractional (α, β, c) resolvent generated by A.

Proof (*Necessity*)**.** Assume that A generates a Riemann–Liouville fractional (α, β, c) resolvent $\{T(t)\}_{t>0}$ such that $\|T(t)\| \leq Me^{\omega t},\ t > 0,\ \omega > 0$. Then $T(\cdot)$ is Laplace transformable. Let

$$R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt, \quad \lambda > \omega > 0. \tag{27}$$

By (c) of Proposition 1, it follows that for any $x \in D(A)$ there holds

$$T(t)x = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x + \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})T(s)Axds. \tag{28}$$

Taking Laplace transform on both sides of (28), we derive that

$$\begin{aligned} R(\lambda)x &= \frac{1}{\lambda^\alpha + 1}x + \frac{\lambda^\beta}{\lambda^\alpha + 1}R(\lambda)Ax \\ &= \frac{1}{\lambda^\alpha + c\lambda^\beta}x + \frac{1}{\lambda^\alpha + c\lambda^\beta}AR(\lambda)x, \quad \forall\, x \in D(A), \end{aligned} \tag{29}$$

where the commutativity of A and $T(\cdot)$ is used. By (d) of Proposition 1, $D(A)$ is dense in X. This implies that

$$R(\lambda) = \frac{1}{\lambda^\alpha + c\lambda^\beta}I + \frac{1}{\lambda^\alpha + c\lambda^\beta}AR(\lambda) \quad \text{on } X. \tag{30}$$

Hence

$$R(\lambda)[\lambda^\alpha + c\lambda^\beta - A]x = x$$

and

$$[\lambda^\alpha + c\lambda^\beta - A]R(\lambda) = I \quad \text{on } X.$$

Therefore $(\omega^\alpha + c\omega^\beta, \infty) \in \rho(A)$ and

$$R(\lambda) = [\lambda^\alpha + c\lambda^\beta - A]^{-1}. \tag{31}$$

This indicates that (26) holds.

(Sufficiency) By (26), we derive that

$$R(\lambda^\alpha + c\lambda^\beta, A) = R(\lambda). \tag{32}$$

Obviously, the following resolvent identify related to operator A holds

$$\begin{aligned} &R(\lambda^\alpha + c\lambda^\beta, A) - R(\mu^\alpha + c\mu^\beta, A) \\ &\quad = [\mu^\alpha + c\mu^\beta - \lambda^\alpha - c\lambda^\beta]R(\lambda^\alpha + c\lambda^\beta, A)R(\mu^\alpha + c\mu^\beta, A). \end{aligned} \tag{33}$$

Then, it follows from (32) and (33) that

$$\frac{1}{\mu^\alpha + c\mu^\beta}\frac{R(\lambda)}{\lambda^\alpha + c\lambda^\beta} - \frac{1}{\lambda^\alpha + c\lambda^\beta}\frac{R(\mu)}{\mu^\alpha + c\mu^\beta} = \frac{R(\lambda)R(\mu)}{\lambda^\alpha + c\lambda^\beta} - \frac{R(\lambda)R(\mu)}{\mu^\alpha + c\mu^\beta}.$$

By virtue of Laplace transform, we have

$$\begin{aligned} &T(s)\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t) - \mathbb{E}_s^{\alpha-\beta,\alpha,-c}T(s)T(t) \\ &\quad = s^{\alpha-1}E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})\mathbb{E}_t^{\alpha-\beta,\alpha,-c}T(t) - t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})\mathbb{E}_s^{\alpha-\beta,\alpha,-c}T(s), \end{aligned}$$

for any $t, s > 0$. The proof is completed. □

4. Solutions to Riemann–Liouville abstract fractional Cauchy problem

In this section we study the existence and the uniqueness of the strong solution of the homogeneous and inhomogeneous Riemann–Liouville fractional equations with damping. Throughout this section, we assume $0 < \beta < \alpha \le 1$.

