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Critical dynamics study on recurrent neural networks: Globally exponential stability ☆

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ABSTRACT

Critical dynamics research of recurrent neural networks (RNNs) is very meaningful in both theoretical importance and practical significance. Due to the essential difficulty in analysis, there were only a few contributions concerning it. In this paper, we devote to study the critical dynamics behaviors for RNNs with general forms. By exploring some intrinsic features processed naturally by the nonlinear activation mappings of RNNs, and by using matrix measure theory, new criteria are found to ascertain the globally exponential stability of RNNs under the critical conditions. The results obtained here either yield new, or sharpen, extend or unify, to a large extent, most of the existing non-critical conclusions as well as the latest critical results.

1. Introduction

Recurrent neural networks (RNNs) are dynamic systems that can be implemented by physical means, and they are mainly used to model dynamic process associated with control process, perform associative memory and solve optimization problems. In the present paper, we consider the generic continuous-time RNNs modeled by the following nonlinear differential equation:

$$\frac{dy(t)}{dt} = -Dy(t)+WF(Ay(t)+b)+q, \quad y_0 \in \mathcal{R}^N, \tag{1}$$

where $y=(y_1,y_2,\ldots,y_N)^T$ is the neural network state, $D=diag(d_1,d_2,\ldots,d_N)$ is a positive matrix with each d_i being the state feedback coefficients, $W=(\omega_{ij})_{N\times N}$ is the connective weight matrix, A is an $N\times N$ diagonal matrix, b,q are two fixed external bias vector and $F:\mathcal{R}^N\to\mathcal{R}^N$ is the nonlinear activation mapping.

As we know, there are two fundamental models which can summarize most of the existing RNNs specials [23], i.e., depending upon whether neural states or local fields states are taken as basic variables, a RNN can frequently be classified either as a local field neural network model or as a static neural network model. These two basic RNNs models are extensively applied in learning, pattern recognition, associative memory, solving optimization problems, etc. It should be noted that by taking $A=I$ and $b=\vec{0}$, model (1) corresponds the local field neural network model, and by choosing $W=I$ and $q=\vec{0}$, model (1) refers to the static neural network model. Actually, model (1) describes uniformly various continuous-time RNNs models studied in literature, for example, Hopfield-type neural networks, recurrent back-propagation neural networks, mean-field neural networks, recurrent correlation associative memories neural networks, bound-constraints optimization solvers, convex optimization solvers, brain-state-in-a-box neural networks, cellular neural networks, and so on.

The analysis of the dynamical behaviors, such as the global convergence, asymptotic stability and exponential stability is a first and necessary step for any practical design and application of RNNs. In recent years, considerable efforts have been devoted to the analysis on the stability and convergence of RNNs without and with delay (see, e.g., the contributions of [2–4,6–8,11,13,17,18] and the references therein). If we define

$$S(\Lambda,P)=\Lambda DP^{-1}-\frac{\Lambda AW+(\Lambda AW)^T}{2},$$

where both Λ and P are positive definite diagonal matrices, and W is the weight matrix of the network, then by generalizing these existing stability results of RNNs, it should be noticed that most of them are on the exponential stability analysis under the

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conditions that for one positive definite diagonal matrix Λ, $S(\Lambda,L)$ is positive definite, where $L=diag\{L_1,L_2,\ldots,L_N\}$ with each $L_i>0$ being the Lipschitz constant of f_i and $F=(f_1,f_2,\ldots,f_N)^T$ is the activation mapping of the network. On the other hand, [18,14] have proved that a RNN will be globally exponentially unstable if there is a positive definite diagonal matrix Λ such that $S(\Lambda,V)$ is negative definite, where $V=diag\{r_1,r_2,\ldots,r_N\}$ with each $r_i>0$ being the inversely Lipschitz constant of f_i (i.e., for all $s,t\in\mathcal{R}^N$, $|f_i(t)-f_i(s)|\geq r_i|t-s|$). From the definitions of Lipschitz constant and inversely Lipschitz constant, we have $r_i\leq L_i$. For any positive definite diagonal matrix $Q=diag\{q_1,q_2,\ldots,q_N\}$ with each q_i satisfying

