J. Math. Anal. Appl. 418 (2014) 638–675

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness and time-decay for compressible viscoelastic fluids in critical Besov space

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ARTICLE INFO

Article history:
Received 19 March 2014
Available online 12 April 2014
Submitted by H.-M. Yin

Keywords:
Compressible viscoelastic fluids
L^p based Besov space
Global well-posedness
Time-decay

ABSTRACT

In this paper, we construct local solution with highly oscillating initial velocity and then get the global strong solution in the L^p based Besov space which improves the result of J. Qian, Z. Zhang (2010) [25] and X. Hu, D. Wang (2011) [14]. The local existence and uniqueness lies on the Lagrange coordinate transform and the contraction mapping theorem. The global result lies on a decomposition of the system and some commutator estimates. In the last part, we prove a time-decay in the critical Besov space framework which seems to have little investigation. The proof is based on the properties of the Green's matrix and various interpolations between Besov type spaces.

1. Introduction

Many fluids do not satisfy the Newtonian law. A viscoelastic fluid of the Oldroyd type is one of the classical non-Newtonian fluids which exhibits elastic behavior, such as memory effects. The elastic properties of the fluid are described by associating the fluid motions with an energy functional of deformation tensor U. Let us assume the elastic energy is $W(U)$, then the compressible viscoelastic system can be written as

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = \operatorname{div}\big(2\mu\mathcal{D}(v) + \nabla\big(\lambda \operatorname{div}(v)\big)\big) + \operatorname{div}\left(\dfrac{W_U(U)U^T}{\det(U)}\right), \\ \partial_t U + u \cdot \nabla U = \nabla u U. \end{cases} \tag{1.1}$$

Here ρ is the density and $v(x,t)$ is the velocity of the fluid. The pressure $P(\rho)$ is a given state equation with $P'(\rho) > 0$ for any ρ and $\mathcal{D}(v) = \frac{1}{2}(\nabla v + \nabla v^T)$ is the strain tensor. The Lamé coefficients μ and λ are assumed to satisfy

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http://dx.doi.org/10.1016/j.jmaa.2014.04.008

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0. \tag{1.2}$$

Such a condition ensures ellipticity for the operator $-\nabla(2\mu\mathcal{D}\cdot) - \nabla(\lambda\nabla\cdot)$ and is satisfied in the physical case, where $\lambda + 2\mu/N \approx 0$. Moreover, $W_U(U)$ is the Piola–Kirchhoff tensor and $\frac{W_U(U)U^T}{\det(U)}$ the Cauchy–Green tensor, respectively. For a special case of the Hookean linear elasticity, $W(U) = |U|^2$.

For the incompressible viscoelastic fluids, there are many important works recently. In [16], the authors proved the well posedness problem and found the relation

$$\nabla_k F^{ij} - \nabla_j F^{ik} = F^{lj}\nabla_l F^{ik} - F^{lk}\nabla_l F^{ij},$$

with $F = U - I$. This relation indicates that the linear term $\nabla \times F$ is actually a higher order term. F. Lin, C. Liu and P. Zhang [19,20] proved the local well posedness in Hilbert space H^s, and global well posedness with small initial data. In the proof of the global part, they capture the damping mechanism on F through very subtle energy estimates. At last, in [26], the authors proved the global well posedness of the incompressible version of system (1.1) in the critical L^p framework which allows us to construct the unique global solution for highly oscillating initial velocity.

For compressible viscoelastic fluids, in [14,25] the authors proved the local and global well-posedness in the L^2 based critical Besov type space. Their work used the properties of the viscoelastic fluids deeply and their results indicated that the deformation tensor U plays a similar role as the density ρ. It should be mentioned that the global existence of a smooth solutions is still an open problem, even in for incompressible viscoelastic fluids. P. Lions and N. Masmoudi [21] proved the global existence of a weak solution with general initial data in the case that the contribution of the strain rate in the constitutive equation is neglected. So it is important to study the global existence of system (1.1) near equilibrium in a functional space that is as large as possible.

There are two main goals: firstly, we want to establish the local and global well posedness in the L^p based critical Besov space. We use Lagrange coordinate method to obtain local existence and uniqueness with highly oscillating initial velocity. For global results, the main difficulty is that the stress tensor U only has weak dissipation which is revealed in [25] and [14]. In [25], the authors transformed the system into a more complex hyperbolic parabolic system and the Green's matrix is too complex to be studied. There are nineteen equations for the transformed system, so it is hard to generalize this method to the L^p based Besov space. In [14], X. Hu and D. Wang used a different idea. The authors decomposed the whole system into three similar subsystems. Since each subsystem can be handled easily, the authors can get the global existence. So it seems this method can be generalized to the L^p based Besov space framework. However, if we want to ensure the equivalence of transformed system and original system, we need to impose a condition on p which prevents us from getting the global existence with highly oscillating initial velocity. If we have the condition $\operatorname{div}(U^T) = 0$ as in the incompressible case, we may obtain the global existence with highly oscillating initial velocity. The more explicit results can be found in Theorem 2.4.

Secondly, we would like to give a time decay rate in the Besov space framework. Time decay is studied by many authors for the compressible Navier–Stokes equations [10,12,13,17,18,22,24]. For the compressible viscoelastic fluid, X. Hu and G. Wu in [15] give a detailed analysis about time-decay rate in the Sobolev space framework. However, in the Besov space framework, there is little work about time-decay rate for compressible fluid system even for the compressible Navier–Stokes equations. Our second aim is to get a time-decay rate as in [1] which gives a time decay of incompressible Navier–Stokes equations in the critical Besov space framework. Under low regularity assumptions (compared with H^3 used in compressible Navier–Stokes equations), we get that the solution will decay in the critical framework. The explicit results can be found in Theorem 2.4.

This paper will be organized as follows: In Section 2, we give some basic properties of the viscoelastic fluids and the main results. In Section 3, we will give a short introduction to the Littlewood–Paley decomposition and Besov space. In Section 4, we give the proof of local existence and a continuation criterion which is important for us to prove the global existence. Then, in Section 5, we prove the global well posedness in the L^p based Besov space. At last, in Section 6, we give the time-decay through the properties of the Green's matrix and various interpolations between Besov type spaces.

2. Basic properties and main results

In this section, we will give some useful intrinsic properties about viscoelastic fluids and the main results of this paper. Let us recall the following proposition which is proved in [25].

Proposition 2.1. *The density ρ and deformation tensor U in (1.1) satisfy the following relations:*

$$\operatorname{div}\left(\frac{U^T}{\det U}\right) = 0, \qquad \operatorname{div}\left(\rho U^T\right) = 0, \qquad \rho \det U = 1, \quad \textit{and}$$
$$U^{lk}\nabla_l U^{ij} - U^{lj}\nabla_l U^{ik} = 0, \tag{2.1}$$

if the initial data $(\rho, U)|_{t=0} = (\rho_0, U_0)$ satisfies

$$\operatorname{div}\left(\frac{U_0^T}{\det U_0}\right) = 0, \qquad \operatorname{div}\left(\rho_0 U_0^T\right) = 0, \qquad \rho_0 \det U_0 = 1, \quad \textit{and}$$
$$U_0^{lk}\nabla_l U_0^{ij} - U_0^{lj}\nabla_l U_0^{ik} = 0, \tag{2.2}$$

respectively.

Using Proposition 2.1, the last term in the second equation of (1.1) can be rewritten as

$$\nabla_j\left(\frac{\frac{\partial W(U)}{\partial U^{ik}}U^{jk}}{\det U}\right) = \frac{1}{\det U}U^{jk}\nabla_j\left(\frac{\partial W(U)}{\partial U^{ik}}\right) = \rho U^{jk}\nabla_j\left(\frac{\partial W(U)}{\partial U^{ik}}\right). \tag{2.3}$$

To simplify the presentation, as in [25], we consider Hookean linear elasticity, $W(U) = |U|^2$. Note that this does not reduce the essential difficulties. All results can be easily generalized to the case of more general elastic energy functionals. In view of (2.3), we will consider the following system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \rho\partial_t v^i + \rho v \cdot \nabla v^i - \operatorname{div}\left(2\mu\mathcal{D}(v)\right) - \nabla(\lambda \operatorname{div} v) + \nabla P(\rho) = \rho U^{jk}\nabla_j U^{ik}, \\ \partial_t U + v\cdot\nabla U = \nabla v U, \\ (\rho, v, U)|_{t=0} = (\rho_0, v_0, U_0) \end{cases} \tag{2.4}$$

where the initial data satisfies (2.2).

For the reader's convenience, we will list some lemmas proved in [9] and prove some new lemmas which will be used frequently in Appendix A.

Let us give the Lagrangian formulation of the system (2.4). Before we give out the Lagrangian form, we need some notations. We agree that for a C^1 function $F : \mathbb{R}^n \to \mathbb{R}^m$ then $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}^m$ with

$$(\operatorname{div} F)^j := \sum_i \partial_i F_{ij} \quad \text{for } 1 \leqslant j \leqslant m,$$

and that for $A = (A_{ij})_{1\leqslant i,j\leqslant n}$ and $B = (B_{ij})_{1\leqslant i,j\leqslant n}$ two $n\times n$ matrices, we denote

$$A : B = \operatorname{Tr} AB = \sum_{i,j} A_{ij}B_{ji}.$$

The notation adj(A) designates the adjugate matrix that is the transposed cofactor matrix. Of course if A is invertible then we have $\mathrm{adj}(A) = (\det A)A^{-1}$. Finally, given some matrix A, we define

$$\mathcal{D}_A(v) := \frac{1}{2}\big(\mathcal{D}v \cdot A + A^T \cdot \nabla v\big) \quad \text{and} \quad \mathrm{div}_A\, v := A^T : \nabla v.$$

Let X be the flow associated to the vector field u, that is the solution to

$$X_v(t,y) = y + \int_0^t v\big(\tau, X_v(\tau,y)\big)\, d\tau. \tag{2.5}$$

Denoting $J_v = \det(D_y X_v(t,y))$ and

$$\bar{\rho}(t,y) := \rho\big(t, X_v(t,y)\big), \qquad \bar{u}(t,y) := u\big(t, X_v(t,y)\big), \qquad \bar{U}(t,y) := U\big(t, X_v(t,y)\big),$$

with (ρ, u, U) a solution of (2.4), using Lemma A.1 in Appendix A, we deduce that

$$\begin{aligned} \overline{\Delta_x u} &= J_v^{-1}\,\mathrm{div}_y\big(\mathrm{adj}(D_y X_v)\overline{\nabla_x u}\big) = J_v^{-1}\,\mathrm{div}_y\big(\mathrm{adj}(D_y X_v)A^T \nabla_y \bar{u}\big), \\ \overline{\nabla_x\, \mathrm{div}_x\, u} &= J_v^{-1}\,\mathrm{div}_y\big(\mathrm{adj}(\nabla_y X_v)\,\overline{\mathrm{div}_x}\, u\big) = J_v^{-1}\,\mathrm{div}_y\big(\mathrm{adj}(\nabla_y X_v)A^T : \nabla_y \bar{u}\big), \\ \overline{\nabla_x P} &= J_v^{-1}\,\mathrm{div}_y\big(\mathrm{adj}(D_y X_v)\bar{P}\big). \end{aligned} \tag{2.6}$$

There is a term $U^{jk}\nabla_j U^{ik}$ in system (2.4), we then prove that

$$\overline{U^{jk}\nabla_j U^{ik}} = J_v^{-1}\bar{U}^{jk}\nabla_l\big(\big(\mathrm{adj}(D_y X_v)\big)_{lj}\bar{U}^{ik}\big). \tag{2.7}$$

Proof. Choosing a test function $\phi \in C_c^\infty$, we have

$$\begin{aligned} \int \nabla_{x_j} U^{ik}(x)\phi(x)\, dx &= -\int U^{ik}(x)\nabla_{x_j}\phi(x)\, dx \\ &= -\int \bar{U}^{ik}\overline{\nabla_{x_j}\phi}(y)J(y)\, dy \\ &= -\int J(y)\bar{U}^{ik}(y)\partial_{y_l}\bar{\phi}(y)\partial_{x_j}\big(X_v^{-1}(x)\big)^l\, dy \\ &= -\int J(y)\bar{U}^{ik}(y)\partial_{y_l}\bar{\phi}(y)\big(\big(D_y X_v(y)\big)^{-1}\big)_{lj}\, dy \\ &= \int \bar{\phi}(y)\partial_{y_l}\big(\big(\mathrm{adj}(D_y X_v)\big)_{lj}\bar{U}^{ik}\big)(y)\, dy \\ &= \int \phi(x)\partial_{y_l}\big(\big(\mathrm{adj}(D_y X_v)\big)_{lj}\bar{U}^{ik}\big)\big(X_v^{-1}(x)\big)J^{-1}\big(X_v^{-1}(x)\big)\, dx. \end{aligned}$$

So we complete the proof. □

Before we give out the Lagrangian formulation, we need to prove the following equality

$$J_v(t,y) = 1 + \int_0^t \mathrm{div}\, v\big(\tau, X_v(\tau,y)\big)J_v(\tau,y)\, d\tau. \tag{2.8}$$

Proof. Obviously, we know that

$$X_v(t,y) = y + \int_0^t v\big(\tau, X_v(\tau,y)\big)\,d\tau.$$

Applying D_y to both sides of the above equality, we have

$$D_y X_v(t,y) = I + \int_0^t D_x v\big(\tau, X_v(\tau,y)\big) D_y X_v(\tau,y)\,d\tau.$$

Solving this equation, we obtain

$$D_y X_v(t,y) = e^{\int_0^t D_x v(\tau, X_v(\tau,y))\,d\tau}.$$

Hence, we easily get

$$J_v(t,y) = e^{\int_0^t \operatorname{div}_x v(\tau, X_v(\tau,y))\,d\tau}.$$

So the proof is completed. □

Now, we give the following calculation

$$\begin{aligned}
\partial_t(J\bar{\rho}) &= \partial_t\left(\bar{\rho} + \int_0^t \operatorname{div}_x v\big(\tau, X(\tau,y)\big) J(\tau,y)\,d\tau\,\bar{\rho}\right) \\
&= \partial_t \rho\big(t, X(t,y)\big) + \nabla_x \rho\big(t, X(t,y)\big) v\big(t, X(t,y)\big) + \operatorname{div}_x v\big(t, X(t,y)\big) J(t,y) \rho\big(t, X(t,y)\big) \\
&\quad + \int_0^t \operatorname{div}_x v\big(\tau, X(\tau,y)\big) J(\tau,y)\,d\tau\,\partial_t \rho\big(t, X(t,y)\big) \\
&\quad + \int_0^t \operatorname{div}_x v\big(\tau, X(\tau,y)\big) J(\tau,y)\,d\tau\,\nabla_x \rho\big(t, X(t,y)\big) v\big(t, X(t,y)\big) \\
&= \partial_t \bar{\rho} + \nabla_x \bar{\rho}\bar{u} + \operatorname{div}_x \bar{u}\bar{\rho} + \int_0^t \operatorname{div}_x v\big(\tau, X(\tau,y)\big) J(\tau,y)\,d\tau (\partial_t \bar{\rho} + \nabla_x \bar{\rho}\bar{u} + \operatorname{div}_x \bar{u}\bar{\rho}) \\
&= \overline{\partial_t \rho + \operatorname{div}_x(\rho u)}.
\end{aligned} \tag{2.9}$$

Similar to the above calculations, we can also get

$$\overline{\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)} = \partial_t(J_v \bar{\rho}\bar{v}). \tag{2.10}$$

Define

$$L_v := I - \int_0^t \nabla_y \bar{v}(\tau,y)\,d\tau, \tag{2.11}$$

then we have

$$\begin{aligned}\partial_t(L_v U) &= \partial_t\left(U\big(t,X(t,y)\big) - \int_0^t \nabla_x v\big(\tau,X(\tau,y)\big)\nabla_y X(\tau,y)\,d\tau\,U\big(t,X(t,x)\big)\right)\\ &= \partial_t\bar U + \bar v\cdot\nabla_x\bar U - \nabla_x\bar v\nabla_y X(t,y)U\big(t,X(t,y)\big) - \int_0^t \nabla_x v\big(\tau,X(\tau,y)\big)\nabla_y X(\tau,y)\,d\tau\,(\partial_t\bar U + \bar v\cdot\nabla_x\bar U)\\ &= \partial_t\bar U + \bar v\cdot\nabla_x\bar U - \nabla_x\bar v\bar U - \int_0^t \nabla_x v\big(\tau,X(\tau,y)\big)\nabla_y X(\tau,y)\,d\tau\,(\partial_t\bar U + \bar v\cdot\nabla_x\bar U - \nabla_x\bar v\bar U)\\ &= \overline{\partial_t U + v\cdot\nabla_x U - \nabla_x vU},\end{aligned}\tag{2.12}$$

where we used

$$\nabla_y X(t,y) = I + \int_0^t \nabla_x v\big(\tau,X(\tau,y)\big)\nabla_y X(\tau,y)\,d\tau.$$

Hence, combining (2.6), (2.7), (2.9), (2.10) and (2.12), we will obtain the Lagrange formulation as follows

$$\begin{cases}\partial_t(J_v\bar\rho) = 0,\\ J_v\rho_0\partial_t\bar v - \operatorname{div}\big(\operatorname{adj}(DX)\big(2\mu\mathcal{D}_A(\bar v)\big) + \lambda\operatorname{div}_A\bar v I + P(\bar\rho)I\big) = \bar\rho\bar U\operatorname{div}\big(\operatorname{adj}(D_yX)\bar U\big),\\ \partial_t(L_v\bar U) = 0.\end{cases}\tag{2.13}$$

Then, we shall prove the existence and uniqueness of a local in time solution $(\bar\rho,\bar v,\bar U)$ for (2.13), with $\bar a = \bar\rho - 1$, $\bar F = \bar U - I$ in $C([0,T];B_p^{n/p})$ and $\bar v$ in the space

$$E_p(T) := \big\{v\in C\big([0,T];B_p^{n/p}\big),\partial_t v,\nabla^2 v\in L^1\big(0,T;B_p^{n/p}\big)\big\},$$

which will be endowed with the norm

$$\|v\|_{E_p(T)} := \|v\|_{L_T^\infty(B_p^{n/p-1})} + \big\|\partial_t v,\nabla^2 v\big\|_{L_T^1(B_p^{n/p-1})}.$$

At this stage, let us state our local well posedness result as follows.

Theorem 2.2. *Let $1 < p < 2n$ and $n \geqslant 2$. Let v_0 be vector field in $B_p^{n/p-1}$. Assume that ρ_0 satisfies $a_0 := \rho_0 - 1 \in B_p^{n/p}$ and U_0 satisfies $F_0 := U_0 - I \in B_p^{n/p}$ and*

$$\inf_x \rho_0(x) > 0.\tag{2.14}$$

Then system (2.13) has a unique local solution $(\bar\rho,\bar v,\bar U)$ with $(\bar a,\bar F)\in C([0,T];B_p^{n/p})$ and $\bar v\in E_p(T)$. Moreover, the flow map $(a_0,v_0,U_0)\to(\bar a,\bar v,\bar F)$ is Lipschitz continuous from $B_p^{n/p}\times B_p^{n/p-1}\times B_p^{n/p}$ to $C([0,T];B_p^{n/p})\times E_p(T)\times C([0,T];B_p^{n/p})$.