Definition 8. A function $u \in C((0,\infty), X)$ is called a strong solution of $(\text{RLAFCP})_x$, if it satisfies

(a) $u \in C((0,\infty), D(A))$;
(b) $t \mapsto \int_0^t (t-s)^{-\alpha}u(s)ds$ and $t \mapsto \int_0^t (t-s)^{-\beta}u(s)ds$ are continuously differentiable on $(0,\infty)$;
(c) there holds

$$D_t^\alpha u(t) + cD_t^\beta u(t) = Au(t), \quad t > 0.$$

In order to obtain our main results, we firstly introduce the following lemma.

Lemma 1. *For any $\gamma > 0$, one has*

$$E_{\alpha-\beta}(-ct^{\alpha-\beta}) + ct^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-ct^{\alpha-\beta}) = 1, \quad t > 0. \tag{34}$$

Proof. The proof can be obtained directly from the definition of Mittag-Leffler function. □

Inspired by Proposition 3.8 of [7], we consider the following theorem which gives the existence condition of the strong solution of $(\text{RLAFCP})_x$.

Theorem 4. *Assume that the operator A generates a Riemann–Liouville fractional (α, β, c) resolvent $\{T(t)\}_{t>0}$ on Banach space. Then for any $x \in D(A)$, $T(\cdot)x$ is a strong solution.*

Proof. Assume that $x \in D(A)$. By Lemma 1, it follows that

$$\left(E_{\alpha-\beta}(-ct^{\alpha-\beta}) + ct^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-ct^{\alpha-\beta}) - 1\right) * T(t)x = 0, \quad t > 0.$$

Using (c) of Proposition 1, we have

$$\begin{aligned}
&\int_0^t E_{\alpha-\beta}(-c(t-s)^{\alpha-\beta})T(s)ds + c\int_0^t (t-s)^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-c(t-s)^{\alpha-\beta})T(s)xds \\
&= \int_0^t T(s)xds \\
&= \int_0^t \Bigg(s^{\alpha-1}E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})x + \int_0^s (s-\sigma)^{\alpha-1}E_{\alpha-\beta,\alpha} \\
&\quad \times (-c(s-\sigma)^{\alpha-\beta})d\sigma T(\sigma)Ax\Bigg)ds \\
&= t^\alpha E_{\alpha-\beta,\alpha+1}(-ct^{\alpha-\beta})x + \int_0^t (t-s)^\alpha E_{\alpha-\beta,\alpha+1}(-c(t-s)^{\alpha-\beta})T(s)Axds, \\
&t > 0.
\end{aligned} \tag{35}$$

Equality (35) can be rewritten as follows

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})\Bigg[\int_0^s \frac{(s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma \\
&\quad + c\int_0^s \frac{(s-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma\Bigg]ds \\
&= \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})\left[x + \int_0^s AT(\sigma)xd\sigma\right]ds, \quad t > 0.
\end{aligned}$$

By Titchmarsh's theorem,

$$\begin{aligned}
&\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma + c\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma = x + \int_0^t AT(\sigma)xd\sigma, \\
&t > 0.
\end{aligned} \tag{36}$$

By (18), we derive

$$\begin{aligned}
\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma = \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}\Bigg[&\sigma^{\alpha-1}E_{\alpha-\beta,\alpha}(-c\sigma^{\alpha-\beta})x \\
&+ \int_0^\sigma (\sigma-\theta)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(\sigma-\theta)^{\alpha-\beta})T(\theta)Axd\theta\Bigg]d\sigma
\end{aligned}$$

$$= t^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-ct^{\alpha-\beta})x + \int_0^t (t-\sigma)^{\alpha-\beta} \times E_{\alpha-\beta,1+\alpha-\beta}(-c(t-\sigma)^{\alpha-\beta})T(\sigma)Axd\sigma, \quad t>0.$$