$$r_i\leq q_i\leq L_i,\quad i=1,2,\ldots,N,$$

then by using $A\leq B$ to denote the condition that matrix $B-A$ is nonnegative definite, it is clear that in the sense of nonnegative definition, there holds the following inequality relation:

$$S(\Lambda,L)\leq S(\Lambda,Q)\leq S(\Lambda,V),$$

where Λ is a given positive definite diagonal matrix. Since $S(\Lambda,L)>0$ (i.e., $S(\Lambda,L)$ is positive definite) is sufficient for the globally exponential stability of RNNs, and $S(\Lambda,V)\geq 0$ (i.e., $S(\Lambda,V)$ is nonnegative definite) is necessary for RNNs to have globally stable dynamics, the questions then arise: what kinds of asymptotic behavior of RNNs will hold when $S(\Lambda,L)\leq 0$ (i.e., $S(\Lambda,L)$ is negative semi-definite) and $S(\Lambda,V)\geq 0$? In particular, what happens in the case that $S(\Lambda,Q)=0$ (i.e., for any $x\in\mathcal{R}^N$, $x^TS(\Lambda,Q)x=0$)? The dynamics analysis of RNNs under such conditions is referred to as the *critical dynamics analysis*. It should be remarked that it is by no means easy to conduct a meaningful critical dynamics study for RNNs since such exploration has essential difficult in analysis.

In comparison to the general critical condition that $S(\Lambda,Q)=0$, $S(\Lambda,L)=0$ is the primary case of it, and a RNN is globally exponential stability when $S(\Lambda,L)>0$, so recently, special attentions have been focused on the dynamics investigations of RNNs under the particular critical condition that $S(\Lambda,L)\geq 0$. Even so, it is still much more difficult than the dynamics analysis under the non-critical condition that $S(\Lambda,L)>0$. Up to now, there are only a few critical stability and convergence analysis of RNNs in the sense that $S(\Lambda,L)$ is nonnegative definite. For a local field neural network model with hyperbolic tangent activation function, [4,5,12] have achieved the globally asymptotical stability and globally exponential stability of the unique equilibrium point of the network under some specific conditions of $S(\Lambda,L)\geq 0$. Ref. [24] have gotten the globally exponential stability of a static neural network with projection operator under the condition that $I-W$ is nonnegative (which is a special case of $S(\Lambda,L)\geq 0$). Ref. [14] have proved that a local field neural network model with Sigmoidal activation mapping has a globally attractive equilibrium state, and when W is quasi-symmetric (i.e., there exists a positive definite diagonal matrix D, such that DW is symmetric), then a static neural network model with nearest point projection activation mapping is global convergence on a region defined by the network. For a RNN with projection activation mapping, the quasi-symmetric requirement of W in [14] has been removed by [15], and some further study of such RNNs has been conducted in [16].

For all that, there are still many important dynamics questions of RNNs unsettled under the critical conditions. For example, with what additional requirement will the Sigmoidal RNNs be globally exponential stability when $S(\Lambda,L)\geq 0$, and what kinds of asymptotical behavior will be when $S(\Lambda,Q)=0$? For a RNN with general projection mapping, does there exist some convergence results when $S(\Lambda,L)<0$, or, under the general critical condition that $S(\Lambda,Q)=0$? Further, for neural network with a generic activation mapping, what asymptotic behaviors of it will be under the critical conditions that $S(\Lambda,L)\geq 0$, or further, $S(\Lambda,Q)=0$? All these are under our current investigation.