In Eulerian coordinates, this result recasts in:

Theorem 2.3. *Under the same assumption as in Theorem 2.2, system (2.4) has a unique local solution (ρ,v,U) with $v\in E_p(T)$, $U-I\in C([0,T];B_p^{n/p})$, ρ bounded away from 0 and $\rho-1\in C([0,T];B_p^{n/p})$.*

Denote

$$\begin{aligned}\mathcal{E}^s := \big\{(a,u,F) \in &\big(L^1\big(0,\infty;B_{2,p}^{s_p+1,s}\big) \cap \tilde{L}^\infty\big(0,\infty;B_{2,p}^{s_p-1,s}\big)\big)\\ &\times \big(L^1\big(0,\infty;B_{2,p}^{s_p+1,s+1}\big) \cap \tilde{L}^\infty\big(0,\infty;B_{2,p}^{s_p-1,s-1}\big)\big)^n\\ &\times \big(L^1\big(0,\infty;B_{2,p}^{s_p+1,s}\big) \cap \tilde{L}^\infty\big(0,\infty;B_{2,p}^{s_p-1,s}\big)\big)^{n\times n}\big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^{n/2} := \big\{(a,v,F) \in &L^1\big(0,\infty;B^{n/2+1}\big) \cap \tilde{L}^\infty\big(0,\infty;B^{n/2-1,n/2+1}\big)\\ &\times \big(L^1\big(0,\infty;B^{n/2+1,n/2+2}\big) \cap \tilde{L}^\infty\big(0,\infty;B^{n/2-1,n/2}\big)\big)^n\\ &\times \big(L^1\big(0,\infty;B^{n/2+1}\big) \cap \tilde{L}^\infty\big(0,\infty;B^{n/2-1,n/2+1}\big)\big)^{n\times n}\big\},\end{aligned}$$

where $s_p = s - \frac{n}{p} + \frac{n}{2}$.

The global well posedness and time-decay rate in Besov space framework are as follows.

Theorem 2.4. *Let $\bar{\rho} > 0$, $\bar{U}$ be two constants such that $P'(\bar{\rho}) > 0$. Suppose that $n = 3$. There exist two positive constants α_0 and C such that for all (ρ_0, v_0, U_0) with $\rho_0 - \bar{\rho} \in B_{2,p}^{n/2-1,n/p}$, $U_0 - \bar{U} \in B_{2,p}^{n/2-1,n/p}$, $v_0 \in B_{2,p}^{n/2-1,n/p-1}$, and*

$$\|\rho_0 - \bar{\rho}\|_{B_{2,p}^{n/2-1,n/p}} + \|v_0\|_{B_{2,p}^{n/2-1,n/p-1}} + \|U_0 - \bar{U}\|_{B_{2,p}^{n/2-1,n/p}} \leqslant \alpha_0, \tag{2.15}$$

then if $2 \leqslant p < n$, system (2.4) has a unique global solution $(\rho - \bar{\rho}, v, U - \bar{U}) \in \mathcal{E}^{n/p}$ with

$$\big\|(\rho - \bar{\rho}, v, U - \bar{U})\big\|_{\mathcal{E}^{n/p}} \leqslant C\big(\|\rho_0 - \bar{\rho}\|_{B_{2,p}^{n/2-1,n/p}} + \|v_0\|_{B_{2,p}^{n/2-1,n/p-1}} + \|U_0 - \bar{U}\|_{B_{2,p}^{n/2-1,n/p}}\big).$$

If in addition take smaller α_0, the initial data $\rho_0 - \bar{\rho} \in B^{n/2-1,n/2+1}$, $v_0 \in B^{n/2-1,n/2}$, $U_0 - \bar{U} \in B^{n/2-1,n/2+1}$, and

$$\|\rho_0 - \bar{\rho}\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}} + \|U_0 - \bar{U}\|_{B^{n/2-1,n/2+1}} \leqslant \alpha_0,$$

there holds

$$\big\|(\rho - \bar{\rho}, v, U - \bar{U})\big\|_{\mathcal{M}^{n/2}} \leqslant C\big(\|\rho_0 - \bar{\rho}\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}} + \|U_0 - \bar{U}\|_{B^{n/2-1,n/2+1}}\big),$$

and

$$\begin{aligned}\|\rho - \bar{\rho}\|_{B^{n/2-1+\epsilon}} &\leqslant C\alpha_0(1+t)^{-\frac{\epsilon}{2}},\\ \|v\|_{B^{n/2-1+\epsilon}} &\leqslant C\alpha_0(1+t)^{-\frac{\epsilon}{2}},\\ \|U - \bar{U}\|_{B^{n/2-1+\epsilon}} &\leqslant C\alpha_0(1+t)^{-\frac{\epsilon}{2}},\end{aligned}$$

where $\epsilon \in [0, 1]$.

Remark 2.5. If we take $\epsilon = 1$, due to the embedding properties in Besov space, we obtain the time-decay as follows

$$\|\rho - \bar{\rho}\|_{L^\infty} \leqslant C\alpha_0(1+t)^{-\frac{1}{2}},$$

$$\|v\|_{L^\infty} \leqslant C\alpha_0(1+t)^{-\frac{1}{2}},$$
$$\|U-\bar{U}\|_{L^\infty} \leqslant C\alpha_0(1+t)^{-\frac{1}{2}}.$$

This time-decay rate is slower than the optimal time-decay rate, however, we only assume the smallness of initial data in low regularity space. We can obtain optimal time-decay with low regularity assumptions is our future work.

3. Short introduction to Besov type space

In this section, we will give some basic knowledge about Besov space, which can be found in [2,23]. The homogeneous Littlewood–Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\phi \in C^\infty(\mathbb{R}^n)$, supported in $\mathcal{C} := \{\xi \in \mathbb{R}^n, 3/4 \leqslant |\xi| \leqslant 8/3\}$ such that

$$\sum_{q\in\mathbb{Z}} \phi\big(2^{-q}\xi\big) = 1 \quad \text{if } \xi \neq 0.$$

Denote $h = \mathcal{F}^{-1}\phi$, we then define the dyadic blocks as follows

$$\Delta_q u := \phi\big(2^{-q}D\big)u = 2^{qn}\int_{\mathbb{R}^N} h\big(2^q y\big)u(x-y)\,dy, \quad \text{and} \quad S_q u = \sum_{k\leqslant q-1} \Delta_k u.$$

The formal decomposition

$$u = \sum_{q\in\mathbb{Z}} \Delta_q u$$

is called homogeneous Littlewood–Paley decomposition. The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of ϕ, we have

$$\Delta_k\Delta_q u = 0 \quad \text{if } |k-q| \geqslant 2,$$
$$\Delta_k(S_{q-1}\Delta_q u) = 0 \quad \text{if } |k-q| \geqslant 5.$$

Let us now introduce the homogeneous Besov space.

Definition 3.1. We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{j\to-\infty} S_j u = 0 \quad \text{in } \mathcal{S}'.$$

Definition 3.2. Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $B^s_{p,r}$ consists of distributions u in $\mathcal{S}'_h$ such that

$$\|u\|_{B^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\Delta_j u\|^r_{L^p}\right)^{1/r} < +\infty.$$

From now on, the notation B^s_p will stand for $B^s_{p,1}$ and the notation B^s will stand for $B^s_{2,1}$.

The study of nonstationary PDEs usually requires spaces of type $L^r_T(X) := L^r(0,T;X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov spaces, so that it is natural to localize the equations through Littlewood–Paley decomposition. We then get estimates for each dyadic block and perform integration in time. However, in doing so, we obtain bounds in spaces which are not of type $L^r(0,T;B^s_p)$. This approach was initiated in [4] and naturally leads to the following definitions:

Definition 3.3. Let $(r,p) \in [1,+\infty]^2$, $T \in (0,+\infty]$ and $s \in \mathbb{R}$. We set

$$\|u\|_{\tilde{L}^r_T(B^s_p)} := \sum_{q\in\mathbb{Z}} 2^{qs}\left(\int_0^T \big\|\Delta_q u(t)\big\|^r_{L^p}\,dt\right)^{1/r}$$

and

$$\tilde{L}^r_T\big(B^s_p\big) := \big\{u \in L^r_T\big(B^s_p\big), \|u\|_{\tilde{L}^r_T(B^s_p)} < +\infty\big\}.$$

Owing to Minkowski inequality, we have $\tilde{L}^r_T(B^s_p) \hookrightarrow L^r_T(B^s_p)$. That embedding is strict in general if $r > 1$. We will denote by $\tilde{C}_T(B^s_p)$ the set of function u belonging to $\tilde{L}^\infty_T(B^s_p) \cap C([0,T];B^s_p)$.

We will often use the following interpolation inequality:

$$\|u\|_{\tilde{L}^r_T(B^s_p)} \leqslant \|u\|^\theta_{\tilde{L}^{r_1}_T(B^{s_1}_p)}\|u\|^{1-\theta}_{\tilde{L}^{r_2}_T(B^{s_2}_p)},$$

with

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} \quad \text{and} \quad s = \theta s_1 + (1-\theta)s_2,$$

and the following embeddings

$$\tilde{L}^r_T\big(B^{n/p}_p\big) \hookrightarrow L^r_T(\mathcal{C}_0) \quad \text{and} \quad \tilde{C}_T\big(B^{n/p}_p\big) \hookrightarrow C\big([0,T]\times\mathbb{R}^n\big).$$

Another important space is the hybrid Besov space. We give the definitions and collect some properties.

Definition 3.4. Let $s,t \in \mathbb{R}$. We set

$$\|u\|_{B^{s,t}_{q,p}} := \sum_{q\leqslant R_0} 2^{qs}\|\Delta_q u\|_{L^q} + \sum_{q>R_0} 2^{qt}\|\Delta_q u\|_{L^p}$$

and

$$B^{s,t}_{q,p}\big(\mathbb{R}^N\big) := \big\{u \in \mathcal{S}'_h\big(\mathbb{R}^N\big), \|u\|_{B^{s,t}_{q,p}} < +\infty\big\},$$

where R_0 is a fixed large enough number determined in the proof of global existence.

Lemma 3.5.

(1) *We have* $B^{s,s}_{2,2} = B^s$.
(2) *If* $s \leqslant t$ *then* $B^{s,t}_{p,p} = B^s_{p,p} \cap B^t_{p,p}$. *Otherwise,* $B^{s,t}_{p,p} = B^s_{p,p} + B^t_{p,p}$.
(3) *The space* $B^{0,s}_{p,p}$ *coincides with the usual inhomogeneous Besov space.*
(4) *If* $s_1 \leqslant s_2$ *and* $t_1 \geqslant t_2$ *then* $B^{s_1,t_1}_{p,p} \hookrightarrow B^{s_2,t_2}_{p,p}$.
(5) *Interpolation: For* $s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R}$ *and* $\theta \in [0,1]$, *we have*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2,\theta\sigma_1+(1-\theta)\sigma_2}_{2,p}} \leqslant \|f\|^\theta_{B^{s_1,\sigma_1}_{2,p}}\|f\|^{1-\theta}_{B^{s_2,\sigma_2}_{2,p}}.$$

From now on, the notation $B_p^{s,t}$ will stand for $B_{p,p}^{s,t}$ and the notation $B^{s,t}$ will stand for $B_{2,2}^{s,t}$.

Throughout the paper, we shall use some paradifferential calculus. It is a nice way to define a generalized product between distributions which is continuous in functional spaces where the usual product does not make sense. The paraproduct between u and v is defined by

$$T_u v := \sum_{q\in\mathbb{Z}} S_{q-1}u\Delta_q v.$$

We thus have the following formal decomposition:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$R(u,v) := \sum_{q\in\mathbb{Z}} \Delta_q u\tilde{\Delta}_q v, \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

We will sometimes use the notation $T'_u v := T_u v + R(u,v)$.

For more information about Besov space and hybrid Besov space, we give references [2,3,6,7].

4. Local existence and uniqueness

In this section, we will use Lagrangian approach to prove the local existence results. Let us look at the following Lamé system with nonconstant coefficients:

$$\partial_t v - 2a\,\mathrm{div}\big(\mu\mathcal{D}(v)\big) - b\nabla(\lambda\,\mathrm{div}\,v) = f. \tag{4.1}$$

We assume that the following uniform ellipticity condition is satisfied:

$$\alpha := \min\Big(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(a\mu)(t,x), \inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(2a\mu+b\lambda)(t,x)\Big) > 0 \tag{4.2}$$

Concerning this equation, we have the following proposition which is proved in [9].

Proposition 4.1. *Let a, b, λ and μ be bounded functions satisfying (4.2). Assume that $a\nabla\mu$, $b\nabla\lambda$, $\mu\nabla a$ and $\lambda\nabla b$ are in $L^\infty(0,T;B_p^{n/p})$ for some $1<p<\infty$. There exist two constants η and κ such that if for some $m\in\mathbb{Z}$ we have*

$$\min\Big(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n} S_m(2a\mu+b\lambda)(t,x), \inf_{(t,x)\in[0,T]\times\mathbb{R}^n} S_m(a\mu)(t,x)\Big) \geqslant \frac{\alpha}{2},$$
$$\big\|(I-S_m)(\mu\nabla a, a\nabla\mu, \lambda\nabla b, b\nabla\lambda)\big\|_{L_T^\infty(B_p^{n/p-1})} \leqslant \eta\alpha, \tag{4.3}$$

then the solutions to (4.1) satisfy for all $t\in[0,T]$,

$$\|v\|_{L_t^\infty(B_p^s)} + \alpha\|v\|_{L_t^1(B_p^{s+2})} \leqslant C\big(\|v_0\|_{B_p^s} + \|f\|_{L_t^1(B_p^s)}\big)\exp\left(\frac{C}{\alpha}\int_0^t \big\|S_m(\mu\nabla a, a\nabla\mu, \lambda\nabla b, b\nabla\lambda)\big\|_{B_p^{n/p}}^2\,d\tau\right),$$

whenever $-\min(n/p, n/p') < s \leqslant n/p-1$ is satisfied.

Proposition 4.2. *Let p be in $(1,\infty)$. Let a, b, λ and μ be bounded functions satisfying (4.2). Assume in addition that there exist some constants $\bar{a}$, $\bar{b}$, $\bar{\lambda}$ and $\bar{\mu}$ such that*

$$2\bar{a}\bar{\mu}+\bar{b}\bar{\lambda}>0 \quad \text{and} \quad \bar{a}\bar{\mu}>0, \tag{4.4}$$

and such that $a-\bar{a}$, $b-\bar{b}$, $\mu-\bar{\mu}$ *and* $\lambda-\bar{\lambda}$ *are in* $C([0,T];B_p^{n/p})$. *Finally, suppose that*

$$\lim_{m\to+\infty}\big\|(I-S_m)(a-\bar{a},b-\bar{a},\lambda-\bar{\lambda},\mu-\bar{\mu})\big\|_{L_T^\infty(B_p^{n/p})}=0. \tag{4.5}$$

Then for any data $v_0\in B_p^{n/p}$ *and* $f\in L^1(0,T;B_p^{n/p})$ *with* s *satisfying* $-\min(n/p,n/p')<s\leqslant n/p-1$, *system (4.1) admits a unique solution* $v\in C([0,T];B_p^s)\cap L^1(0,T;B_p^{s+2})$. *That solution satisfies the estimates of Proposition 4.1 for all large enough* $m\in\mathbb{Z}$ *is fulfilled.*

For notational simplification, we omit the bars if there is no confusion. From the Lagrangian formulation (2.13), we know that

$$\rho=J_v^{-1}\rho_0, \qquad U=J_v^{-1}U_0. \tag{4.6}$$

So we can recast the velocity equation as follows:

$$L_{\rho_0}(v)=I_1(v,v)+I_2(v,U_0)+\rho_0^{-1}\operatorname{div}\big(I_3(v,v)+I_4(v,v)+I_5(v)\big), \tag{4.7}$$

with

$$\begin{aligned}
L_{\rho_0}(v) &:= \partial_t v-\rho_0^{-1}\operatorname{div}\big(2\mu\mathcal{D}(v)+\lambda\operatorname{div}vI\big),\\
I_1(v,w) &= (1-J_v)\partial_t w,\\
I_2(v,U_0) &= J_v^{-1}L_v^{-1}U_0\operatorname{div}\big(\operatorname{adj}(DX_v)L_v^{-1}U_0\big),\\
I_3(v,w) &= \big(\operatorname{adj}(DX_v)-I\big)\big(\mu\big(Dw\cdot A_v+A_v^T\cdot\nabla w\big)+\lambda A_v^T:\nabla w\big),\\
I_4(v,w) &= \mu\big(Dw\cdot(A_v-I)+(A_v-I)^T\cdot\nabla w\big)+\lambda(A_v-I)^T:\nabla wI,\\
I_5(v) &= \operatorname{adj}(DX_v)P\big(J_v^{-1}\rho_0\big).
\end{aligned}$$

In order to solve (2.13) locally, it suffices to show that the map

$$\Phi: u\to v \tag{4.8}$$

with v the solution to

$$\begin{cases} L_{\rho_0}(v)=I_1(u,u)+I_2(u,U_0)+\rho_0^{-1}\operatorname{div}\big(I_3(u,u)+I_4(u,u)+I_5(u)\big),\\ u|_{t=0}=u_0,\end{cases} \tag{4.9}$$

has a fixed point in $E_p(T)$ for small enough T.

Then we begin to prove Theorem 2.2.