This implies that $t \mapsto \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma$ is continuously differentiable on $(0,\infty)$. Obviously, $x + \int_0^t AT(\sigma)xd\sigma$ is continuously differentiable on $(0,\infty)$. Hence $t \mapsto \int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma$ is continuously differentiable on $(0,\infty)$ and we can obtain that for $t>0$,

$$\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma + c\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma = \frac{d}{dt}\int_0^t AT(\sigma)xd\sigma.$$

This implies that $t \mapsto D_t^\alpha T(t)x$, $t \mapsto AD_t^\beta T(t)x$ are continuously on $(0,\infty)$, and

$$D_t^\alpha T(t)x + cD_t^\beta T(t)x = AT(t)x.$$

The proof is completed. □

Lemma 2. *Let $x \in X$. Assume that $(g_{1-\alpha} * u)(0) = x$. Then $(g_{1-\beta} * u)(0) = 0$.*

Proof. By the dominated convergence theorem, we can compute

$$\begin{aligned}(g_{1-\beta} * u)(0) &= (g_{\alpha-\beta} * g_{1-\alpha} * u)(0)\\ &= \lim_{t\to 0^+} \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}(g_{1-\alpha} * u)(s)ds\\ &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\lim_{t\to 0^+}\left(t^{\alpha-\beta}(g_{1-\alpha} * u)(ts)\right)ds\\ &= 0.\end{aligned}$$

The proof is completed. □

Theorem 5. *Assume that A generates a Riemann–Liouville fractional (α,β,c) resolvent $\{T(t)\}_{t>0}$ on Banach space X. Assume that u is a strong solution of* (RLAFCP)$_x$. *Then $u(t) = T(t)x,\ t>0$.*

Proof. Assume that u is a strong solution of (AFRE)$_{x,0}$. Then

$$(g_{1-\alpha} * u)(0) = x \tag{37}$$

and

$$\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}u(\sigma)d\sigma + c\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}u(\sigma)d\sigma = Au(t). \tag{38}$$

Observe that (37) implies $g * u \in C([0,\infty), X)$.

Combining (37), (38) and Lemma 2, we have

$$\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}u(\sigma)d\sigma + c\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}u(\sigma)d\sigma = x + \int_0^t Au(\sigma)d\sigma, \quad t>0. \tag{39}$$

Taking convolution of $t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})$ on both sides of (39), we derive

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})\Bigg[\int_0^s \frac{(s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}u(\sigma)d\sigma$$

$$+c\int_0^s \frac{(s-\sigma)^{-\beta}}{\Gamma(1-\beta)}u(\sigma)d\sigma\Bigg]ds$$

$$=\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})\left[x+\int_0^s Au(\sigma)d\sigma\right]ds,\quad t>0. \tag{40}$$

By (11) and (40), we have

$$\left(E_{\alpha-\beta}(-ct^{\alpha-\beta})+ct^{\alpha-\beta-1}E_{\alpha-\beta,1+\alpha-\beta}(-ct^{\alpha-\beta})\right)*u(t)$$

$$=\int_0^t\left(s^{\alpha-1}E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})x+\mathbb{E}_s^{\alpha-\beta,\alpha,-c}Au(s)x\right)ds,\quad t>0. \tag{41}$$

The combination of (1) and (41) implies that

$$\int_0^t u(s)ds=\int_0^t\left(s^{\alpha-1}E_{\alpha-\beta,\alpha}(-cs^{\alpha-\beta})x+\mathbb{E}_s^{\alpha-\beta,\alpha,-c}Au(s)x\right)ds,\quad t>0. \tag{42}$$

Taking the derivative on both sides of (42), we obtain that

$$u(t)=t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x+\mathbb{E}_t^{\alpha-\beta,\alpha,-c}Au(t)x,\quad t>0.$$