In the current paper, we devote to answer the question that what dynamics behavior will happen for RNNs when $S(\Lambda,L)\geq 0$. By exploring some intrinsic features processed naturally by nonlinear activation mappings, e.g., the decreasing anti-monotone as well as the uniformly anti-monotone properties, and by applying the energy function method, it is shown that a RNN has a unique equilibrium state and which is globally asymptotically stable if there exists a positive definite diagonal matrix Λ, such that $S(\Lambda,L)$ is nonnegative definite. What is more important, based on matrix measure theory, we get that a RNN is globally exponentially stable under the conditions that one matrix (which is defined by the network and is very similar to $S(\Lambda,L)$) is positive definite at the unique equilibrium state. The results obtained here sharpen and generalize, to a large extent, most of the existing non-critical conclusions (see, e.g.,[1–3,6–8,10,11,13,17,18,22,23] and the references quoted there) as well as the latest critical results given by [4,5,12,14,24]. Furthermore, they can provide a wider application range for RNNs and can be applied directly to many concrete RNN models, e.g., the well-known recurrent back-propagation neural networks, Hopfield-type neural networks, recurrent correlation associative memories with exponential activation mapping, etc.

2. Preliminaries

In the present investigation, we denote by $\mathcal{R}^N$ the N-dimensional real vector space with norm $\|\cdot\|$. Given any $N\times N$ matrix A, let A^T be its transpose, $\sigma(A)$ be its spectral set (i.e., all eigenvalues of A), $\|A\|$ and $\mu(A)$ be its matrix norm and matrix measure that are, respectively, defined by

$$\|A\|=\sup_{\|x\|=1}\|Ax\|\quad\text{and}\quad \mu(A)=\lim_{\lambda\to 0^+}\frac{\|I+\lambda A\|-1}{\lambda},$$

where I is the identity matrix. Use $\alpha(A)=\max\{\text{Re }\lambda:\lambda\in\sigma(A)\}$ to denote the maximum abscissa of $\sigma(A)$. Associated with a mapping F, denoted by $\mathbf{D}(F)$ and $\mathbf{R}(F)$ its domain and range, respectively. F is said to be *diagonally nonlinear* if F is defined componentwise by

$$F(x)=(f_1(x_1),f_2(x_2),\ldots,f_N(x_N))^T,$$

where each f_i is a one-dimensional nonlinear function. Assume that f_i is Lipschitz continuous. L_i, the *minimum Lipschitz constant* of f_i, is defined as follows

$$L_i=\sup_{t,s\in\mathcal{R},t\neq s}\frac{|f_i(t)-f_i(s)|}{|t-s|}.\tag{2}$$

Without loss of generality, through out this paper, we let $L=diag\{L_1,L_2,\ \ldots,L_N\}$ and assume that each $L_i>0$. The equilibrium state set of system (1) is denoted by Ω.

Definition 1 (*Qiao et al. [18]*). F is said to be β-uniformly anti-monotone (β-UAM) if there is a positive constant $\beta>0$, such that for any $x,y\in\mathbf{D}(F)$, it satisfies

$$\|F(x)-F(y)\|^2\leq\beta\cdot\langle F(x)-F(y),x-y\rangle.\tag{3}$$

Obviously, a β-UAM mapping is a monotone mapping.

Definition 2. For a given point $x^*\in\mathbf{D}(F)$, F is called C_F-decreasing anti-monotone (C_F-DAM) at x^*, if there exists a constant $C_F>0$ and a monotonically decreasing function $\beta_{x^*}:[0,+\infty)\to(0,C_F]$ which attains its maximum only at zero (i.e., $\beta_{x^*}(t)=C_F$ if and only if $t=0$), such that

$$\|F(x)-F(x^*)\|^2\leq\beta_{x^*}(\|x-x^*\|)\cdot\langle F(x)-F(x^*),x-x^*\rangle,\quad\forall x\in\mathbf{D}(F).$$

Definition 3. For a given point $x^*\in\mathbf{D}(F)$, F is called diagonally C_F-decreasing anti-monotone (diagonally C_F-DAM) at x^*, if it is diagonally nonlinear, and there exists a positive matrix $C_F = diag\{C_{f_1}, C_{f_2}, \ldots, C_{f_N}\}$ such that each f_i is C_{f_i}-DAM at x_i^*.