Proof of Theorem 2.2. We introduce the free solution to the Lamé system corresponding to $\rho=1$, that is the vector field v_L in $E_p(T)$, satisfying

$$L_1v_L=0, \qquad v_L|_{t=0}=v_0. \tag{4.10}$$

We claim that the Banach fixed point theorem applies to the map Φ defined in (4.8) in some closed ball $\bar{B}_{E_p(T)}(v_L,R)$ with suitably small T and R. Denoting $\tilde{v}:=v-v_L$, we see that $\tilde{v}$ has to satisfy

$$\begin{cases} L_{\rho_0}\tilde{v} = I_1(u,u) + I_2(u,U_0) + \rho_0^{-1}\operatorname{div}\big(I_3(u,u) + I_4(u,u) + I_5(u)\big) + (L_1 - L_{\rho_0})v_L, \\ \tilde{v}|_{t=0} = 0. \end{cases} \tag{4.11}$$

If we choose m large enough, the coefficients on the left hand side obviously satisfy (4.3). If the right hand side of (4.11) in $L^1(0,T;B_p^{n/p})$, then, there exists $\tilde{v}$ in $E_p(T)$ by Proposition 4.2. So we now start to estimate the right hand side. Here, we assume a priori that for sufficiently small $c > 0$

$$\int_0^T \|Du\|_{B_p^{n/p}}\,dt \leqslant c. \tag{4.12}$$

Step 1: Stability of $\bar{B}_{E_p(T)}(v_L, R)$ for small enough R and T. Using Proposition 4.1, we obtain

$$\begin{aligned} \|\tilde{v}\|_{E_p(T)} \leqslant{}& Ce^{C\int_0^T \|S_m(\frac{a_0}{1+a_0})\|_{B_p^{n/p}}\,d\tau}\big(\big\|I_1(u,u)\big\|_{L_T^1(B_p^{n/p-1})} \\ &+ \big\|I_2(u,U_0)\big\|_{L_T^1(B_p^{n/p-1})} + \big\|(L_1 - L_{\rho_0})v_L\big\|_{L_T^1(B_p^{n/p-1})} \\ &+ \big(1 + \|a_0\|_{B_p^{n/p}}\big)\big(\big\|I_3(u,u)\big\|_{L_T^1(B_p^{n/p})} + \big\|I_4(u,u)\big\|_{L_T^1(B_p^{n/p})} + \big\|I_5(u)\big\|_{L_T^1(B_p^{n/p})}\big)\big). \end{aligned} \tag{4.13}$$

Since

$$\int_0^T \|Du\|_{B_p^{n/p}}\,d\tau \leqslant \int_0^T \|Dv_L\|_{B_p^{n/p}}\,d\tau + R,$$

we can take T and R small so that (4.12) is satisfied. Firstly, we know that

$$\int_0^T \left\|S_m\nabla\left(\frac{a_0}{1+a_0}\right)\right\|_{B_p^{n/p}} d\tau \leqslant C2^m\|a_0\|_{B_p^{n/p}}T. \tag{4.14}$$

For the estimates about terms $I_1(u,w)$, $I_3(u,w)$, $I_4(u,w)$, $I_5(u)$ and $(L_1 - L_{\rho_0})v_L$ appeared in [9], we only give the results about these terms.

$$\begin{aligned} \big\|I_1(u,w)\big\|_{L_T^1(B_p^{n/p})} &\lesssim \|Du\|_{L_T^1(B_p^{n/p})}\|\partial_t w\|_{L_T^1(B_p^{n/p-1})}, \\ \big\|I_i(u,w)\big\|_{L_T^1(B_p^{n/p})} &\lesssim \|Du\|_{L_T^1(B_p^{n/p})}\|Dw\|_{L_T^1(B_p^{n/p})}, \quad i = 3,4, \\ \big\|I_5(u)\big\|_{L_T^1(B_p^{n/p})} &\lesssim T\big(1 + \|a_0\|_{B_p^{n/p}}\big)\big(1 + \|Du\|_{L_T^1(B_p^{n/p})}\big), \\ \big\|(L_1 - L_{\rho_0})v_L\big\|_{L_T^1(B_p^{n/p-1})} &\lesssim \|a_0\|_{B_p^{n/p}}\big(1 + \|a_0\|_{B_p^{n/p}}\big)\|Dv_L\|_{L_T^1(B_p^{n/p})}. \end{aligned} \tag{4.15}$$

Now we estimate the term $I_2(u,U_0)$. Changing $\operatorname{adj}(DX_u)$, J_u^{-1} and L_u^1 to $(\operatorname{adj}(DX_u) - I) + I$, $(J_u^{-1} - 1) + 1$ and $(L_u^1 - 1) + 1$ respectively, using (4.12), Lemma A.2 and Lemma A.3, we can get

$$\big\|I_2(u,U_0)\big\|_{L_T^1(B_p^{n/p-1})} \lesssim T\big(1 + \|F_0\|_{B_p^{n/p}}\big)^2. \tag{4.16}$$

Plugging (4.14), (4.15) and (4.16) into (4.13), we have

$$\begin{aligned} \|\tilde{v}\|_{E_p(T)} \leqslant{}& Ce^{C2^m\|a_0\|_{B_p^{n/p}}T}\big(\|Du\|_{L_T^1(B_p^{n/p})}\|\partial_t u\|_{L_T^1(B_p^{n/p-1})} \\ &+ T\big(1 + \|F_0\|_{B_p^{n/p}}\big)^2 + \|a_0\|_{B_p^{n/p}}\big(1 + \|a_0\|_{B_p^{n/p}}\big)\|Dv_L\|_{L_T^1(B_p^{n/p})} \\ &+ \big(1 + \|a_0\|_{B_p^{n/p}}\big)\big(\|Du\|^2_{L_T^1(B_p^{n/p})} + T\big(1 + \|a_0\|_{B_p^{n/p}}\big)\big(1 + \|Du\|_{L_T^1(B_p^{n/p})}\big)\big)\big) \end{aligned}$$

Remembering that $u=\tilde u+v_L$ and $\tilde u\in B_{E_p(T)}(v_L,R)$, we could easily get

$$\|\tilde v\|_{E_p(T)}\leqslant Ce^{C2^m\|a_0\|_{B_p^{n/p}}T}\big(1+\|a_0\|_{B_p^{n/p}}\big)^2\big(1+\|F_0\|_{B_p^{n/p}}\big)^2\big(T+R^2+\|Dv_L\|^2_{L^1_T(B_p^{n/p})}+\|\partial_t v_L\|^2_{L^1_T(B_p^{n/p})}+\|a_0\|_{B_p^{n/p}}\|Dv_L\|_{L^1_T(B_p^{n/p})}\big)$$

At this stage, we first choose R so that for a small enough constant η,

$$\big(1+\|a_0\|_{B_p^{n/p}}\big)^2\big(1+\|F_0\|_{B_p^{n/p}}\big)^2R\leqslant\eta \tag{4.17}$$

and then take T so that

$$C2^m\|a_0\|_{B_p^{n/p}}T\leqslant\log 2,\qquad T\leqslant R^2,\qquad \|a_0\|_{B_p^{n/p}}\|Dv_L\|_{L^1_T(B_p^{n/p})}\leqslant R^2,$$
$$\|\partial_t v_L\|_{L^1_T(B_p^{n/p-1})}+\|Dv_L\|_{L^1_T(B_p^{n/p})}\leqslant R, \tag{4.18}$$

then we conclude that Φ maps $\bar B_{E_p(T)}(v_L,R)$ into itself.

Step 2: Contraction estimates. Now, let us prove Φ is contractive under the conditions (4.17) and (4.18). We consider two vector fields u^1 and u^2 in $\bar B_{E_p(T)}(v_L,R)$ and set $v^1:=\Phi(u^1)$ and $v^2:=\Phi(u^2)$. Let $\delta v:=v^2-v^1$ and $\delta u:=u^2-u^1$ and we have

$$L_{\rho_0}\delta v=I_1\big(u^1,\delta u\big)+(J_{u^1}-J_{u^2})\partial_t u^2+\big(I_2\big(u^2,U_0\big)-I_2\big(u^1,U_0\big)\big)+\rho_0^{-1}\operatorname{div}\big(\big(I_3\big(u^2,u^2\big)-I_3\big(u^1,u^1\big)\big)+\big(I_4\big(u^2,u^2\big)-I_3\big(u^1,u^1\big)\big)+\big(I_5\big(u^2\big)-I_5\big(u^1\big)\big)\big) \tag{4.19}$$

Given that $C2^m\|a_0\|_{B_p^{n/p}}T\leqslant\log 2$, we have

$$\|\delta v\|_{E_p(T)}\leqslant C\big(1+\|a_0\|_{B_p^{n/p}}\big)\big(\big\|I_1\big(u^1,\delta u\big)\big\|_{L^1_T(B_p^{n/p-1})}+\big\|(J_{u^1}-J_{u^2})\partial_t u^2\big\|_{L^1_T(B_p^{n/p-1})}+\big\|I_2\big(u^2,U_0\big)-I_2\big(u^1,U_0\big)\big\|_{L^1_T(B_p^{n/p-1})}+\big\|I_3\big(u^2,u^2\big)-I_3\big(u^1,u^1\big)\big\|_{L^1_T(B_p^{n/p})}+\big\|I_4\big(u^2,u^2\big)-I_4\big(u^1,u^1\big)\big\|_{L^1_T(B_p^{n/p})}+\big\|I_5\big(u^2\big)-I_5\big(u^1\big)\big\|_{L^1_T(B_p^{n/p})}\big). \tag{4.20}$$

The first, second and the last three terms in the bracket can be estimated as in the Navier–Stokes case, so here we only give the results as follows

$$\big\|I_1\big(u^1,\delta u\big)\big\|_{L^1_T(B_p^{n/p-1})}\lesssim\big\|Du^1\big\|_{L^1_T(B_p^{n/p})}\|\partial_t\delta u\|_{L^1_T(B_p^{n/p-1})},$$
$$\big\|(J_{u^1}-J_{u^2})\partial_t u^2\big\|_{L^1_T(B_p^{n/p-1})}\lesssim\|D\delta u\|_{L^1_T(B_p^{n/p})}\big\|\partial_t u^2\big\|_{L^1_T(B_p^{n/p-1})},$$
$$\big\|I_3\big(u^2,u^2\big)-I_3\big(u^1,u^1\big)\big\|_{L^1_T(B_p^{n/p})}\lesssim\big\|\big(Du^1,Du^2\big)\big\|_{L^1_T(B_p^{n/p})}\|D\delta u\|_{L^1_T(B_p^{n/p})},$$
$$\big\|I_4\big(u^2,u^2\big)-I_4\big(u^1,u^1\big)\big\|_{L^1_T(B_p^{n/p})}\lesssim\big\|\big(Du^1,Du^2\big)\big\|_{L^1_T(B_p^{n/p})}\|D\delta u\|_{L^1_T(B_p^{n/p})},$$
$$\big\|I_5\big(u^2\big)-I_5\big(u^1\big)\big\|_{L^1_T(B_p^{n/p})}\lesssim\big(1+\|a_0\|_{B_p^{n/p}}\big)T\|D\delta u\|_{L^1_T(B_p^{n/p})}. \tag{4.21}$$

Since

$$\begin{aligned}I_2\big(u^2,U_0\big)-I_2\big(u^1,U_0\big)&=J_{u^2}^{-1}L_{u^2}^{-1}U_0\operatorname{div}\big(\operatorname{adj}(DX_{u^2})L_{u^2}^{-1}U_0\big)-J_{u^1}^{-1}L_{u^1}^{-1}U_0\operatorname{div}\big(\operatorname{adj}(DX_{u^1})L_{u^1}^{-1}U_0\big)\\&=\big(J_{u^2}^{-1}-J_{u^1}^{-1}\big)L_{u^2}^{-1}U_0\operatorname{div}\big(\operatorname{adj}(DX_{u^2})L_{u^2}^{-1}U_0\big)\\&\quad+J_{u^1}^{-1}\big(L_{u^2}^{-1}-L_{u^1}^{-1}\big)U_0\operatorname{div}\big(\operatorname{adj}(DX_{u^2})L_{u^2}^{-1}U_0\big)\\&\quad+J_{u^1}^{-1}L_{u^1}^{-1}U_0\operatorname{div}\big(\big(\operatorname{adj}(DX_{u^2})-\operatorname{adj}(DX_{u^1})\big)L_{u^2}^{-1}U_0\big)\\&\quad+J_{u^1}^{-1}L_{u^1}^{-1}U_0\operatorname{div}\big(\operatorname{adj}(DX_{u^1})\big(L_{u^2}^{-1}-L_{u^1}^{-1}\big)U_0\big),\end{aligned}$$

through Lemma A.2 and Lemma A.3, we obtain

$$\big\|I_2\big(u^2,U_0\big)-I_2\big(u^2,U_0\big)\big\|_{L^1_T(B^{n/p-1}_p)}\lesssim T\|D\delta u\|_{L^1_T(B^{n/p}_p)}\big(1+\|F_0\|_{B^{n/p}_p}\big)^2. \tag{4.22}$$

Plugging (4.21), (4.22) into (4.20) and noticing that $u^1,u^2\in\bar B_{E_p(T)}(v_L,R)$, we have

$$\begin{aligned}\|\delta v\|_{E_p(T)}&\leqslant C\big(1+\|a_0\|_{B^{n/p}_p}\big)^2\big(1+\|F_0\|_{B^{n/p}_p}\big)^2R\|\delta u\|_{E_p(T)}\\&\leqslant C\eta\|\delta u\|_{E_p(T)}.\end{aligned}$$

If taking η smaller, we will have

$$\|\delta v\|_{E_p(T)}\leqslant\frac{1}{2}\|\delta u\|_{E_p(T)}.$$

So we conclude that the map Φ admits a unique fixed point in $\bar B_{E_p(T)}(v_L,R)$.

Step 3: Regularity of the density and the deformation tensor. We set $\rho:=J_v^{-1}\rho_0$ and $U:=L_v^{-1}U_0$. Then we have

$$\begin{aligned}a&=\big(J_v^{-1}-1\big)a_0+a_0+\big(J_v^{-1}-1\big),\\F&=\big(L_v^{-1}-I\big)F_0+F_0+\big(L_v^{-1}-I\big).\end{aligned}$$

Applying Lemma A.2 and the fact that $Dv\in L^1_T(B^{n/p}_p)$, it is clear that $J_v^{-1}-1$ and $L_v^{-1}-I$ all belong to $C([0,T];B^{n/p}_p)$. Hence, a and F belong to $C([0,T];B^{n/p}_p)$ too. For the space $B^{n/p}_p$ being continuously embedded in L^∞, we know that

$$\rho(t)\geqslant(1+a_0)-\big(1+\|a_0\|_{L^\infty}\big)\big(\|v_L\|_{L^1_T(B^{n/p}_p)}+R\big). \tag{4.23}$$

Hence we can take R and T smaller to ensure that ρ is positive on $[0,T]$.

Step 4: Uniqueness and continuity of the flow map. We now consider two couples (ρ^1_0,v^1_0,U^1_0) and (ρ^2_0,v^2_0,U^2_0) of data satisfying the assumptions in Theorem 2.2. We denote by (ρ^1,v^1,U^1) and (ρ^2,v^2,U^2) two solutions corresponding to the above initial data. Setting $\delta v:=v^2-v^1$, we see that

$$\begin{aligned}L_{\rho^1_0}(\delta v)&=(L_{\rho^1_0}-L_{\rho^2_0})\big(v^2\big)+(1-J_{v^1})\partial_t\delta v+(J_{v^1}-J_{v^2})\partial_tv^2+\big(I_2\big(v^2,U^2_0\big)-I_2\big(v^1,U^1_0\big)\big)\\&\quad+\big(\rho^1_0\big)^{-1}\operatorname{div}\big(I_3\big(v^2,v^2\big)-I_3\big(v^1,v^1\big)+I_4\big(v^2,v^2\big)-I_4\big(v^1,v^1\big)+I_5\big(v^2\big)-I_5\big(v^1\big)\big)\\&\quad+\big(\big(\rho^2_0\big)^{-1}-\big(\rho^1_0\big)^{-1}\big)\operatorname{div}\big(I_3\big(v^2,v^2\big)+I_4\big(v^2,v^2\big)+I_5\big(v^2\big)\big).\end{aligned}$$

We now give the estimation about $I_2(v^2,U^2_0)-I_2(v^1,U^1_0)$ as follows

$$I_2\big(v^2,U^2_0\big)-I_2\big(v^1,U^1_0\big)=M_1+M_2+M_3+M_4+M_5+M_6,$$

where

$$\begin{aligned}M_1&=\big(J^{-1}_{v^2}-J^{-1}_{v^1}\big)L^{-1}_{v^2}U^2_0\operatorname{div}\big(\operatorname{adj}(DX_{v^2})L^{-1}_{v^2}U^2_0\big),\\M_2&=J^{-1}_{v^1}\big(L^{-1}_{v^2}-L^{-1}_{v^1}\big)U^2_0\operatorname{div}\big(\operatorname{adj}(DX_{v^2})L^{-1}_{v^2}U^2_0\big),\\M_3&=J^{-1}_{v^1}L^{-1}_{v^1}\big(U^2_0-U^1_0\big)\operatorname{div}\big(\operatorname{adj}(DX_{v^2})L^{-1}_{v^2}U^2_0\big),\\M_4&=J^{-1}_{v^1}L^{-1}_{v^1}U^1_0\operatorname{div}\big(\big(\operatorname{adj}(DX_{v^2})-\operatorname{adj}(DX_{v^1})\big)L^{-1}_{v^2}U^2_0\big),\\M_5&=J^{-1}_{v^1}L^{-1}_{v^1}U^1_0\operatorname{div}\big(\operatorname{adj}(DX_{v^1})\big(L^{-1}_{v^2}-L^{-1}_{v^1}\big)U^2_0\big),\\M_6&=J^{-1}_{v^1}L^{-1}_{v^1}U^1_0\operatorname{div}\big(\operatorname{adj}(DX_{v^1})L^{-1}_{v^1}\big(U^2_0-U^1_0\big)\big).\end{aligned}$$

Using Lemma A.2 and Lemma A.3, we have

$$\begin{aligned}
&\|I_2(v^2,U_0^2)-I_2(v^1,U_0^1)\|_{L_t^1(B_p^{n/p})}\\
&\quad\leqslant Ct\|\delta v\|_{L_t^1(B_p^{n/p})}\big(1+\|(Dv^1,Dv^2)\|_{L_t^1(B_p^{n/p})}\big)^3\big(1+\|(F_0^1,F_0^2)\|_{B_p^{n/p}}\big)^2\\
&\qquad+Ct\|\delta U_0\|_{B_p^{n/p}}\big(1+\|(F_0^1,F_0^2)\|_{B_p^{n/p}}\big)\big(1+\|(Dv^1,Dv^2)\|_{L_t^1(B_p^{n/p})}\big)^4. \qquad (4.24)
\end{aligned}$$

The other terms can be estimated as in the Navier–Stokes case, so we only give the results as follows

$$\begin{aligned}
\|(L_{\rho_0^1}-L_{\rho_0^2})(v^2)\|_{L_t^1(B_p^{n/p-1})}&\leqslant C_{\rho_0^1}\|\delta\rho_0\|_{B_p^{n/p}}\|Dv^2\|_{L_t^1(B_p^{n/p})},\\
\|(1-J_{v^1})\partial_t\delta v\|_{L_t^1(B_p^{n/p-1})}&\leqslant C\|Dv^1\|_{L_t^1(B_p^{n/p})}\|\partial_t\delta v\|_{L_t^1(B_p^{n/p-1})},\\
\|(J_{v^1}-J_{v^2})\partial_t v^2\|_{L_t^1(B_p^{n/p-1})}&\leqslant C\|D\delta v\|_{L_t^1(B_p^{n/p})}\|\partial_t v^2\|_{L_t^1(B_p^{n/p-1})},\\
&\big\|(\rho_0^1)^{-1}\operatorname{div}\big(I_3(v^2,v^2)-I_3(v^1,v^1)\big)\big\|_{L_t^1(B_p^{n/p-1})}\\
&\quad\leqslant C\big(1+\|a_0^1\|_{B_p^{n/p}}\big)^2\big(t+\|(Dv^1,Dv^2)\|_{L_t^1(B_p^{n/p})}\big)\|D\delta v\|_{L_t^1(B_p^{n/p})},\\
&\big\|\big((\rho_0^2)^{-1}-(\rho_0^1)^{-1}\big)\operatorname{div}\big(I_3(v^2,v^2)\big)\big\|_{L_t^1(B_p^{n/p-1})}\\
&\quad\leqslant C_{\rho_0^1}\|\delta\rho_0\|_{B_p^{n/p}}\big(\|Dv^2\|_{L_t^1(B_p^{n/p})}^2+t+t\|Dv^2\|_{L_t^1(B_p^{n/p})}\big), \qquad (4.25)
\end{aligned}$$

where the terms containing I_4, I_5 have the same estimation as the term containing I_3.