By Proposition 1, for any $t>0$, we have

$$\begin{aligned}t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})*u(t)&=\left(T(t)-t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})*AT(t)\right)*u(t)\\&=T(t)*u(t)-t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha}\\&\quad\times(-ct^{\alpha-\beta})*AT(t)*u(t)\\&=T(t)*\left(u(t)-t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})*Au(t)\right)\\&=t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})*T(t)x.\end{aligned}$$

By Titchmarsh's theorem, we have $u(t)=T(t)x,\ t>0.$ □

Remark 9. The combination of Remark 8 and Theorem 5 implies that for any $x\in D(A)$, the strong solution of $(\text{RLAFCP})_x$ is unique.

Remark 10. The final part of proof of Theorem 5 implies that for $x\in D(A)$, $u(t)=T(t)x,\ t>0$ is the unique strong solution of the Volterra equation

$$u(t)=t^{\alpha-1}E_{\alpha-\beta,\alpha}(-ct^{\alpha-\beta})x+\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-c(t-s)^{\alpha-\beta})Au(s)ds,\quad t>0. \tag{43}$$

Here a function $u\in C((0,\infty),X)$ is called a strong solution of the Volterra equation (43), if $u\in C((0,\infty),D(A))$ and the equality (43) holds.

By Theorem 4, Theorem 5 and Remark 10, the Riemann–Liouville Abstract Fractional Cauchy Problem $(\text{RLAFCP})_x$ is equivalent to the Volterra equation (43) in the sense of strong solution. So it is reasonable to define the mild solution of $(\text{RLAFCP})_x$ as follows.

Definition 9. Assume that A generates a Riemann–Liouville fractional (α, β, c) resolvent $\{T(t)\}_{t>0}$. For any $x \in X$, we call $u(t) = T(t)x,\ t > 0$, the mild solution of $(\text{RLAFCP})_x$.

Remark 10 implies that the mild solution of $(\text{RLAFCP})_x$ exists and is unique.

Example 1. As is stated in the introduction, fractional differential equations are of essential importance to model anomalous diffusion. Concretely, anomalous diffusion can be modeled by fractional differential equation $D_t^\alpha u(t,x) = k^2 \frac{\partial^2}{\partial x^2} u(t,x)$, see [15,22]. However, in the real material, the existence of damping is inevitable. Therefore, anomalous diffusion should be described by fractional differential equations with damping. We consider the following fractional partial differential equations with fractional damping and Dirichlet boundary conditions

$$\begin{cases} D_t^\alpha u(t,x) + c D_t^\beta u(t,x) = k^2 \dfrac{\partial^2}{\partial x^2} u(t,x), \quad t > 0,\ x \in (0,1) \\ u(t,0) = u(t,1) = 0, \\ \lim\limits_{t\to 0^+} J_t^{1-\alpha} u(t,x) = f(x). \end{cases} \tag{44}$$

Example 3.1 in [19] is the special case $(c = 0)$ of system (44). In order to write the system (44) as the abstract form of system $(\text{FLAFCP})_x$, we take

- $X = L^2(0, \pi)$;
- $A = k^2 \frac{\partial^2}{\partial x^2}$ with domain $D(A) = \{g \in W^{2,2}(0,1) : g(0) = g(1) = 0\}$.

Observe that A is closed, densely defined and has eigenvalues $\{\lambda_n = -k^2 n^2 \pi^2\}_{n\in N}$ with eigenfunctions $\{\sin(n\pi x)\}_{n\in N}$. Moreover, we can obtain $\rho(A) = \mathbb{C} \setminus \{-k^2n^2\pi^2,\ n \in N\}$. For $g(x) = \sum_{n=1}^\infty g_n \sin(n\pi x)$, we define the family $\{T(t)\}_{t>0}$ by

$$(T(t)g)(x) = \sum_{n=1}^\infty \left(\sum_{p=0}^\infty (-1)^p k^{2p} n^{2p} \pi^{2p} t^{\alpha(p+1)-1} E_{\alpha-\beta,\alpha(p+1)}(-ct^{\alpha-\beta}) \right) g_n \sin(n\pi x).$$

We shall show that $\{T(t)\}_{t>0}$ is a Riemann–Liouville fractional (α, β, c) resolvent.