Remark 1. Many typical activation mappings naturally process the decreasing anti-monotone property, e.g., Sigmoidal mappings [14, Lemma 1], the nearest point projection mappings, etc. The following example shows the DAM property possessed by 1-D nearest point projection. Consider:

$$f(x) = \tfrac{1}{2}(|x+1| - |x-1|).$$

Define:

$$\beta_{x^*}(t) = \begin{cases} 2(1-\frac{t}{2}), & 0 \le t \le 1, \\ \frac{1}{t}, & t > 1. \end{cases}$$

Let $x^* = 0$ and $C_f = 2$. Obviously, $\beta_{x^*} : [0, +\infty) \to (0, C_f]$ is monotonically decreasing and satisfies $\beta_{x^*}(0) = C_f$. Further, for any $t > 0$, $\beta_{x^*}(t) \in (0, C_f)$ holds always. This is, f is C_f-DAM.

In fact, the RNNs whose activation mappings owning the decreasing anti-monotone property are very usual, and the typical models include: Hopfield-type neural networks, recurrent back-propagation neural networks, bidirectional associative memory neural networks, bound-constraints optimization neural networks, brain-state-in-a-box /domain type neural networks, cellular neural networks and so on. All these RNNs models have been widely applied in various fields of science and engineering.

3. Main results

The following lemma gives the globally asymptotic stable result of network (1). To begin with, it should be noted that network (1) has at least one equilibrium state. In fact, when $\mathbf{R}(F)$ is bounded, the operator G defined by $G(y) = D^{-1}(y + WF(Ay + b) + q)$ is compact. Hence, by the well-known Brouwer fixed point theorem, G has at least one fixed point; that is, network (1) has at least one equilibrium state y^*. If we can prove that any trajectory of (1) will converge to y^* as $t \to +\infty$, then, by the arbitrariness of y^* and the boundedness of $\mathbf{R}(F)$, it follows that y^* is the unique equilibrium state of (1) and it is globally asymptotically stable.

Lemma 1. *Suppose that $A = diag\{a_1, a_2, \ldots, a_N\}$, f_i is continuous and strictly monotonically increasing, $\mathbf{R}(F)$ is bounded. Let $L = diag\{L_1, L_2, \ldots, L_N\}$, where each L_i is the minimum Lipschitz constant of f_i, and $y^* \in \Omega$. If F is diagonal L-DAM at $x^*(:= Ay^* + b)$, then y^* is the unique equilibrium state of network (1), and it is globally asymptotically stable whenever there is a positive definite diagonal matrix $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ such that*

$$S(\Lambda, L) = \Lambda D L^{-1} - \frac{\Lambda A W + (\Lambda A W)^T}{2} \tag{4}$$

is nonnegative definite.

Proof. The main object of the proof is to show that any trajectory of (1) converges to y^* as $t \to +\infty$.

First, by the assumption that F is diagonal L-DAM at x^*, we have F is diagonally nonlinear and there exist monotonically decreasing functions $\beta_{x_i^*} : [0, +\infty) \to (0, L_i]$ $(i = 1, 2, \ldots, N)$ satisfying $\beta_{x_i^*}(t) \in (0, L_i)$ for all $t > 0$, $\beta_{x_i^*}(0) = L_i$ and

$$|f_i(s) - f_i(x_i^*)|^2 \le \beta_{x_i^*}(|s - x_i^*|) \cdot \langle f_i(s) - f_i(x_i^*), s - x_i^* \rangle, \quad \forall s \in \mathbf{D}(f_i).$$

Define

$$E(y(t)) = \sum_{i=1}^{N} \lambda_i \int_{a_i y_i^* + b_i}^{a_i y_i(t) + b_i} (f_i(s) - f_i(a_i y_i^* + b_i))\, ds, \tag{5}$$

then

$$\begin{aligned} \frac{dE(y(t))}{dt} &= \sum_{i=1}^{N} \lambda_i (f_i(a_i y_i(t) + b_i) - f_i(a_i y_i^* + b_i)) a_i \frac{d(y_i(t) - y_i^*)}{dt} \\ &= \langle \Lambda A(F(Ay(t)+b) - F(Ay^*+b)), -D(y(t) - y^*) \\ &\quad + W(F(Ay(t)+b) - F(Ay^*+b)) \rangle. \end{aligned} \tag{6}$$