Considering (4.24) and (4.25), using Proposition 4.1, we may have

$$\|\delta v\|_{E_p(t)}\leqslant C_{\rho_0^1,U_0^1}\big(N_1\|\delta\rho_0\|_{B_p^{n/p}}+N_2\|\delta v\|_{E_p(t)}+N_3\|\delta U_0\|_{B_p^{n/p}}\big),$$

where

$$\begin{aligned}
N_1&=\|Dv^1\|_{L_t^1(B_p^{n/p})}+\|Dv^1\|_{L_t^1(B_p^{n/p})}^2+\|D\delta v\|_{L_t^1(B_p^{n/p})}+\|D\delta v\|_{L_t^1(B_p^{n/p})}^2+T+T^2,\\
N_2&=\|Dv^1\|_{L_t^1(B_p^{n/p})}+\|\partial_t v^1\|_{L_t^1(B_p^{n/p-1})}+\|\partial_t\delta v\|_{L_t^1(B_p^{n/p-1})}+t\big(1+\|Dv^1\|_{L_t^1(B_p^{n/p})}+\|D\delta v\|_{L_t^1(B_p^{n/p})}\big)^3,\\
N_3&=t\big(1+\|Dv^1\|_{L_t^1(B_p^{n/p})}+\|D\delta v\|_{L_t^1(B_p^{n/p})}\big)^4.
\end{aligned}$$

Noticing the form of N_1, N_2 and N_3, a bootstrap argument shows that if t, δv_0, $\delta\rho_0$ and δU_0 are small enough then

$$\|\delta v\|_{E_p(t)}\leqslant 2C_{\rho_0^1,U_0^1}\big(\|\delta\rho_0\|_{B_p^{n/p}}+\|\delta U_0\|_{B_p^{n/p}}+\|\delta v_0\|_{B_p^{n/p}}\big). \qquad (4.26)$$

As regards the density and the deformation tensor, we notice that

$$\begin{aligned}
\delta a&=J_{v^1}^{-1}\delta a_0+\big(J_{v^2}^{-1}-J_{v^1}^{-1}\big)a_0^2,\\
\delta F&=J_{v^1}^{-1}\delta F_0+\big(L_{v^2}^{-1}-L_{v^1}^{-1}\big)F_0^2
\end{aligned}$$

Hence for all $t\in[0,T]$, we have

$$\begin{aligned}
\|\delta a(t)\|_{B_p^{n/p}}&\leqslant C\big(1+\|Dv^1\|_{L_t^1(B_p^{n/p})}\big)\|\delta a_0\|_{B_p^{n/p}}+C\|a_0^2\|_{B_p^{n/p}}\|D\delta v\|_{L_t^1(B_p^{n/p})},\\
\|\delta F(t)\|_{B_p^{n/p}}&\leqslant C\big(1+\|Dv^1\|_{L_t^1(B_p^{n/p})}\big)\|\delta F_0\|_{B_p^{n/p}}+C\|F_0^2\|_{B_p^{n/p}}\|D\delta v\|_{L_t^1(B_p^{n/p})}.
\end{aligned}$$

So we finally get uniqueness and continuity of the flow map on a small time interval. Then iterating the proof yields uniqueness on the initial time interval $[0, T]$. Note that it also yields Lipschitz continuity of the flow map. At this stage, we complete the proof of Theorem 2.2. $\square$

Next, we define

$$X_v(t, y) := y + \int_0^t \bar{v}(\tau, y)\, d\tau, \tag{4.27}$$

and

$$\rho(t, \cdot) := \rho \circ X^{-1}(t, \cdot), \qquad U(t, \cdot) := U \circ X^{-1}(t, \cdot), \qquad v(t, \cdot) := v \circ X^{-1}(t, \cdot).$$

Then using similar arguments as in the proof of Proposition 7 in [9], we will obtain Theorem 2.3.

Before going to the last part of this section, we give the following conventional lemma. We consider the following parabolic system

$$\begin{cases} \partial_t v + u \cdot \nabla v - b \operatorname{div}\bigl(2\mu \mathcal{D}(v)\bigr) - b\nabla(\lambda \operatorname{div} v) = f, \\ v|_{t=0} = v_0. \end{cases} \tag{4.28}$$

Above v is the unknown function. We assume that $v_0 \in B_p^{n/p}$ and $f \in L_T^1(B_p^{n/p})$, that u is time dependent vector-fields with coefficients in $L_T^1(B_p^{n/p+1})$, that b is bounded below by a positive constant $\underline{b}$ and that $\sigma := b - 1$ belongs to $L_T^\infty(B_p^{n/p})$.

Lemma 4.3. *Let $\underline{\nu} := \underline{b} \min(\mu, \lambda + 2\mu)$ and $\bar{\nu} := \mu + |\lambda + \mu|$. Assume that $s \in (-n/p, n/p]$. Let $m \in \mathbb{Z}$ be such that $b_m := 1 + S_m\sigma$ satisfies*

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^n} b_m(t, x) \geqslant \underline{b}/2. \tag{4.29}$$

There exist three constants c, C, and κ such that if in addition we have

$$\|\sigma - S_m\sigma\|_{L_T^\infty(B_p^{n/p})} \leqslant c\underline{\nu}/\bar{\nu}, \tag{4.30}$$

then setting

$$V(t) := \int_0^t \|u\|_{B_p^{n/p+1}}\, d\tau, \qquad Z_m(t) := 2^{2m}\bar{\nu}^2\underline{\nu}^{-1} \int_0^t \|\sigma\|_{B_p^{n/p}}^2\, d\tau,$$

we have for all $t \in [0, T]$,

$$\|v\|_{L_t^\infty(B_p^s)} + \kappa\underline{\nu}\|v\|_{L_t^1(B_p^{s+2})} \leqslant e^{C(V+Z_m)(t)} \left(\|v_0\|_{B_p^s} + \int_0^t \bigl\|f(\tau)\bigr\|_{B_p^s}\, d\tau \right).$$

In addition if $u = v$ and $s > 0$, $V(T)$ can be replaced by $\int_0^t \|\nabla v\|_{L^\infty}\, d\tau$.

The proof of this lemma may be found in [2,8] so we omit the proof here.

In the last part of this section, we will give the following continuation criterion.

Theorem 4.4. *Assume that system (2.4) has a unique solution $(a, v, F) \in C([0, T]; B_p^{n/p}) \times E_p(T) \times C([0, T]; B_p^{n/p})$ on the time interval $[0, T)$ which satisfies the following four conditions:*

(1) *the function a belongs to $L_T^\infty(B_p^{n/p})$,*
(2) *the function F belongs to $L_T^\infty(B_p^{n/p})$,*
(3) *the function $\rho = 1 + a$ is bounded away from zero,*
(4) *we have $\int_0^T \|v\|_{B_p^{n/p+1}}\,dt < \infty$ (if $p < n$, it can be replaced by $\int_0^T \|\nabla v\|_{L^\infty}\,dt < \infty$).*

Then (a, v, F) may be continued beyond T.

Proof. The velocity equation in system (2.4) can be written as follows

$$\partial_t v^i + v \cdot \nabla v^i - \left(1 - \frac{a}{1+a}\right) \operatorname{div}\bigl(2\mu \mathcal{D}(v)\bigr) - \left(1 - \frac{a}{1+a}\right) \nabla(\lambda \operatorname{div} v)$$
$$= -\frac{P'(1+a)}{1+a} \nabla_i a + \nabla_k F^{ik} + F^{jk} \nabla_j F^{ik}.$$

Taking $\sigma = -\frac{a}{1+a}$, $b = 1 - \frac{a}{1+a}$ and $\underline{b} = \frac{1}{1+\|a\|_{L_T^\infty(L^\infty)}}$ in Lemma 4.3, we have

$$\|v\|_{L_t^\infty(B_p^{n/p})} \leqslant e^{C(2^{2m} T \|a\|^2_{L_T^\infty(B_p^{n/p})} + \int_0^T \|v(\tau)\|_{B_p^{n/p+1}}\,d\tau)} \bigl(\|v_0\|_{B_p^{n/p-1}} + M_1(T) + M_2(T) + M_3(T)\bigr),$$

where

$$M_1(T) = \int_0^T \left\| \frac{P'(1+a)}{1+a} \nabla a \right\|_{B_p^{n/p-1}} d\tau,$$

$$M_2(T) = \int_0^T \bigl\|\nabla_k F^{ik}\bigr\|_{B_p^{n/p-1}}\,d\tau,$$

$$M_3(T) = \int_0^T \bigl\|F^{ik} \nabla_j F^{ik}\bigr\|_{B_p^{n/p-1}}\,d\tau.$$

The conditions (4.29), (4.30) can be easily verified by taking m large enough and using composition rules in Besov space for term $\frac{a}{1+a}$. Using product laws in Besov space for $M_1(T)$, $M_2(T)$ and $M_3(T)$, we finally obtain

$$\|v\|_{L_t^\infty(B_p^{n/p})} \leqslant C e^{C(2^{2m} T \|a\|^2_{L_T^\infty(B_p^{n/p})} + \int_0^T \|v(\tau)\|_{B_p^{n/p+1}}\,d\tau)} \bigl(\|v_0\|_{B_p^{n/p-1}} + T\|a\|_{L_T^\infty(B_p^{n/p})}$$
$$+ T\|F\|_{L_T^\infty(B_p^{n/p})} + T\|F\|^2_{L_T^\infty(B_p^{n/p})}\bigr).$$

Conditions (1) to (4) yield a bound on $\|v\|_{L_T^\infty(B_p^{n/p})}$. By replacing $\|a_0\|_{B_p^{n/p}}$, $\|F_0\|_{B_p^{n/p}}$ to $\|a\|_{L_T^\infty(B_p^{n/p})}$ and $\|F\|_{L_T^\infty(B_p^{n/p})}$ separately in (4.17), (4.18) and (4.23), we obtain an $\epsilon > 0$ such that system (2.4) with data $a(T-\epsilon)$, $v(T-\epsilon)$ and $F(T-\epsilon)$ has a solution on $[0, 2\epsilon]$. Since the solution (a, v, F) is unique on $[0, T)$, this provides a continuation of (a, v, F) beyond T. For $p < n$, the proof is similar, so we omit it. □

5. Global well-posedness

In this section, we state the global existence result. We first reformulate system (2.4). Without loss of generality, we assume $P'(1) = 1$ and set $\nu = \lambda + 2\mu$, $\mathcal{A} = \mu\Delta + (\lambda + \mu)\nabla \operatorname{div}$. Define

$$K(a)=\frac{P'(1+a)}{1+a}-1,\qquad d=\Lambda^{-1}\operatorname{div}v,$$
$$\Omega=\Lambda^{-1}\operatorname{curl}v\quad\text{with }(\operatorname{curl}v)_{ij}=\partial_{x_j}v^i-\partial_{x_i}v^j,$$
$$\mathcal{E}_{ij}=\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\big(F^{ij}+F^{ji}\big),$$
$$\mathcal{W}=\Lambda^{-1}\partial_{x_k}\big(F^{lk}\nabla_l F^{ij}-F^{lj}\nabla_l F^{ik}\big)-\Lambda^{-1}\partial_{x_k}\big(F^{lk}\nabla_l F^{ji}-F^{li}\nabla_l F^{jk}\big),$$

where

$$\Lambda^s f=\mathcal{F}^{-1}\big(|\xi|^s\hat{f}\big)\quad\text{for }s\in\mathbb{R}.$$

Performing the same procedure as in [14], we will obtain

$$\begin{cases}\partial_t a+\Lambda d=L-v\cdot\nabla a,\\ \partial_t d-\mu\Delta d-2\Lambda a=G-v\cdot\nabla d,\\ \partial_t\mathcal{E}+2\Lambda d=J-v\cdot\nabla\mathcal{E},\\ \partial_t\big(F^T-F\big)+\Lambda\Omega=I-v\cdot\nabla\big(F^T-F\big),\\ \partial_t\Omega-\mu\Delta\Omega-\Lambda(F^T-F)=H-v\cdot\nabla\Omega,\end{cases}\tag{5.1}$$

where the equation about d has the following equivalent form

$$\partial_t d-\nu\Delta d-\Lambda\mathcal{E}=K-v\cdot\nabla d,\tag{5.2}$$

where

$$\begin{aligned}
&L=-a\operatorname{div}v,\\
&G=v\cdot\nabla d+\Lambda^{-1}\operatorname{div}\Big(-v\cdot\nabla v+F\nabla F-K(a)\nabla a-\frac{a}{1+a}\mathcal{A}v-\operatorname{div}(aF)\Big),\\
&H=v\cdot\nabla\Omega+\Lambda^{-1}\operatorname{curl}\Big(-v\cdot\nabla v+F\nabla F-K(a)\nabla a-\frac{a}{1+a}\mathcal{A}v\Big)+\mathcal{W},\\
&I=(\nabla vF)^T-\nabla vF,\\
&J=-\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j},v^k\big]\partial_{x_k}\big(F^{ij}+F^{ji}\big)+\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\big((\nabla vF)^{ij}+(\nabla vF)^{ji}\big),\\
&K=v\cdot\nabla d+\Lambda^{-1}\operatorname{div}\Big(-v\cdot\nabla v+F\nabla F-K(a)\nabla a-\frac{a}{1+a}\mathcal{A}v+\operatorname{div}(aF)\Big).
\end{aligned}$$

5.1. Estimation of linearized system with convection terms

In this part, we need some estimations about linearized system with convection terms.

$$\begin{cases}\partial_t a+\Lambda d=L-v\cdot\nabla a,\\ \partial_t d-\mu\Delta d-2\Lambda a=G-v\cdot\nabla d,\\ \partial_t\mathcal{E}+2\Lambda d=J-v\cdot\nabla\mathcal{E},\\ \partial_t\big(F^T-F\big)+\Lambda\Omega=I-v\cdot\nabla\big(F^T-F\big),\\ \partial_t\Omega-\mu\Delta\Omega-\Lambda(F^T-F)=H-v\cdot\nabla\Omega.\end{cases}\tag{5.3}$$

We can decompose the above system into three subsystems.

$$\begin{cases} \partial_t a + \Lambda d = L - v \cdot \nabla a, \\ \partial_t d - \mu \Delta d - 2\Lambda a = G - v \cdot \nabla d. \end{cases} \tag{5.4}$$

$$\begin{cases} \partial_t \mathcal{E} + 2\Lambda d = J - v \cdot \nabla \mathcal{E}, \\ \partial_t d - \nu \Delta d - \Lambda \mathcal{E} = K - v \cdot \nabla d. \end{cases} \tag{5.5}$$

$$\begin{cases} \partial_t \big(F^T - F\big) + \Lambda \Omega = I - v \cdot \nabla \big(F^T - F\big), \\ \partial_t \Omega - \mu \Delta \Omega - \Lambda \big(F^T - F\big) = H - v \cdot \nabla \Omega. \end{cases} \tag{5.6}$$

It is easily observed that the above three systems are similar, so we now study the following linear system

$$\begin{cases} \partial_t c + \alpha \Lambda u = 0, \\ \partial_t u - \kappa \Delta u - \beta \Lambda c = 0, \end{cases} \tag{5.7}$$

where c, u are scalar functions and α, β, κ are positive constants. We first give some important properties of the Green's matrix for the above system.

Lemma 5.1. *Let $\mathcal{G}$ be the Green's matrix of system (5.7). Then we have the following explicit expression of $\hat{\mathcal{G}}$:*

$$\hat{\mathcal{G}}(\xi,t) = \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\alpha\big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\big)|\xi| \\ -\beta\big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\big)|\xi| & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix}$$

where

$$\lambda_\pm = -\frac{1}{2}\kappa|\xi|^2 \pm \frac{1}{2}\sqrt{\kappa^2|\xi|^4 - 4\alpha\beta|\xi|^2}.$$

Lemma 5.2.

(1) *Given $R > 0$, there is a positive number θ depending on R such that, for any multi-indices γ and $|\xi| \leqslant R$,*

$$\big|D_\xi^\gamma \hat{\mathcal{G}}(\xi,t)\big| \leqslant Ce^{-\theta|\xi|^2 t}\big(1 + |\xi|\big)^{|\gamma|}(1+t)^{|\gamma|}$$

where $C = C(R, |\gamma|)$.

(2) *There exist positive constants R, θ such that the following expansion is valid for $|\xi| \geqslant R$:*

$$\hat{\mathcal{G}}(\xi,t) = e^{-\kappa^{-1}t}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + e^{-\kappa|\xi|^2 t}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\mathcal{G}}^1(\xi,t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{\mathcal{G}}^2(\xi,t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{\mathcal{G}}^1$ and $\hat{\mathcal{G}}^2$ satisfy the estimates

$$\big|\partial_\xi^\gamma \mathcal{G}^1\big| \leqslant C|\xi|^{-|\gamma|-1}\big(e^{-ct} + e^{-\theta|\xi|^2 t}\big),$$
$$\big|\partial_\xi^\gamma \mathcal{G}^2\big| \leqslant C|\xi|^{-|\gamma|-2}\big(e^{-ct} + e^{-\theta|\xi|^2 t}\big),$$

for a positive constant C depending on $|\gamma|$, κ, α and β and a small constant c depending on κ, α and β.

Remark 5.3. In fact, $\hat{\mathcal{G}}^1$ and $\hat{\mathcal{G}}^2$ are diagonal matrices (different to [6]). Since both nonzero elements in $\hat{\mathcal{G}}^1$ and $\hat{\mathcal{G}}^2$ have the same estimates, we do not care about its explicit expression, and view it as a scalar function.

The proof of the above two lemmas follows from Lemma 3.1 and Theorem 3.2 in [12], so we omit the proof for simplicity. As in [6], we also have

Lemma 5.4. *Let $\mathcal{C}$ be a ring centered at 0 in $\mathbb{R}^n$. Then there exist positive constants R_0, C and c depending on κ, α and β such that, if $\operatorname{supp}\hat{u} \subset \lambda\mathcal{C}$, we have the following:*

(1) *If $\lambda \leqslant R_0$, then*

$$\|\mathcal{G} * u\|_{L^2} \leqslant Ce^{-c\lambda^2 t}\|u\|_{L^2}. \tag{5.8}$$

(2) *If $b \leqslant \lambda \leqslant R_0$, then for any $1 \leqslant p \leqslant \infty$,*

$$\|\mathcal{G} * u\|_{L^p} \leqslant C\big(1 + b^{-n-1}\big)e^{-c\lambda^2 t}\|u\|_{L^p}. \tag{5.9}$$

(3) *If $\lambda > R_0$, then for any $1 \leqslant p \leqslant \infty$,*

$$\begin{aligned} \big\|\mathcal{G}^1 * u\big\|_{L^p} &\leqslant C\lambda^{-1}e^{-ct}\|u\|_{L^p}, \\ \big\|\mathcal{G}^2 * u\big\|_{L^p} &\leqslant C\lambda^{-2}e^{-ct}\|u\|_{L^p}. \end{aligned} \tag{5.10}$$

Then we give an estimation of the following linear system with convection terms.

$$\begin{cases} \partial_t c + \alpha\Lambda u = -u\cdot\nabla c + Q, \\ \partial_t u - \kappa\Delta u - \beta\Lambda c = -v\cdot\nabla u + P. \end{cases} \tag{5.11}$$

Let $T > 0$, $s \in \mathbb{R}$, $p \geqslant 2$, and $s_p := s - \frac{n}{p} + \frac{n}{2}$. We introduce the functional space $\tilde{\mathcal{E}}_T^s$, which is defined as

$$\tilde{\mathcal{E}}_T^s := \big\{(c,u) \in \big(L_T^1\big(B_{2,p}^{s_p+1,s}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{s_p-1,s}\big)\big) \times \big(L_T^1\big(B_{2,p}^{s_p+1,s+2}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{s_p-1,s-1}\big)\big)^n\big\}$$

with the norm

$$\big\|(c,u)\big\|_{\tilde{\mathcal{E}}_T^s} := \|c\|_{\tilde{L}_T^\infty(B_{2,p}^{s_p-1,s})} + \|u\|_{\tilde{L}_T^\infty(B_{2,p}^{s_p-1,s-1})} + \|c\|_{L_T^1(B_{2,p}^{s_p+1,s})} + \|u\|_{L_T^1(B_{2,p}^{s_p+1,s+1})}.$$

We denote

$$\bar{V}(t) := \|v\|_{L_t^1(B_{2,p}^{n/2+1,n/p+1})} + \|v\|_{\tilde{L}_T^\infty(B_{2,p}^{n/2-1,n/p-1})}.$$

With Lemma 5.1, Lemma 5.2 and Lemma 5.4 at hand, we can get the following two propositions by using similar methods as in [6].