(a) Obviously, $T(\cdot)g \in C((0,\infty), X)$, and

$$\begin{aligned} \lim_{t\to 0^+} \Gamma(\alpha) t^{1-\alpha} T(t) g &= \lim_{t\to 0^+} \Gamma(\alpha) \sum_{n=1}^\infty \Bigg(\sum_{p=0}^\infty (-1)^p k^{2p} n^{2p} \pi^{2p} t^{\alpha p} \\ &\qquad \times E_{\alpha-\beta,\alpha(p+1)}(-ct^{\alpha-\beta}) \Bigg) g_n \sin(n\pi \cdot) \\ &= g. \end{aligned}$$

(b) The commutativity of $T(t)$ and $T(s)$, $t, s > 0$ is obtained directly by the definition of $\{T(t)\}_{t>0}$.

(c) By (10), the Laplace transform of $\{T(t)\}_{t>0}$ is obtained by

$$\begin{aligned} \hat{T}(\lambda) g &= \sum_{n=1}^\infty \left(\sum_{p=0}^\infty (-1)^p k^{2p} n^{2p} \pi^{2p} \frac{\lambda^{(\alpha-\beta)(p+1)-\alpha(p+1)}}{(\lambda^{\alpha-\beta} + c)^{p+1}} \right) g_n \sin(n\pi \cdot) \\ &= \sum_{n=1}^\infty \frac{1}{\lambda^\alpha + c\lambda^\beta} \left(\sum_{p=0}^\infty (-1)^p k^{2p} n^{2p} \pi^{2p} \frac{1}{(\lambda^\alpha + c\lambda^\beta)^p} \right) g_n \sin(n\pi \cdot) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda^\alpha + c\lambda^\beta} \frac{1}{1 + \frac{k^2n^2\pi^2}{\lambda^\alpha + c\lambda^\beta}} g_n \sin(n\pi\cdot)$$
$$= \sum_{n=1}^{\infty} \frac{1}{\lambda^\alpha + c\lambda^\beta + k^2n^2\pi^2} g_n \sin(n\pi\cdot).$$

Assume that $w \in D(A)$ such that

$$\lambda w - Aw = g, \quad \lambda \in \rho(A).$$

Then, $w = \sum_{n=0}^{\infty} w_n \sin(n\pi\cdot)$, and

$$\sum_{n=0}^{\infty}(\lambda + k^2n^2\pi^2)w_n \sin(n\pi\cdot) = \sum_{n=0}^{\infty} g_n \sin(n\pi\cdot), \quad \lambda \in \rho(A).$$

This indicates that

$$R(\lambda, A)g = \sum_{n=0}^{\infty} \frac{g_n}{\lambda + k^2n^2\pi^2} \sin(n\pi\cdot), \quad \lambda \in \rho(A).$$

Hence

$$R(\lambda^\alpha + c\lambda^\beta, A)g = \sum_{n=0}^{\infty} \frac{g_n}{\lambda^\alpha + c\lambda^\beta + k^2n^2\pi^2} \sin(n\pi\cdot)$$
$$= \int_0^\infty e^{-\lambda t} T(t) g dt, \quad \lambda > 0, \ x \in X. \tag{45}$$

The combination of the denseness of $D(A)$, equality (45) and Theorem 3 indicates that $\{T(t)\}_{t>0}$ is a Riemann–Liouville fractional (α, β, c) resolvent generated by A. By Theorems 4 and 5, for any $f \in D(A)$, fractional differential equation (44) has a unique strong solution. Moreover, for any $f \in X$, $(T(t)f)(x)$ is the unique mild solution of system (44).

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