Proposition 5.5. *Let $2 \leqslant p < 2n$, $p \leqslant \min(4, \frac{2n}{n-2})$ and*

$$1 - \frac{n}{p} < s \leqslant 1 + \frac{2n}{p} - \frac{n}{2}.$$

Assume that $v \in L_T^\infty(B_{2,p}^{n/2-1,n/p-1}) \cap L_T^1(B_{2,p}^{n/2+1,n/p+1})$, $Q \in L_T^1(B_{2,p}^{s_p-1,s})$, $P \in L_T^1(B_{2,p}^{s_p-1,s-1})$. Let (c,u) be a solution of (5.11) on $[0,T]$. Then there exists a constant C independent of T such that

$$\begin{aligned} \big\|(c,u)\big\|_{\tilde{\mathcal{E}}_T^s} \leqslant Ce^{C\bar{V}(T)}\big\{&\big\|(c_0,u_0)\big\|_{\tilde{\mathcal{E}}_0^s} + \big(\bar{V}(T) + \bar{V}(T)^{\frac{1}{2}}\big)\big\|(c,u)\big\|_{\tilde{\mathcal{E}}_T^s} \\ &+ \|Q\|_{L_T^1(B_{2,p}^{s_p-1,s})} + \|P\|_{L_T^1(B_{2,p}^{s_p-1,s-1})}\big\}. \end{aligned}$$

Here $\|(c_0,u_0)\|_{\tilde{\mathcal{E}}_0^s} := \|c_0\|_{B_{2,p}^{s_p-1,s}} + \|u_0\|_{B_{2,p}^{s_p-1,s-1}}$.

Since the subsystems (5.4), (5.5) and (5.6) (estimate each component separately and then sum up the estimates of every component) are the same as (5.11) with different κ, α and β, we can use Proposition 5.5 to get estimations about $\|(a,d)\|_{\tilde{\mathcal{E}}_T^s}$, $\|(\mathcal{E},d)\|_{\tilde{\mathcal{E}}_T^s}$ and $\|(F^T-F,\Omega)\|_{\tilde{\mathcal{E}}_T^s}$. Then combining the obtained estimation we will get the estimation about a, v and F. From now on, we assume that $n=3$. To proceed the proof, we need some new notations. Define

$$\mathcal{E}_T^s := \big\{(a,u,F) \in \big(L_T^1\big(B_{2,p}^{s_p+1,s}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{s_p-1,s}\big)\big) \times \big(L_T^1\big(B_{2,p}^{s_p+1,s+1}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{s_p-1,s-1}\big)\big)^n \times \big(L_T^1\big(B_{2,p}^{s_p+1,s}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{s_p-1,s}\big)\big)^{n\times n}\big\},$$

and

$$E_T^{n/p} = \big(L_T^1\big(B_{2,p}^{n/2+1,n/p}\big) \cap \tilde{L}_T^\infty\big(B_{2,p}^{n/2-1,n/p}\big)\big)^{n\times n}.$$

We will have the following inequality

$$\begin{aligned}
&\big\|(a,v)\big\|_{\mathcal{E}_T^{n/p}} + \|\mathcal{E}\|_{E_T^{n/p}} + \big\|F-F^T\big\|_{E_T^{n/p}} \\
&\leqslant Ce^{C\bar{V}(T)}\big\{\big\|(a_0,v_0)\big\|_{\mathcal{E}_0^{n/p}} + \|\mathcal{E}_0\|_{E_0^{n/p}} + \big\|F_0-F_0^T\big\|_{E_0^{n/p}} \\
&\quad + \big(\bar{V}(T)+\bar{V}(T)^{\frac{1}{2}}\big)\big(\big\|(a,v)\big\|_{\mathcal{E}_T^{n/p}} + \|\mathcal{E}\|_{E_T^{n/p}} + \big\|F-F^T\big\|_{E_T^{n/p}}\big) \\
&\quad + \big\|(L,J,I)\big\|_{L_T^1(B_{2,p}^{n/2-1,n/p})} + \big\|(G,K,H)\big\|_{L_T^1(B_{2,p}^{n/2-1,n/p-1})}\big\}.
\end{aligned} \tag{5.12}$$

Now, we need to give a detailed explanation about (3.6) which appeared in [14]. In that paper, there seems to be no clear explanation about (3.6). We give a possible explanation, however, we need to impose a condition due to the product laws in Besov space which prevents us to get the global existence with highly oscillating initial data. Since

$$F+F^T = -\Lambda^{-1}\nabla\big(-\Lambda^{-1}\nabla\mathcal{E} - \Lambda^{-1}\operatorname{div}\big(\Lambda^{-1}\operatorname{curl}\big(\Lambda^{-1}\operatorname{div}\big(E+E^T\big)\big)\big)\big) - \Lambda^{-1}\operatorname{div}\big(\Lambda^{-1}\operatorname{curl}\big(E+E^T\big)\big),$$

due to the fourth identity in (2.1), we have

$$\begin{aligned}
\Lambda^{-1}\operatorname{div}\big(\Lambda^{-1}\operatorname{curl}\big(\Lambda^{-1}\operatorname{div}\big(F+F^T\big)\big)\big) &= \Lambda^{-1}\partial_{x_k}\big(\Lambda^{-2}\partial_{x_j}\big(F^{lj}\nabla_l F^{ik} - F^{lk}\nabla_l F^{ij}\big) \\
&\quad + \Lambda^{-2}\partial_{x_j}\big(F^{li}\nabla_l F^{jk} - F^{lk}\nabla_l F^{jk}\big)\big).
\end{aligned}$$

Obviously, it is higher order terms. Hence we easily have

$$\|\mathcal{E}\|_{E_T^{n/p}} + \big\|F-F^T\big\|_{E_T^{n/p}} \lesssim \|F\|_{E_T^{n/p}}, \tag{5.13}$$

and

$$\|F\|_{E_T^{n/p}} \lesssim \|\mathcal{E}\|_{E_T^{n/p}} + \big\|F-F^T\big\|_{E_T^{n/p}} + \big\|\Lambda^{-1}(F\nabla F)\big\|_{L_T^1(B_{2,p}^{n/2+1,n/p})\cap\tilde{L}_T^\infty(B_{2,p}^{n/2-1,n/p})}.$$

The last term $\|\Lambda^{-1}(F\nabla F)\|_{L_T^1(B_{2,p}^{n/2+1,n/p})\cap\tilde{L}_T^\infty(B_{2,p}^{n/2-1,n/p})}$ can be estimated as follows

$$\begin{aligned}
\big\|\Lambda^{-1}(F\nabla F)\big\|_{L_T^1(B_{2,p}^{n/2+1,n/p})\cap\tilde{L}_T^\infty(B_{2,p}^{n/2-1,n/p})} &\lesssim \|F\nabla F\|_{L_T^1(B_{2,p}^{n/2,n/p-1})\cap\tilde{L}_T^\infty(B_{2,p}^{n/2-2,n/p-1})} \\
&\lesssim \|F\|_{L_T^1(B_{2,p}^{n/2+1,n/p})\cap\tilde{L}_T^\infty(B_{2,p}^{n/2-1,n/p})}.
\end{aligned} \tag{5.14}$$

There is $\frac{n}{2}-2$ appeared in the low frequency exponent of the Besov space, so we need $n>p$ here by using Lemma A.4. Hence, finally we obtain

$$\|F\|_{E_T^{n/p}} \lesssim \|\mathcal{E}\|_{E_T^{n/p}} + \left\|F - F^T\right\|_{E_T^{n/p}} + \|F\|^2_{E_T^{n/p}}. \tag{5.15}$$

Combing (5.13), (5.15) and (5.12), we finally obtain

$$\begin{aligned}\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} \leqslant Ce^{C\bar{V}(T)}\big\{&\left\|(a_0,v_0,F_0)\right\|_{\mathcal{E}_0^{n/p}} + \big(\bar{V}(T) + \bar{V}(T)^{\frac{1}{2}}\big)\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} + \|F\|^2_{E_T^{n/p}} \\ &+ \left\|(L,J,I)\right\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} + \left\|(G,K,H)\right\|_{L^1_T(B^{n/2-1,n/p-1}_{2,p})}\big\}.\end{aligned} \tag{5.16}$$

5.2. Global existence

In this subsection, we will give the proof of global existence. Firstly, we need to prove the following a priori estimates.

Proposition 5.6. *Let $2 \leqslant p < n$. Assume that (a,v,F) is a smooth solution of system (2.4) on $[0,T]$ with*

$$\|a\|_{L^\infty([0,T]\times\mathbb{R}^n)} \leqslant \frac{1}{2}.$$

Then we have

$$\begin{aligned}\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} \leqslant Ce^{C\|(a,v,F)\|_{\mathcal{E}_T^{n/p}}}\big\{&\left\|(a_0,v_0,F_0)\right\|_{\mathcal{E}_T^{n/p}} \\ &+ \left\|(a,v,F)\right\|^{3/2}_{\mathcal{E}_T^{n/p}}\big(1 + \left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}}\big)^{n/2+2}\big\}.\end{aligned} \tag{5.17}$$

Proof. Thanks to (5.16), we get

$$\begin{aligned}\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} \leqslant Ce^{C\|(a,v,F)\|_{\mathcal{E}_T^{n/p}}}\big\{&\left\|(a_0,v_0,F_0)\right\|_{\mathcal{E}_0^{n/p}} \\ &+ \big(\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} + \left\|(a,v,F)\right\|^{1/2}_{\mathcal{E}_T^{n/p}}\big)\left\|(a,v,F)\right\|_{\mathcal{E}_T^{n/p}} \\ &+ \left\|(L,J,I)\right\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} + \left\|(G,K,H)\right\|_{L^1_T(B^{n/2-1,n/p-1}_{2,p})}\big\}.\end{aligned} \tag{5.18}$$

Now, we give the estimation of L, J, I, G, K, H. The most tricky term is

$$J_1 := \big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v^k\big]\partial_{x_k}\big(F^{ij} + F^{ji}\big).$$

In order to treat this term, we need the following lemma.

Lemma 5.7. *For two vector fields $v \in L^1_T(B^{n/2+1,n/p+1}_{2,p})$ and $u \in L^\infty_T(B^{n/2-1,n/p}_{2,p})$ with $n = 3$, we have*

$$\left\|\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v\cdot\nabla\big]u\right\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} \lesssim \|u\|_{L^\infty_T(B^{n/2-1,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}.$$

For the reader to get a big picture of the proof, we postpone the proof of this lemma in the end of this subsection. Hence, we have

$$\left\|\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v\cdot\nabla\big]\big(F^{ij} + F^{ji}\big)\right\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} \lesssim \|F\|_{L^\infty_T(B^{n/2-1,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}. \tag{5.19}$$

For other terms, the methods in [6] can give us the appropriate estimations. So we only give the results of typical terms as follows

$$\begin{aligned}
\|a\operatorname{div} v\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} &\lesssim \|a\|_{\tilde L^{\tilde p}_T(B^{n/p,n/p}_{2,p})}\|v\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/2}_{2,p})} + \|v\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\|a\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})} \\
&\quad + \|a\|_{\tilde L^\infty_T(B^{n/2-1,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}, \\
\|v\cdot\nabla v\|_{L^1_T(B^{n/2-1,n/p-1}_{2,p})} &\lesssim \|v\|_{\tilde L^{\tilde p}_T(B^{n/p,2n/p-n/2}_{2,p})}\|v\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/2}_{2,p})} + \|v\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\|v\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}, \\
\|F\nabla v\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} &\lesssim \|F\|_{\tilde L^{\tilde p}_T(B^{n/p,n/p}_{2,p})}\|v\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/2}_{2,p})} + \|v\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\|F\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})} \\
&\quad + \|F\|_{\tilde L^\infty_T(B^{n/2-1,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})},
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
&\sum_{2^j\leqslant R_0} 2^{j(n/2-1)}\big\|\Delta_j\big(K(a)\nabla a + F\nabla F + a\nabla F + F\nabla a\big)\big\|_{L^1_T(L^2)} \\
&\quad\lesssim \big(1+\|a\|_{\tilde L^\infty_T(B^{n/p,n/p}_{2,p})}\big)^{n/2+1}\big(\big\|(a,F)\big\|_{\tilde L^{\tilde p}_T(B^{n/p,n/p}_{2,p})}\big\|(a,F)\big\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/p}_{2,p})} \\
&\qquad + \big\|(a,F)\big\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\big\|(a,F)\big\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\big),
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
&\sum_{2^j\leqslant R_0} 2^{j(n/2-1)}\left\|\Delta_j\left(\left(\frac{a}{1+a}\right)\big(\mu\Delta v + (\lambda+\mu)\nabla\operatorname{div} v\big)\right)\right\|_{L^1_T(L^2)} \\
&\quad\lesssim \big(1+\|a\|_{\tilde L^\infty_T(B^{n/p,n/p}_{2,p})}\big)^{n/2+1}\big(\|a\|_{\tilde L^{\tilde p}_T(B^{n/p,n/p}_{2,p})}\|v\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/2}_{2,p})} \\
&\qquad + \|a\|_{\tilde L^\infty_T(B^{n/2,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}\big),
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&\sum_{2^j> R_0} 2^{j(n/p-1)}\bigg(\big\|\Delta_j\big(K(a)\nabla a + F\nabla F + a\nabla F + F\nabla a\big)\big\|_{L^1_T(L^p)} \\
&\qquad + \left\|\Delta_j\left(\frac{a}{1+a}\big(\mu\Delta v + (\lambda+\mu)\nabla\operatorname{div} v\big)\right)\right\|_{L^1_T(L^p)}\bigg) \\
&\quad\lesssim \big(1+\|a\|_{\tilde L^\infty_T(B^{n/p,n/p}_{2,p})}\big)^{n/2+1}\big(\big\|(a,F)\big\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})}\big\|(a,F)\big\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})} \\
&\qquad + \|a\|_{\tilde L^\infty_T(B^{n/2,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}\big),
\end{aligned} \tag{5.23}$$

where $\frac{1}{\tilde p} = \frac{n}{2p} - \frac{n}{4} + \frac{1}{2}$. Notice that

$$\begin{aligned}
\|v\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})} &\leqslant \|v\|^{1/2}_{\tilde L^\infty_T(B^{n/2-1,n/p-1}_{2,p})}\|v\|^{1/2}_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}, \\
\|v\|_{\tilde L^{\tilde p}_T(B^{n/p,2n/p-n/2}_{2,p})} &\leqslant \|v\|^{1/\tilde p'}_{\tilde L^\infty_T(B^{n/2-1,n/p-1}_{2,p})}\|v\|^{1/\tilde p}_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}, \\
\|v\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/2}_{2,p})} &\leqslant \|v\|^{1/\tilde p}_{\tilde L^\infty_T(B^{n/2-1,n/p-1}_{2,p})}\|v\|^{1/\tilde p'}_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}, \\
\big\|(a,F)\big\|_{\tilde L^2_T(B^{n/2,n/p}_{2,p})} &\leqslant \big\|(a,F)\big\|^{1/2}_{\tilde L^\infty_T(B^{n/2-1,n/p}_{2,p})}\big\|(a,F)\big\|^{1/2}_{L^1_T(B^{n/2+1,n/p}_{2,p})}, \\
\big\|(a,F)\big\|_{\tilde L^{\tilde p'}_T(B^{n/p',n/p}_{2,p})} &\leqslant \big\|(a,F)\big\|^{1/\tilde p}_{\tilde L^\infty_T(B^{n/2-1,n/p}_{2,p})}\big\|(a,F)\big\|^{1/\tilde p'}_{\tilde L^1_T(B^{n/2+1,n/p}_{2,p})}, \\
\big\|(a,F)\big\|_{\tilde L^{\tilde p}_T(B^{n/p,n/p}_{2,p})} &\leqslant \big\|(a,F)\big\|^{1/\tilde p'}_{\tilde L^\infty_T(B^{n/2-1,n/p}_{2,p})}\big\|(a,F)\big\|^{1/\tilde p}_{L^1_T(B^{n/2+1,n/p}_{2,p})}.
\end{aligned} \tag{5.24}$$

Then adding (5.19), (5.20), (5.21), (5.22) and (5.23) into (5.18), we finally obtain (5.17). □

With Proposition 5.6 at hand, we can use the same method as in [6] or [5] to complete the proof. For it is now a conventional method, we omit the details.

In the last part of this section, let us give the proof of Lemma 5.7.

Proof of Lemma 5.7. We make a decomposition as follows

$$\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v\cdot\nabla\big]u = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$\mathrm{I} = \sum_{q\in\mathbb{Z}}\mathrm{I}_q = \sum_{q\in\mathbb{Z}}\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, S_{q-1}v\cdot\nabla\big]\Delta_q u,$$
$$\mathrm{II} = \sum_{q\in\mathbb{Z}}\mathrm{II}_q = \sum_{q\in\mathbb{Z}}\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, \Delta_q v\cdot\nabla\big]S_{q-1} u,$$
$$\mathrm{III} = \sum_{q\in\mathbb{Z}}\mathrm{III}_q = \sum_{q\in\mathbb{Z}}\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, \Delta_q v\cdot\nabla\big]\tilde{\Delta}_q u.$$

Through Proposition 3.1 in [11], we know that there exist Schwartz functions ϕ_i and ϕ_j such that

$$\begin{aligned}
\mathrm{I}_q &= 2^{2qn}\int_{\mathbb{R}^n}\phi_i*\phi_j\big(2^q(x-y)\big)\big(S_{q-1}v(y)-S_{q-1}v(x)\big)\Delta_q\nabla u(y)\,dy\\
&= 2^{2qn}\int_{\mathbb{R}^n}\phi_i*\phi_j\big(2^q(x-y)\big)\int_0^1\frac{d}{dt}S_{q-1}v\big(x+t(y-x)\big)\,dt\,\Delta_q\nabla u(y)\,dy\\
&= 2^{-q}2^{2qn}\int_{\mathbb{R}^n}2^q(y-x)\phi_i*\phi_j\big(2^q(x-y)\big)\int_0^1\nabla S_{q-1}v\big(x+t(y-x)\big)\,dt\,\Delta_q\nabla u(y)\,dy
\end{aligned}$$

So, we have

$$\|\mathrm{I}_q\|_{L^2}\lesssim 2^{-q}\|\nabla v\|_{L^\infty}\|\nabla\Delta_q u\|_{L^2},$$
$$\|\mathrm{I}_q\|_{L^p}\lesssim 2^{-q}\|\nabla v\|_{L^\infty}\|\nabla\Delta_q u\|_{L^p}.$$

Hence, we obtain

$$\begin{aligned}
\|\mathrm{I}\|_{L^1_T(B^{n/2-1,n/p}_{2,p})} &\lesssim \sum_{2^j\leqslant R_0}2^{j(n/2-1)}2^{-j}\|\nabla v\|_{L^1_T(L^\infty)}\|\nabla\Delta_j u\|_{L^\infty_T(L^2)}\\
&\quad+\sum_{2^j>R_0}2^{j\frac{n}{p}}2^{-j}\|\nabla v\|_{L^1_T(L^\infty)}\|\nabla\Delta_j u\|_{L^\infty_T(L^p)}\\
&\lesssim \sum_{2^j\leqslant R_0}2^{j(\frac{n}{2}-1)}\|\Delta_j u\|_{L^\infty_T(L^2)}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}\\
&\quad+\sum_{2^j>R_0}2^{j\frac{n}{p}}\|\Delta_j u\|_{L^\infty_T(L^p)}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}\\
&\lesssim \|u\|_{\tilde{L}^\infty_T(B^{n/2-1,n/p}_{2,p})}\|v\|_{L^1_T(B^{n/2+1,n/p+1}_{2,p})}.
\end{aligned}$$

As above, there exist two Schwartz functions ϕ_i and ϕ_j such that

$$\begin{aligned}
\mathrm{II}_q = 2^{-q}2^{2qn}\int_{\mathbb{R}^n}2^q(y-x)\phi_i*\phi_j\big(2^q(x-y)\big)\\
+\int_0^1\nabla\Delta_q v\big(x+t(y-x)\big)\,dt\,\nabla S_{q-1}u(y)\,dy.
\end{aligned}$$

For low frequency that is $2^j \leqslant R_0$, we have

$$\|\mathrm{II}_q\|_{L^2} \lesssim 2^{-q}\|v\|_{B_{2,p}^{n/2+1,n/p+1}} \sum_{k\leqslant q-2} 2^k \|\Delta_q u\|_{L^2}. \tag{5.25}$$

For high frequency that is $2^j > R_0$, we have

$$\begin{aligned}\|\mathrm{II}_q\|_{L^p} &\lesssim 2^{-q}\|\nabla\Delta_q v\|_{L^p}\|S_{q-1}\nabla u\|_{L^\infty}\\ &\lesssim \|\nabla\Delta_q v\|_{L^p}\|u\|_{L^\infty}.\end{aligned} \tag{5.26}$$

Combining (5.25) and (5.26), we easily get

$$\begin{aligned}\|\mathrm{II}\|_{L^1_T(B_{2,p}^{n/2-1,n/p})} &\lesssim \sum_{2^j\leqslant R_0} 2^{j(\frac{n}{2}-1)}2^{-j}\|v\|_{L^1_T(B_{2,p}^{n/2+1,n/p+1})}\sum_{k\leqslant j-2}2^k\|\Delta_k u\|_{L^\infty_T(L^2)}\\ &\quad+\sum_{2^j>R_0}2^{j\frac{n}{p}}\|\nabla\Delta_j v\|_{L^1_T(L^p)}\|u\|_{L^\infty_T(L^\infty)}\\ &\lesssim \|v\|_{L^1_T(B_{2,p}^{n/2+1,n/p+1})}\sum_{2^j\leqslant R_0}\sum_{k-j\leqslant -2}2^{(k-j)(2-\frac{n}{2})}2^{k(\frac{n}{2}-1)}\|\Delta_k u\|_{L^2}\\ &\quad+\|u\|_{\tilde L^\infty_T(B_{2,p}^{n/2-1,n/p})}\|v\|_{L^1_T(B_{2,p}^{n/2+1,n/p+1})}\\ &\lesssim \|v\|_{L^1_T(B_{2,p}^{n/2+1,n/p+1})}\|u\|_{\tilde L^\infty_T(B_{2,p}^{n/2-1,n/p})}.\end{aligned}$$

The last term III is more tricky for the lack of quasi-orthogonality. After simple calculations, we have

$$\begin{aligned}\|\mathrm{III}_q\|_{L^2} &\lesssim \|\Delta_q v\|_{L^\infty}\|\nabla\tilde\Delta_q u\|_{L^2},\\ \|\mathrm{III}_q\|_{L^p} &\lesssim \|\Delta_q v\|_{L^\infty}\|\nabla\tilde\Delta_q u\|_{L^p},\\ \|\mathrm{III}_q\|_{L^2} &\lesssim \|\Delta_q v\|_{L^{\frac{2p}{p-2}}}\|\nabla\tilde\Delta_q u\|_{L^p}.\end{aligned} \tag{5.27}$$

So in addition, we can get

$$\begin{aligned}\|\mathrm{III}\|_{B_{2,p}^{n/2-1,n/p}} &\lesssim \sum_{2^j\leqslant R_0}2^{j(\frac{n}{2}-1)}\sum_{q\geqslant j-4}\|\Delta_j\mathrm{III}_q\|_{L^2}+\sum_{2^j>R_0}2^{j\frac{n}{p}}\sum_{q\geqslant j-4}\|\Delta_j\mathrm{III}_q\|_{L^p}\\ &\lesssim J_1+J_2.\end{aligned}$$

The term J_1 can be in addition decomposed as follows

$$\begin{aligned}J_1 &= \sum_{2^j\leqslant R_0}\sum_{2^{j-4}\leqslant 2^q\leqslant R_0}2^{j(\frac{n}{2}-1)}\|\Delta_j\mathrm{III}_q\|_{L^2}+\sum_{2^j\leqslant R_0}\sum_{2^q>R_0}2^{j(\frac{n}{2}-1)}\|\Delta_j\mathrm{III}_q\|_{L^2}\\ &= J_{11}+J_{12}.\end{aligned}$$

For J_{11}, we have

$$\begin{aligned}J_{11} &\lesssim \sum_{2^j\leqslant R_0}\sum_{2^{j-4}\leqslant 2^q\leqslant R_0}2^{(j-q)(\frac{n}{2}-1)}\|\nabla\Delta_q v\|_{L^\infty}2^{q(\frac{n}{2}-1)}\|\tilde\Delta_q u\|_{L^2}\\ &\lesssim \|u\|_{B_{2,p}^{n/2-1,n/p}}\|v\|_{B_{2,p}^{n/2+1,n/p+1}}.\end{aligned}$$

For the term J_{12}, we have

$$\begin{aligned}\|J_{12}\| &\lesssim \sum_{2^j\leqslant R_0}\sum_{2^q>R_0} 2^{j(\frac{n}{2}-1)}\|\nabla\Delta_q v\|_{L^6}^{\frac{3(p-2)}{p}}\|\nabla\Delta_q v\|_{L^\infty}^{\frac{6-2p}{p}}\|\tilde\Delta_q u\|_{L^p}\\
&\lesssim \sum_{2^j\leqslant R_0}\sum_{2^q>R_0} 2^{j(\frac{n}{2}-1)}2^{-q\frac{n}{p}}2^{-q\frac{n}{2}}\left(1-\frac{2}{p}\right)2^{q\frac{n}{p}}\|\nabla\Delta_q v\|_{L^p}2^{q\frac{n}{p}}\|\tilde\Delta_q u\|_{L^p}\\
&\lesssim \|v\|_{B_{2,p}^{n/2+1,n/p+1}}\|u\|_{B_{2,p}^{n/2-1,n/p}}.\end{aligned}$$

As J_1, we have the following decomposition

$$\begin{aligned}J_2 &= \sum_{2^j>R_0}\sum_{2^q\leqslant R_0,\, q\geqslant j-4} 2^{j\frac{n}{p}}\|\Delta_j \mathrm{III}_q\|_{L^p} + \sum_{2^j>R_0}\sum_{2^q>R_0} 2^{j\frac{n}{p}}\|\Delta_j \mathrm{III}_q\|_{L^p}\\
&= J_{21}+J_{22}.\end{aligned}$$

For J_{21}, we have

$$\begin{aligned}J_{21} &\lesssim \sum_{2^j>R_0}\sum_{2^q\leqslant R_0,\, q\geqslant j-4} 2^{j\frac{n}{p}}\|\nabla\Delta_q v\|_{L^\infty}2^{q(\frac{n}{2}-\frac{n}{p})}\|\tilde\Delta_q u\|_{L^2}\\
&\lesssim \sum_{2^j>R_0}\sum_{2^q\leqslant R_0,\, q\geqslant j-4} 2^{(j-q)\frac{n}{p}}2^{q(\frac{n}{p}-\frac{n}{2}+1)}2^{q(\frac{n}{2}-1)}\|\tilde\Delta_q u\|_{L^2}\|v\|_{B_{2,p}^{n/2+1,n/p+1}}\\
&\lesssim \|u\|_{B_{2,p}^{n/2-1,n/p}}\|v\|_{B_{2,p}^{n/2+1,n/p+1}}.\end{aligned}$$

For J_{22}, we have

$$\begin{aligned}J_{22} &\lesssim \sum_{2^j>R_0}\sum_{2^q>R_0} 2^{j\frac{n}{p}}\|\nabla\Delta_q v\|_{L^\infty}\|\tilde\Delta_q u\|_{L^p}\\
&\lesssim \sum_{2^j>R_0}\sum_{2^q>R_0} 2^{(j-q)\frac{n}{p}}2^{q\frac{n}{p}}\|\tilde\Delta_q u\|_{L^p}\|v\|_{B_{2,p}^{n/2+1,n/p+1}}\\
&\lesssim \|u\|_{B_{2,p}^{n/2-1,n/p}}\|v\|_{B_{2,p}^{n/2+1,n/p+1}}.\end{aligned}$$

Combining all the estimates above, we get the desired results. □

6. The time decay estimates

In this section, we give a time decay rate in the Besov space framework with dimension $n=3$. Firstly, we need to prove the existence of solution with higher order regularity.

6.1. Solution with high regularity

Denote

$$\begin{aligned}\mathcal{M}_T^{n/2} = \big\{(a,v,F) \in\, & L_T^1\big(B^{n/2+1}\big)\cap \tilde L_T^\infty\big(B^{n/2-1,n/2+1}\big)\\
&\times \big(L_T^1\big(B^{n/2+1,n/2+2}\big)\cap \tilde L_T^\infty\big(B^{n/2-1,n/2}\big)\big)^n\\
&\times \big(L_T^1\big(B^{n/2+1}\big)\cap \tilde L_T^\infty\big(B^{n/2-1,n/2+1}\big)\big)^{n\times n}\big\},\end{aligned}$$

and

$$M_T^{n/2} = \big(L_T^1\big(B^{n/2+1}\big)\cap \tilde L_T^\infty\big(B^{n/2-1,n/2+1}\big)\big)^{n\times n}.$$

Since

$$\begin{aligned}\big\|\Lambda^{-1}(F\nabla F)\big\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})} &\lesssim \|F\|^2_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})} \lesssim \|F\|^2_{M^{n/2}_T},\\ \big\|\Lambda^{-1}(F\nabla F)\big\|_{L^1_T(B^{n/2+1})} &\lesssim \|F\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})}\|F\|_{L^1_T(B^{n/2+1})} \lesssim \|F\|^2_{M^{n/2}_T}. \end{aligned} \tag{6.1}$$

Taking $s=\frac{n}{2}$ and $s=\frac{n}{2}+1$ respectively, as in the derivation of (5.16), we can get two inequalities similar to (5.16), then adding the two inequalities together we will obtain

$$\begin{aligned}\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T} \leqslant Ce^{C\bar{V}(T)}\big\{&\big\|(a_0,v_0,F_0)\big\|_{\mathcal{M}^{n/2}_0} + \|F\|^2_{M^{n/2}_T} + \big(\bar{V}(T)+\bar{V}(T)^2\big)\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T}\\ &+ \big\|(L,J,I)\big\|_{L^1_T(B^{n/2-1,n/2+1})} + \big\|(G,K,H)\big\|_{L^1_T(B^{n/2-1,n/2})}\big\},\end{aligned}$$

where $\bar{V}(t) := \|v\|_{L^1_t(B^{n/2+1})} + \|v\|_{\tilde{L}^\infty_T(B^{n/2-1})}$ and $\|(a_0,v_0,F_0)\|_{\mathcal{M}^{n/2}_0} = \|a_0\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}} + \|F_0\|_{B^{n/2-1,n/2+1}}$. Hence, we can easily get

$$\begin{aligned}\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T} \leqslant Ce^{C\|(a,v,F)\|_{\mathcal{M}^{n/2}_T}}\big\{&\big\|(a_0,v_0,F_0)\big\|_{\mathcal{M}^{n/2}_0}\\ &+ \big(\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T} + \big\|(a,v,F)\big\|^{1/2}_{\mathcal{M}^{n/2}_T}\big)\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T}\\ &+ \big\|(L,J,I)\big\|_{L^1_T(B^{n/2-1,n/2+1})} + \big\|(G,K,H)\big\|_{L^1_T(B^{n/2-1,n/2})}\big\}.\end{aligned} \tag{6.2}$$

As in the proof of Proposition 5.6, we need to estimate the term $[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v\cdot\nabla](F^{ij}+F^{ji})$. Following the proof of Lemma 5.7 (simpler), we can obtain

$$\big\|\big[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v\cdot\nabla\big]\big(F^{ij}+F^{ji}\big)\big\| \lesssim \|v\|_{L^1_T(B^{n/2+1,n/2+2})}\|F\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})}.$$

The other typical terms can be estimated as follows

$$\begin{aligned}
\|a\operatorname{div} v\|_{L^1_T(B^{n/2-1,n/2+1})} &\lesssim \|a\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})}\|v\|_{L^1_T(B^{n/2+1,n/2+2})},\\
\|v\cdot\nabla v\|_{L^1_T(B^{n/2-1,n/2})} &\lesssim \|v\|^2_{\tilde{L}^2_T(B^{n/2,n/2+1})} + \|v\|_{L^1_T(B^{n/2+1})}\|v\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2})},\\
\|F\nabla v\|_{L^1_T(B^{n/2-1,n/2+1})} &\lesssim \|F\|_{\tilde{L}^\infty_T(B^{n/2-1,n/2+1})}\|v\|_{L^1_T(B^{n/2+1,n/2+2})},
\end{aligned}$$

$$\sum_{2^j\leqslant R_0} 2^{j(\frac{n}{2}-1)}\big\|\Delta_j\big(K(a)\nabla a + F\nabla F + a\nabla F + F\nabla a\big)\big\|_{L^1_T(L^2)} \lesssim \big(1+\|a\|_{\tilde{L}^\infty_T(B^{n/2})}\big)^{\frac{n}{2}+1}\big\|(a,F)\big\|^2_{\tilde{L}^2_T(B^{n/2})},$$

$$\begin{aligned}\sum_{2^j\leqslant R_0} 2^{j(\frac{n}{2}-1)}&\left\|\Delta_j\left(\frac{a}{1+a}\big(\mu\Delta v + (\lambda+\mu)\nabla\operatorname{div} v\big)\right)\right\|_{L^1_T(L^2)}\\ &\lesssim \big(1+\|a\|_{\tilde{L}^\infty_T(B^{n/2})}\big)^{\frac{n}{2}+1}\|a\|_{\tilde{L}^\infty_T(B^{n/2})}\|v\|_{L^1_T(B^{n/2+1})},\end{aligned}$$

$$\sum_{2^j> R_0} 2^{j\frac{n}{2}}\big\|\Delta_j\big(K(a)\nabla a + F\nabla F + a\nabla F + F\nabla a\big)\big\|_{L^1_T(L^2)} \lesssim \big(1+\|a\|_{\tilde{L}^\infty_T(B^{n/2})}\big)^{\frac{n}{2}+1}\big\|(a,F)\big\|^2_{\tilde{L}^2_T(B^{n/2,n/2+1})},$$

$$\sum_{2^j> R_0} 2^{j\frac{n}{2}}\left\|\Delta_j\left(\frac{a}{1+a}\big(\mu\Delta v + (\lambda+\mu)\nabla\operatorname{div} v\big)\right)\right\|_{L^1_T(L^2)} \lesssim \|a\|_{\tilde{L}^\infty_T(B^{n/2})}\|v\|_{L^1_T(B^{n/2+2})}.$$

Combining all the estimates together as in the proof of Proposition 5.6, we finally obtain

$$\begin{aligned}\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T} \leqslant Ce^{C\|(a,v,F)\|_{\mathcal{M}^{n/2}_T}}\big\{&\big\|(a_0,v_0,F_0)\big\|_{\mathcal{M}^{n/2}_0}\\ &+ \big\|(a,v,F)\big\|^{3/2}_{\mathcal{M}^{n/2}_T}\big(1+\big\|(a,v,F)\big\|_{\mathcal{M}^{n/2}_T}\big)^{\frac{n}{2}+2}\big\}.\end{aligned} \tag{6.3}$$

With this a priori estimate at hand, we know that if the initial data $(a_0, v_0, F_0) \in \mathcal{M}_0^{n/2}$ the solution built in the previous section also belongs to $\mathcal{M}^{n/2}$ and satisfies

$$\|(a, v, F)\|_{\mathcal{M}^{n/2}} \leqslant C\|(a_0, v_0, F_0)\|_{\mathcal{M}_0^{n/2}}. \tag{6.4}$$

6.2. Decay estimates

Our proof will be decomposed into four steps.

Proof. Step 1: Reformulate the system. In Section 5, we get three subsystems. The linear part of each subsystem is very similar. Denote $\mathcal{G}_1(x,t)$, $\mathcal{G}_2(x,t)$ and $\mathcal{G}_3(x,t)$ as the Green's matrices of systems (5.4), (5.5) and (5.6) separately. Then we can write the three subsystems as

$$\begin{pmatrix} a \\ d \end{pmatrix} = \mathcal{G}_1 * \begin{pmatrix} a_0 \\ d_0 \end{pmatrix} + \int_0^t \mathcal{G}_1 * \begin{pmatrix} L - v\cdot\nabla a \\ G - v\cdot\nabla d \end{pmatrix} d\tau, \tag{6.5}$$

$$\begin{pmatrix} \mathcal{E} \\ d \end{pmatrix} = \mathcal{G}_2 * \begin{pmatrix} \mathcal{E}_0 \\ d_0 \end{pmatrix} + \int_0^t \mathcal{G}_2 * \begin{pmatrix} J - v\cdot\nabla \mathcal{E} \\ K - v\cdot\nabla d \end{pmatrix} d\tau, \tag{6.6}$$

$$\begin{pmatrix} F^T - F \\ \Omega \end{pmatrix} = \mathcal{G}_3 * \begin{pmatrix} F_0^T - F_0 \\ \Omega_0 \end{pmatrix} + \int_0^t \mathcal{G}_3 * \begin{pmatrix} I - v\cdot\nabla(F^T - F) \\ H - v\cdot\nabla \Omega \end{pmatrix} d\tau. \tag{6.7}$$

Applying the operator Δ_q on both sides of the above three systems, we have

$$\begin{pmatrix} \Delta_q a \\ \Delta_q d \end{pmatrix} = \mathcal{G}_1 * \begin{pmatrix} \Delta_q a_0 \\ \Delta_q d_0 \end{pmatrix} + \int_0^t \mathcal{G}_1 * \begin{pmatrix} \Delta_q(L - v\cdot\nabla a) \\ \Delta_q(G - v\cdot\nabla d) \end{pmatrix} d\tau, \tag{6.8}$$

$$\begin{pmatrix} \Delta_q \mathcal{E} \\ \Delta_q d \end{pmatrix} = \mathcal{G}_2 * \begin{pmatrix} \Delta_q \mathcal{E}_0 \\ \Delta_q d_0 \end{pmatrix} + \int_0^t \mathcal{G}_2 * \begin{pmatrix} \Delta_q(J - v\cdot\nabla \mathcal{E}) \\ \Delta_q(K - v\cdot\nabla d) \end{pmatrix} d\tau, \tag{6.9}$$

$$\begin{pmatrix} \Delta_q(F^T - F) \\ \Delta_q(\Omega) \end{pmatrix} = \mathcal{G}_3 * \begin{pmatrix} \Delta_q(F_0^T - F_0) \\ \Delta_q \Omega_0 \end{pmatrix} + \int_0^t \mathcal{G}_3 * \begin{pmatrix} \Delta_q(I - v\cdot\nabla(F^T - F)) \\ \Delta_q(H - v\cdot\nabla \Omega) \end{pmatrix} d\tau. \tag{6.10}$$

Denote

$$\begin{aligned} \Lambda(t) = {} & \sup_{0\leqslant\tau\leqslant t} \big((1+\tau)^{\frac{1}{2}}\|a(\tau)\|_{B^{n/2}} + \|a(\tau)\|_{B^{n/2-1,n/2+1}}\big) \\ & + \sup_{0\leqslant\tau\leqslant t} \big((1+\tau)^{\frac{1}{2}}\|F(\tau)\|_{B^{n/2}} + \|F(\tau)\|_{B^{n/2-1,n/2+1}}\big) \\ & + \sup_{0\leqslant\tau\leqslant t} \big((1+\tau)^{\frac{1}{2}}\|v(\tau)\|_{B^{n/2}} + \|v(\tau)\|_{B^{n/2-1,n/2}}\big) + \|v\|_{L_t^1(B^{n/2+1,n/2+2})}. \end{aligned} \tag{6.11}$$

Step 2: Estimates of the low frequency part. For the low frequency part that is $2^q \leqslant R_0$, applying Lemma 5.2, Lemma 5.4, there exists a small positive number $c > 0$ such that

$$\begin{aligned} \|\Delta_q a(t)\|_{L^2} + \|\Delta_q d(t)\|_{L^2} &\lesssim \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3, \\ \|\Delta_q \mathcal{E}\|_{L^2} + \|\Delta_q d(t)\|_{L^2} &\lesssim \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3, \\ \|\Delta_q(F^T - F)\|_{L^2} + \|\Delta_q \Omega(t)\|_{L^2} &\lesssim \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3 \end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_1 &= e^{-c2^{2q}t}\big(\|\Delta_q a_0\|_{L^2} + \|\Delta_q d_0\|_{L^2}\big),\\
\mathrm{I}_2 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q(v\cdot\nabla a)\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla d)\big\|_{L^2}\big)\,d\tau,\\
\mathrm{I}_3 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q L(\tau)\big\|_{L^2} + \big\|\Delta_q G(\tau)\big\|_{L^2}\big)\,d\tau,\\
\mathrm{II}_1 &= e^{-c2^{2q}t}\big(\|\Delta_q \mathcal{E}_0\|_{L^2} + \|\Delta_q d_0\|_{L^2}\big),\\
\mathrm{II}_2 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q(v\cdot\nabla \mathcal{E})\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla d)\big\|_{L^2}\big)\,d\tau,\\
\mathrm{II}_3 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q J(\tau)\big\|_{L^2} + \big\|\Delta_q K(\tau)\big\|_{L^2}\big)\,d\tau,\\
\mathrm{III}_1 &= e^{-c2^{2q}t}\big(\big\|\Delta_q\big(F_0^T - F_0\big)\big\|_{L^2} + \|\Delta_q \varOmega_0\|_{L^2}\big),\\
\mathrm{III}_2 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q\big(v\cdot\nabla\big(F^T - F\big)\big)\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla \varOmega)\big\|_{L^2}\big)\,d\tau,\\
\mathrm{III}_3 &= \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q I(\tau)\big\|_{L^2} + \big\|\Delta_q H(\tau)\big\|_{L^2}\big)\,d\tau.
\end{aligned}$$

Considering that there are a lot of similar terms in I_1 to III_3, so we only list the estimates of some typical terms. For I_1, we have

$$\begin{aligned}
\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\mathrm{I}_1 &\lesssim \sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}t}\big(\|\Delta_q a_0\|_{L^2} + \|\Delta_q d_0\|_{L^2}\big)\\
&\lesssim \sum_{2^q\leqslant R_0} 2^{q(\frac{n}{2}-1)} e^{-c2^{2q}t} 2^q\big(\|\Delta_q a_0\|_{L^2} + \|\Delta_q d_0\|_{L^2}\big)\\
&\lesssim (1+t)^{-\frac{1}{2}}\big(\|a_0\|_{B^{n/2-1,n/2+1}} + \|d_0\|_{B^{n/2-1,n/2}}\big)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\mathrm{II}_1 &\lesssim (1+t)^{-\frac{1}{2}}\big(\|\mathcal{E}_0\|_{B^{n/2-1,n/2+1}} + \|d_0\|_{B^{n/2-1,n/2}}\big),\\
\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\mathrm{III}_1 &\lesssim (1+t)^{-\frac{1}{2}}\big(\big\|F_0^T - F_0\big\|_{B^{n/2-1,n/2+1}} + \|\varOmega_0\|_{B^{n/2-1,n/2}}\big).
\end{aligned}$$

For a small positive number $\epsilon > 0$, we have

$$\begin{aligned}
\big\|a(t)\big\|_{B^{n/2-\epsilon}} &\lesssim \big\|a(t)\big\|_{B^{n/2-1}}^{\epsilon}\big\|a(t)\big\|_{B^{n/2}}^{1-\epsilon} \lesssim (1+t)^{-\frac{1-\epsilon}{2}}\varLambda(t),\\
\big\|F(t)\big\|_{B^{n/2-\epsilon}} &\lesssim \big\|F(t)\big\|_{B^{n/2-1}}^{\epsilon}\big\|F(t)\big\|_{B^{n/2}}^{1-\epsilon} \lesssim (1+t)^{-\frac{1-\epsilon}{2}}\varLambda(t),\\
\big\|v(t)\big\|_{B^{n/2-\epsilon}} &\lesssim \big\|v(t)\big\|_{B^{n/2-1}}^{\epsilon}\big\|v(t)\big\|_{B^{n/2}}^{1-\epsilon} \lesssim (1+t)^{-\frac{1-\epsilon}{2}}\varLambda(t). \qquad (6.12)
\end{aligned}$$

For term $\int_0^t e^{-c2^{2q}(t-\tau)}\|\Delta_q(v\cdot\nabla a)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}
&\int_0^t \sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}(t-\tau)}\big\|\Delta_q(v\cdot\nabla a)\big\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} \sum_{2^q\leqslant R_0} 2^{q(\frac{n}{2}-1-\epsilon)}2^{q(1+\epsilon)} e^{-c2^{2q}(t-\tau)}\big\|\Delta_q(v\cdot\nabla a)\big\|_{L^2}\,d\tau\\
&\quad+\int_{t/2}^t \sum_{2^q\leqslant R_0} 2^{q(\frac{n}{2}-1)}2^{q} e^{-c2^{2q}(t-\tau)}\big\|\Delta_q(v\cdot\nabla a)\big\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2}(t-\tau)^{-\frac12-\frac{\epsilon}{2}}\|v\|_{B^{n/2-\epsilon}}\|\nabla a\|_{B^{n/2-1}}\,d\tau+\int_{t/2}^t(t-\tau)^{-\frac12}\|v\|_{B^{n/2}}\|a\|_{B^{n/2}}\,d\tau\\
&\lesssim \int_0^{t/2}(t-\tau)^{-\frac12-\frac{\epsilon}{2}}(1+\tau)^{-1+\frac{\epsilon}{2}}\,d\tau\Lambda^2(t)+\int_{t/2}^t(t-\tau)^{-\frac12}(1+\tau)^{-1}\,d\tau\Lambda^2(t)\\
&\lesssim (1+t)^{-\frac12}\Lambda^2(t),
\end{aligned}$$

where we used (6.12). In the following, we will use (6.12) frequently without mentioning for it is similar to the above case. For term $\int_0^t e^{-c2^{2q}(t-\tau)}\|\Delta_q(\frac{a}{1+a}\mathcal{A}v)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}
&\int_0^t \sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}(t-\tau)}\left\|\Delta_q\left(\frac{a}{1+a}\mathcal{A}v\right)\right\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} \sum_{2^q\leqslant R_0} 2^{q(\frac{n}{2}-2)}2^{q(1+\epsilon)} e^{-c2^{2q}(t-\tau)}2^{q(1-\epsilon)}\left\|\Delta_q\left(\frac{a}{1+a}\mathcal{A}v\right)\right\|_{L^2}\,d\tau\\
&\quad+\int_{t/2}^t \sum_{2^q\leqslant R_0} 2^{q(\frac{n}{2}-2)} e^{-c2^{2q}(t-\tau)}2^{q}\left\|\Delta_q\left(\frac{a}{1+a}\mathcal{A}v\right)\right\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2}(t-\tau)^{-\frac12-\frac{\epsilon}{2}}\big\|a(\tau)\big\|_{B^{n/2}}\big\|v(\tau)\big\|_{B^{n/2}}\,d\tau+\int_{t/2}^t(t-\tau)^{-\frac12}\big\|a(\tau)\big\|_{B^{n/2}}\big\|v(\tau)\big\|_{B^{n/2}}\,d\tau\\
&\lesssim \int_0^{t/2}(t-\tau)^{-\frac12-\frac{\epsilon}{2}}(1+\tau)^{-1}\,d\tau\Lambda^2(t)+\int_{t/2}^t(t-\tau)^{-\frac12}(1+\tau)^{-1}\,d\tau\Lambda^2(t)\\
&\lesssim (1+t)^{-\frac12}\Lambda^2(t).
\end{aligned}$$

Other terms can be estimated similarly, so we now just give the results as follows

$$\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\big(\big\|\Delta_q a(t)\big\|_{L^2}+\big\|\Delta_q d(t)\big\|_{L^2}\big)\lesssim(1+t)^{-\frac12}\big(\|a_0\|_{B^{n/2-1,n/2+1}}+\|d_0\|_{B^{n/2-1,n/2}}\big)+(1+t)^{-\frac12}\Lambda^2(t),$$

$$\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\big(\big\|\Delta_q \mathcal{E}(t)\big\|_{L^2}+\big\|\Delta_q d(t)\big\|_{L^2}\big)\lesssim(1+t)^{-\frac12}\big(\|\mathcal{E}_0\|_{B^{n/2-1,n/2+1}}+\|d_0\|_{B^{n/2-1,n/2}}\big)+(1+t)^{-\frac12}\Lambda^2(t),$$

$$\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\big(\big\|\Delta_q\big(F^T-F\big)\big\|_{L^2}+\big\|\Delta_q\Omega(t)\big\|_{L^2}\big)$$
$$\lesssim (1+t)^{-\frac{1}{2}}\big(\big\|F_0^T-F_0\big\|_{B^{n/2-1,n/2+1}}+\|\Omega_0\|_{B^{n/2-1,n/2}}\big)+(1+t)^{-\frac{1}{2}}\Lambda^2(t).$$

Summing up the above three inequalities, we obtain

$$(1+t)^{\frac{1}{2}}\sum_{2^q\leqslant R_0} 2^{q\frac{n}{2}}\big(\big\|\Delta_q a(t)\big\|_{L^2}+\big\|\Delta_q v(t)\big\|_{L^2}+\big\|\Delta_q F(t)\big\|_{L^2}\big)$$
$$\lesssim \big\|(a_0,F_0)\big\|_{B^{n/2-1,n/2+1}}+\|v_0\|_{B^{n/2-1,n/2}}+\Lambda^2(t). \tag{6.13}$$

Step 3: Estimates of the high frequency part. For the high frequency part that is $2^q > R_0$, applying Lemma 5.2, Lemma 5.4, there exists a small positive number $c > 0$ such that

$$\big\|\Delta_q a(t)\big\|_{L^2}\lesssim \mathrm{I}_1^1+\mathrm{I}_2^1+\mathrm{I}_3^1,\qquad \big\|\Delta_q d(t)\big\|_{L^2}\lesssim \mathrm{I}_1^2+\mathrm{I}_2^2+\mathrm{I}_3^2,$$
$$\big\|\Delta_q\mathcal{E}(t)\big\|_{L^2}\lesssim \mathrm{II}_1^1+\mathrm{II}_2^1+\mathrm{II}_3^1,\qquad \big\|\Delta_q d(t)\big\|_{L^2}\lesssim \mathrm{II}_1^2+\mathrm{II}_2^2+\mathrm{II}_3^2,$$
$$\big\|\Delta_q\big(F^T(t)-F(t)\big)\big\|_{L^2}\lesssim \mathrm{III}_1^1+\mathrm{III}_2^1+\mathrm{III}_3^1,\qquad \big\|\Delta_q\Omega(t)\big\|_{L^2}\lesssim \mathrm{III}_1^2+\mathrm{III}_2^2+\mathrm{III}_3^2,$$

where

$$\mathrm{I}_1^1=e^{-ct}\|\Delta_q a_0\|_{L^2}+2^{-q}e^{-ct}\|\Delta_q d_0\|_{L^2},$$
$$\mathrm{I}_2^1=\int_0^t e^{-c(t-\tau)}\big(\big\|\Delta_q L(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla a)(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{I}_3^1=\int_0^t 2^{-q}e^{-c(t-\tau)}\big(\big\|\Delta_q G(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla d)(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{I}_1^2=2^{-q}e^{-ct}\|\Delta_q a_0\|_{L^2}+e^{-c2^{2q}t}\|\Delta_q d_0\|_{L^2},$$
$$\mathrm{I}_2^2=\int_0^t e^{-c(t-\tau)}2^{-q}\big(\big\|\Delta_q L(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla a)(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{I}_3^2=\int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q G(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla d)(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{II}_1^1=e^{-ct}\|\Delta_q\mathcal{E}_0\|_{L^2}+2^{-q}e^{-ct}\|\Delta_q d_0\|_{L^2},$$
$$\mathrm{II}_2^1=\int_0^t e^{-c(t-\tau)}\big(\big\|\Delta_q J(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla\mathcal{E})(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{II}_3^1=\int_0^t 2^{-q}e^{-c(t-\tau)}\big(\big\|\Delta_q K(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla d)(\tau)\big\|_{L^2}\big)\,d\tau,$$
$$\mathrm{II}_1^2=2^{-q}e^{-ct}\|\Delta_q\mathcal{E}_0\|_{L^2}+e^{-c2^{2q}t}\|\Delta_q d_0\|_{L^2},$$
$$\mathrm{II}_2^2=\int_0^t e^{-c(t-\tau)}2^{-q}\big(\big\|\Delta_q J(\tau)\big\|_{L^2}+\big\|\Delta_q(v\cdot\nabla\mathcal{E})(\tau)\big\|_{L^2}\big)\,d\tau,$$

$$\mathrm{II}_3^2 = \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q K(\tau)\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla d)(\tau)\big\|_{L^2}\big)\,d\tau,$$

$$\mathrm{III}_1^1 = e^{-ct}\big\|\Delta_q\big(F_0^T - F_0\big)\big\|_{L^2} + 2^{-q}e^{-ct}\|\Delta_q \Omega_0\|_{L^2},$$

$$\mathrm{III}_2^1 = \int_0^t e^{-c(t-\tau)}\big(\big\|\Delta_q I(\tau)\big\|_{L^2} + \big\|\Delta_q\big(v\cdot\nabla\big(F^T - F\big)\big)(\tau)\big\|_{L^2}\big)\,d\tau,$$

$$\mathrm{III}_3^1 = \int_0^t 2^{-q}e^{-c(t-\tau)}\big(\big\|\Delta_q H(\tau)\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla d)(\tau)\big\|_{L^2}\big)\,d\tau,$$

$$\mathrm{III}_1^2 = 2^{-q}e^{-ct}\big\|\Delta_q\big(F_0^T - F_0\big)\big\|_{L^2} + e^{-c2^{2q}t}\|\Delta_q \Omega_0\|_{L^2},$$

$$\mathrm{III}_2^2 = \int_0^t e^{-c(t-\tau)}2^{-q}\big(\big\|\Delta_q I(\tau)\big\|_{L^2} + \big\|\Delta_q\big(v\cdot\nabla\big(F^T - F\big)\big)(\tau)\big\|_{L^2}\big)\,d\tau,$$

$$\mathrm{III}_3^2 = \int_0^t e^{-c2^{2q}(t-\tau)}\big(\big\|\Delta_q H(\tau)\big\|_{L^2} + \big\|\Delta_q(v\cdot\nabla \Omega)(\tau)\big\|_{L^2}\big)\,d\tau.$$

As in the low frequency case, there are also a lot of similar terms, so we only give estimates about some typical terms. For the term I_1^1, we have

$$\begin{aligned}\sum_{2^q>R_0} 2^{q\frac{n}{2}}\mathrm{I}_1^1 &\lesssim \sum_{2^q>R_0} 2^{q\frac{n}{2}}e^{-ct}\|\Delta_q a_0\|_{L^2} + \sum_{2^q>R_0} 2^{q(\frac{n}{2}-1)}e^{-ct}\|\Delta_q d_0\|_{L^2}\\ &\lesssim e^{-ct}\big(\|a_0\|_{B^{n/2-1,n/2+1}} + \|d_0\|_{B^{n/2-1,n/2}}\big).\end{aligned}$$

Similarly, we have

$$\sum_{2^q>R_0} 2^{q\frac{n}{2}}\big(\mathrm{I}_1^2 + \mathrm{II}_1^1 + \mathrm{II}_1^2 + \mathrm{III}_1^1 + \mathrm{III}_1^2\big) \lesssim e^{-ct}\big(\|a_0\|_{B^{n/2-1,n/2+1}} + \|d_0\|_{B^{n/2-1,n/2}}\big).$$

For the term $\int_0^t e^{-ct(t-\tau)}\|\Delta_q(v\cdot\nabla a)(\tau)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}\int_0^t \sum_{2^q>R_0} 2^{q\frac{n}{2}}\big\|\Delta_q(v\cdot\nabla a)(\tau)\big\|_{L^2}\,d\tau &\lesssim \int_0^t e^{-c(t-\tau)}\big\|v(\tau)\big\|_{B^{n/2}}\big\|a(\tau)\big\|_{B^{n/2-1,n/2+1}}\,d\tau\\ &\lesssim (1+t)^{-\frac{1}{2}}\Lambda^2(t).\end{aligned}$$

For the term $\int_0^t e^{-c(t-\tau)}\|\Delta_q(a\operatorname{div} v)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}\int_0^t \sum_{2^q>R_0} 2^{q\frac{n}{2}}e^{-c(t-\tau)}\big\|\Delta_q(a\operatorname{div} v)\big\|_{L^2}\,d\tau &\lesssim \int_0^t e^{-c(t-\tau)}\big\|a(\tau)\big\|_{B^{n/2}}\big\|v(\tau)\big\|_{B^{n/2+1}}\,d\tau\\ &\lesssim \bigg(\int_0^t e^{-2c(t-\tau)}\big\|a(\tau)\big\|_{B^{n/2}}^2\,d\tau\bigg)^{\frac{1}{2}}\|v\|_{L_t^2(B^{n/2+1})}\end{aligned}$$

$$\lesssim \left(\int_0^t e^{-2c(t-\tau)}(1+\tau)^{-1}\, d\tau \right)^{\frac{1}{2}} \Lambda^2(t)$$
$$\lesssim (1+t)^{-\frac{1}{2}} \Lambda^2(t).$$

For the term $\int_0^t e^{-c(t-\tau)} 2^{-q} \|\Delta_q(F\nabla F)\|\, d\tau$, we have

$$\begin{aligned}
\int_0^t \sum_{2^q > R_0} 2^{q\frac{n}{2}} e^{-c(t-\tau)} 2^{-q} \big\|\Delta_q(F\nabla F)\big\|_{L^2}\, d\tau &\lesssim \int_0^t e^{-c(t-\tau)} \|F\nabla F\|_{B^{n/2-1}}\, d\tau \\
&\lesssim \int_0^t e^{-c(t-\tau)} \|F\|^2_{B^{n/2}}\, d\tau \\
&\lesssim (1+t)^{-1} \Lambda^2(t).
\end{aligned}$$

For the term $\int_0^t 2^{-q} e^{-c(t-\tau)} \|\Delta_q(\frac{a}{1+a}\Delta v)\|_{L^2}\, d\tau$, we have

$$\begin{aligned}
\int_0^t \sum_{2^q > R_0} 2^{q\frac{n}{2}} e^{-c(t-\tau)} 2^{-q} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2} d\tau &\lesssim \int_0^t e^{-c(t-\tau)} \big\|a(\tau)\big\|_{B^{n/2}} \big\|v(\tau)\big\|_{B^{n/2+1}}\, d\tau \\
&\lesssim \left(\int_0^t e^{-2c(t-\tau)} \big\|a(\tau)\big\|^2_{B^{n/2}}\, d\tau \right)^{\frac{1}{2}} \|v\|_{L^2_T(B^{n/2+1})} \\
&\lesssim \left(\int_0^t e^{-2c(t-\tau)} (1+\tau)^{-1} \right)^{\frac{1}{2}} \Lambda^2(t) \\
&\lesssim (1+t)^{-\frac{1}{2}} \Lambda^2(t).
\end{aligned}$$

For the term $\int_0^t e^{-c2^{2q}(t-\tau)} \|\Delta_q(v\cdot\nabla v)\|_{L^2}\, d\tau$, we have

$$\begin{aligned}
&\int_0^t \sum_{2^q > R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(v\cdot\nabla v)\big\|_{L^2}\, d\tau \\
&\lesssim \int_0^{t/2} \sum_{2^q > R_0} 2^{q(\frac{n}{2}-1-\epsilon)} 2^{q(1+\epsilon)} e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(v\cdot\nabla v)\big\|_{L^2}\, d\tau \\
&\quad + \int_{t/2}^t \sum_{2^q > R_0} 2^{q(\frac{n}{2}-1)} 2^q e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(v\cdot\nabla v)\big\|_{L^2}\, d\tau \\
&\lesssim \int_0^{t/2} (t-\tau)^{-\frac{1}{2}-\frac{\epsilon}{2}} \big\|v(\tau)\big\|_{B^{n/2-\epsilon}} \big\|v(\tau)\big\|_{B^{n/2}}\, d\tau + \int_{t/2}^t (t-\tau)^{-\frac{1}{2}} \big\|v(\tau)\big\|^2_{B^{n/2}}\, d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{\epsilon}{2}} (1+\tau)^{-\frac{1-\epsilon}{2}} (1+\tau)^{-\frac{1}{2}}\, d\tau \Lambda^2(t) + \int_{t/2}^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-1}\, d\tau \Lambda^2(t) \\
&\lesssim (1+t)^{-\frac{1}{2}} \Lambda^2(t).
\end{aligned}$$

For the term $\int_0^t e^{-c2^{2q}(t-\tau)}\|\Delta_q(F\nabla F)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}
&\int_0^t \sum_{2^q>R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(F\nabla F)\big\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} \sum_{2^q>R_0} 2^{q(\frac{n}{2}-1-\epsilon)}2^{q(1+\epsilon)} e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(F\nabla F)\big\|_{L^2}\,d\tau\\
&\quad+\int_{t/2}^t \sum_{2^q>R_0} 2^{q(\frac{n}{2}-1)}2^{q} e^{-c2^{2q}(t-\tau)} \big\|\Delta_q(F\nabla F)\big\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} (t-\tau)^{-\frac12-\frac{\epsilon}{2}}\big\|F(\tau)\big\|_{B^{n/2-\epsilon}}\big\|F(\tau)\big\|_{B^{n/2}}\,d\tau+\int_{t/2}^t (t-\tau)^{-\frac12}\big\|F(\tau)\big\|^2_{B^{n/2}}\,d\tau\\
&\lesssim \int_0^{t} (t-\tau)^{-\frac12-\frac{\epsilon}{2}}(1+\tau)^{-\frac{1-\epsilon}{2}}(1+\tau)^{-\frac12}\,d\tau\Lambda^2(t)+\int_{t/2}^t (t-\tau)^{-\frac12}(1+\tau)^{-1}\,d\tau\Lambda^2(t)\\
&\lesssim (1+t)^{-\frac12}\Lambda^2(t).
\end{aligned}$$

For the term $\int_0^t e^{-c2^{2q}(t-\tau)}\|\Delta_q(\frac{a}{1+a}\Delta v)\|_{L^2}\,d\tau$, we have

$$\begin{aligned}
&\int_0^t \sum_{2^q>R_0} 2^{q\frac{n}{2}} e^{-c2^{2q}(t-\tau)} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} \sum_{2^q>R_0} 2^{q(\frac{n}{2}-2)}2^{2q} e^{-c2^{2q}(t-\tau)} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2}\,d\tau\\
&\quad+\int_{t/2}^t \sum_{2^q>R_0} 2^{q(\frac{n}{2}-1+\frac{\epsilon}{2})}2^{q(1-\epsilon)} e^{-c2^{2q}(t-\tau)} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2}\,d\tau\\
&\lesssim \mathrm{M}_1+\mathrm{M}_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{M}_1&=\int_0^{t/2} \sum_{2^q>R_0} 2^{q(\frac{n}{2}-2)}2^{2q} e^{-c2^{2q}(t-\tau)} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2}\,d\tau\\
&\lesssim \int_0^{t/2} (t-\tau)^{-1}\big\|a(\tau)\big\|_{B^{n/2}}\big\|v(\tau)\big\|_{B^{n/2}}\,d\tau\\
&\lesssim (1+t)^{-\frac12}\Lambda^2(t),\\
\mathrm{M}_2&=\int_{t/2}^t \sum_{2^q>R_0} 2^{q(\frac{n}{2}-1+\frac{\epsilon}{2})}2^{q(1-\epsilon)} e^{-c2^{2q}(t-\tau)} \left\|\Delta_q\left(\frac{a}{1+a}\Delta v\right)\right\|_{L^2}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}+\frac{\epsilon}{2}} e^{-c(t-\tau)} \left\| \frac{a}{1+a} \Delta v \right\|_{B^{n/2-1+\epsilon/2}} d\tau \\
&\lesssim \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}+\frac{\epsilon}{2}} e^{-c(t-\tau)} \left\| a(\tau) \right\|_{B^{n/2}} \left\| v(\tau) \right\|_{B^{n/2+1+\epsilon/2}} d\tau \\
&\lesssim \left(\int_{t/2}^{t} (t-\tau)^{-\frac{1-\epsilon}{2}\frac{4}{2-\epsilon}} (1+\tau)^{-\frac{2}{2-\epsilon}} e^{-c(t-\tau)} \, d\tau \right)^{\frac{2-\epsilon}{4}} \\
&\lesssim (1+t)^{-\frac{1}{2}} \left(\int_{0}^{t/2} \tau^{-\frac{2-2\epsilon}{2-\epsilon}} e^{-\frac{4}{2-\epsilon}} \, d\tau \right)^{\frac{2-\epsilon}{4}} \Lambda^2(t) \\
&\lesssim (1+t)^{-\frac{1}{2}} \Lambda^2(t),
\end{aligned}
$$

with ϵ to be a small enough positive number. For other terms appeared in I_1^1 to III_3^2, we can get estimates similar to the above typical terms. Combining all the estimates and (6.1), we will get

$$
\begin{aligned}
&(1+t)^{\frac{1}{2}} \sum_{2^q > R_0} 2^{q\frac{n}{2}} \left(\left\| \Delta_q a(t) \right\|_{L^2} + \left\| \Delta_q v(t) \right\|_{L^2} + \left\| \Delta_q F(t) \right\|_{L^2} \right) \\
&\lesssim \left\| (a_0, F_0) \right\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}} + \Lambda^2(t). \qquad (6.14)
\end{aligned}
$$

Step 4: Bootstrap argument. Combining (6.13) and (6.14), we will obtain

$$
(1+t)^{\frac{1}{2}} \left(\left\| a(t) \right\|_{B^{n/2}} + \left\| v(t) \right\|_{B^{n/2}} + \left\| F(t) \right\|_{B^{n/2}} \right) \leqslant C \left(\left\| (a_0, F_0) \right\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}} \right) + \Lambda^2(t).
$$

Form (6.4), we know that if $\alpha_0 = \|(a_0, F_0)\|_{B^{n/2-1,n/2+1}} + \|v_0\|_{B^{n/2-1,n/2}}$ small enough, there holds

$$
\Lambda(t) \leqslant C\alpha_0 + C\Lambda^2(t).
$$

Suppose that

$$
T := \sup\left\{ t > 0 : \Lambda(t) \leqslant 2C\alpha_0 \right\}. \qquad (6.15)
$$

If $T = \infty$, the proof is completed. If $T < \infty$, for any $0 < t < T$, we have

$$
\Lambda(t) \leqslant C\alpha_0 + C4C^2\alpha_0^2.
$$

For α_0 small enough such that $4C^2\alpha_0 \leqslant \frac{1}{2}$, we obtain

$$
\Lambda(t) \leqslant \frac{3}{2} C\alpha_0,
$$

which contradicts to (6.15). Hence, we finally obtain

$$
\Lambda(t) \leqslant 2C\alpha_0.
$$

By interpolation, the proof is completed. □

Acknowledgments

The authors thank the anonymous referee for his/her careful reading of the paper and valuable suggestions. J. Jia would like to thank China Scholarship Council that has provided a scholarship for his research work in the United States. This work was partially supported by National Natural Science Foundation of China under the grant No. 11131006, by the National Basic Research Program of China under the grant No. 2013CB329404, and by the Doctoral Program of Higher Education of China under the grant No. 20130201120053.

Appendix A

Let us first recall some algebraic relations involving changes of coordinates. We are given a C^1 diffeomorphism X over $\mathbb{R}^n$. For $H : \mathbb{R}^n \to \mathbb{R}^m$, we agree that $\bar{H}(y) = H(x)$ with $x = X(y)$. With this convention, the chain rule is

$$D_y\bar{H}(y) = D_xH\big(X(y)\big) \cdot D_yX(y), \tag{A.1}$$

with

$$(D_xH)_{ij} = \partial_{x_j}H^i \quad \text{and} \quad (D_yX)_{ij} = \partial_{y_j}X^i.$$

Denoting $\nabla_y = D_y^T$ and $\nabla_x = D_x^T$,

$$\nabla_y\bar{H}(y) = \nabla_yX(y) \cdot \nabla_xH\big(X(y)\big).$$

Hence we have

$$D_xH(x) = D_y\bar{H}(y) \cdot A(y) \quad \text{with } A(y) = \big(D_yX(y)\big)^{-1} = D_xX^{-1}(x). \tag{A.2}$$

Lemma A.1. *Let K be a C^1 scalar function over $\mathbb{R}^n$ and H, a C^1 vector field. Let X be a C^1 diffeomorphism such that $J := \det(D_yX) > 0$. Then the following relations hold true:*

$$\overline{\nabla_xK} = J^{-1}\operatorname{div}_y\big(\operatorname{adj}(D_yX)\bar{K}\big), \tag{A.3}$$

$$\overline{\operatorname{div}_x H} = J^{-1}\operatorname{div}_y\big(\operatorname{adj}(\nabla_yX)\bar{H}\big), \tag{A.4}$$

where $\operatorname{adj}(D_yX)$ stands for the adjugate of D_yX.

Lemma A.2. *Let $p \in [1, +\infty)$ and $\bar{v}$ be in $E_p(T)$ satisfying (4.12). Let X_v be defined by (4.27). Then we have for all $t \in [0, T]$,*

$$\big\|I - \operatorname{adj}\big(DX_v(t)\big)\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})},$$

$$\big\|I - A_v(t)\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})},$$

$$\big\|J_v^{\pm}(t) - 1\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})},$$

$$\big\|I - L_v^{\pm}\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})}.$$

Furthermore, if $\bar{w}$ is a vector field such that $D\bar{w} \in L^1(0, T; B_p^{n/p})$ then

$$\big\|\big(\operatorname{adj}(DX_v)D_{A_v}(\bar{w}) - D(\bar{w})\big)(t)\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})}\|D\bar{w}\|_{L_t^1(B_p^{n/p})},$$

$$\big\|\big(\operatorname{adj}(DX_v)\operatorname{div}_{A_v}(\bar{w}) - \operatorname{div}\bar{w}I\big)(t)\big\|_{B_p^{n/p}} \lesssim \|D\bar{v}\|_{L_t^1(B_p^{n/p})}\|D\bar{w}\|_{L_t^1(B_p^{n/p})}.$$

Proof. We just give the proof about L_v, the other inequalities are proved in [9]. Let $C_v = \int_0^t \nabla_y \bar{v}(\tau, y)\, d\tau$, then we have

$$L_v^{-1} = (I - C_v)^{-1} = \sum_{k\in\mathbb{Z}} \big(C_v(t)\big)^k.$$

Due to (4.12), we get the result easily. Since

$$L_v - I = -\int_0^t \nabla_y \bar{v}(\tau, y)\, d\tau,$$

the result is obvious. □

Lemma A.3. *Let $\bar{v}_1$ and $\bar{v}_2$ be two vector fields satisfying (4.12), and $\delta v := \bar{v}_2 - \bar{v}_1$. Then we have for all $p \in [1, +\infty)$ and all $t \in [0, T]$*

$$\begin{aligned}
\big\|A_{v_2}(t) - A_{v_1}(t)\big\|_{B_p^{n/p}} &\lesssim \|D\delta v\|_{L_t^1(B_p^{n/p})}, \\
\big\|\mathrm{adj}\big(DX_{v_2}(t)\big) - \mathrm{adj}\big(DX_{v_1}(t)\big)\big\|_{B_p^{n/p}} &\lesssim \|D\delta v\|_{L_t^1(B_p^{n/p})}, \\
\big\|J_{v_2}^{\pm}(t) - J_{v_1}^{\pm}(t)\big\|_{B_p^{n/p}} &\lesssim \|D\delta v\|_{L_t^1(B_p^{n/p})}, \\
\big\|L_{v_2}^{\pm} - L_{v_1}^{\pm}\big\|_{B_p^{n/p}} &\lesssim \|D\delta v\|_{L_t^1(B_p^{n/p})}.
\end{aligned}$$

Proof. For the same reason as for Lemma A.2, we only give the proof about L_{v_i} $(i = 1, 2)$. Since $L_{v_2} - L_{v_1} = -\int_0^t \nabla_y \delta v(\tau, y)\, d\tau$, we easily obtain the result. Let us denote $C_{v_i} = \int_0^t \nabla_y \bar{v}_i(\tau, y)\, d\tau$ $(i = 1, 2)$. Due to

$$\begin{aligned}
L_{v_2}^{-1} - L_{v_1}^{-1} &= \sum_{k\geqslant 1} \big(C_{v_2}^k - C_{v_1}^k\big) \\
&= \int_0^t D\delta v\, d\tau \sum_{k\geqslant 1} \sum_{j=0}^{k-1} C_1^j C_2^{k-1-j},
\end{aligned}$$

and (4.12), we can get the desired result. □

For the reader's convenience, we list some properties about Besov space.

Lemma A.4. *Let $s, t, \tilde{s}, \tilde{t}, \sigma, \tau \in \mathbb{R}$, $2 \leqslant p \leqslant 4$, and $1 \leqslant r, r_1, r_2 \leqslant \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then we have the following:*

(1) *If $\sigma, \tau \leqslant \frac{n}{p}$ and $\sigma + \tau > 0$, then*

$$\sum_{2^j > R_0} 2^{j(\sigma+\tau-n/p)} \big\|\Delta_j(fg)\big\|_{L_T^r(L^p)} \leqslant C\|f\|_{\tilde{L}_T^{r_1}(B_{2,p}^{n/2-n/p+\sigma,\sigma})} \|g\|_{\tilde{L}_T^{r_2}(B_{2,p}^{n/2-n/p+\tau,\tau})}.$$

(2) *If $s, \tilde{s} \leqslant \frac{n}{p}$ and $s + t > n - \frac{2n}{p}$ with $s + t = \tilde{s} + \tilde{t}$, and $\gamma \in \mathbb{R}$, then*

$$\begin{aligned}
\sum_{2^j \leqslant R_0} 2^{j(s+t-n/2)} \big\|\Delta_j(fg)\big\|_{L_T^r(L^2)} &\leqslant C\big(\|f\|_{\tilde{L}_T^{r_1}(B_{2,p}^{s,s-n/2+n/p})} \|g\|_{\tilde{L}_T^{r_2}(B_{2,p}^{t,t-n/2+n/p+\gamma})} \\
&\quad + \|g\|_{\tilde{L}_T^{r_2}(B_{2,p}^{\tilde{d},\tilde{s}-n/2+n/p})} \|f\|_{\tilde{L}_T^{r_1}(B_{2,p}^{\tilde{t},\tilde{t}-n/2+n/p})}\big).
\end{aligned}$$

(3) *If* $s, \tilde{s} \leqslant \frac{n}{2}$ *and* $s + t > \frac{n}{2} - \frac{n}{p}$ *with* $s + t = \tilde{s} + \tilde{t}$, *then*

$$\sum_{j\in\mathbb{Z}} 2^{j(s+t-n/2)} \|\Delta_j(fg)\|_{L_T^r(L^2)} \leqslant C\big(\|f\|_{\tilde{L}_T^{r_1}(B_{2,p}^{s,s-n/2+n/p})}\|g\|_{\tilde{L}_T^{r_2}(B^t)} + \|g\|_{\tilde{L}_T^{r_2}(B_{2,p}^{\tilde{s},\tilde{s}-n/2+n/p})}\|f\|_{\tilde{L}_T^{r_1}(B^{\tilde{t}})}\big).$$

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