Let $B_{x^*} = diag\{\beta_{x_1^*}(|x_1(t) - x_1^*|), \beta_{x_2^*}(|x_2(t) - x_2^*|), \ldots, \beta_{x_N^*}(|x_N(t) - x_N^*|)\}$, where $x(t) = Ay(t) + b$. Since $S(\Lambda, L)$ is nonnegative definite, thus from (6), we have

$$\begin{aligned} \frac{dE(y(t))}{dt} &= -\langle F(x(t)) - F(x^*), \Lambda D(x(t) - x^*) \rangle \\ &\quad + (F(x(t)) - F(x^*))^T \Lambda D L^{-1} (F(x(t)) - F(x^*)) \\ &\quad - (F(x(t)) - F(x^*))^T S(\Lambda, L)(F(x(t)) - F(x^*)) \\ &\le -\langle F(x(t)) - F(x^*), \Lambda D B_{x^*}^{-1}(F(x(t)) - F(x^*)) \rangle \\ &\quad + (F(x(t)) - F(x^*))^T \Lambda D L^{-1} (F(x(t)) - F(x^*)) \\ &\quad - (F(x(t)) - F(x^*))^T S(\Lambda, L)(F(x(t)) - F(x^*)) \\ &\le -\langle F(x(t)) - F(x^*), \Lambda D(B_{x^*}^{-1} - L^{-1})(F(x(t)) - F(x^*)) \rangle. \end{aligned} \tag{7}$$

By the definition of B_{x^*}, it follows that $E(y(t))$ is monotonically decreasing and $\lim_{t \to +\infty} E(y(t))$ exists since $E(y(t))$ is nonnegative (which can be deduced from Lemma 3 in [15] directly). In addition, we will show that the limit of $E(y(t))$ equals to 0. If we can prove that there is a subsequence $\{t_n\}$ such that $\lim_{n \to +\infty} \|F(x(t_n)) - F(x^*)\| = 0$, then from the boundedness of $x(t)$, we may assume that there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\lim_{k \to +\infty} x(t_{n_k}) = \lim_{k \to +\infty} (Ay(t_{n_k}) + b) := s^*$, and by the continuity of F, we have $F(x^*) = F(s^*)$. Hence $x^* = s^*$ because of the strictly increasing property of each f_i. From (5), one know $\lim_{k \to +\infty} E(y(t_{n_k})) = 0$, and combining with the result that $\lim_{t \to +\infty} E(y(t))$ exists, implies that $\lim_{t \to +\infty} E(y(t)) = 0$. Applying Lemma 3 in [15] to (5), we have

$$E(y(t)) \ge \sum_{i=1}^{N} \frac{\lambda_i}{2L_i} (f_i(a_i y_i(t) + b_i) - f_i(a_i y_i^* + b_i))^2 = \sum_{i=1}^{N} \frac{\lambda_i}{2L_i} (f_i(x_i(t)) - f_i(x_i^*))^2.$$

Thus

$$0 = \lim_{t \to +\infty} E(y(t)) \ge 2^{-1} r \limsup_{t \to +\infty} \|F(x(t)) - F(x^*)\|^2 \ge 0,$$

in which $r = \min_{1 \le i \le N} \{\lambda_i L_i^{-1}\} > 0$. Consequently,

$$\lim_{t \to +\infty} \|F(x(t)) - F(x^*)\| = 0. \tag{8}$$

On the other hand, we know that $y(t)$, as a trajectory of (1), solves the following integral equation:

$$y(t) - y^* = e^{-(t-t_0)D}(y_0 - y^*) + \int_{t_0}^{t} e^{-(t-s)D} \cdot W(F(x(s)) - F(x^*))\, ds.$$

Obviously, it holds that

$$\|y(t) - y^*\| \le e^{-(t-t_0)d_{min}} \|y_0 - y^*\| + \int_{t_0}^{t} e^{-(t-s)d_{min}} \|W\| \cdot \|F(x(s)) - F(x^*)\|\, ds, \tag{9}$$

in which $d_{min} = \min_{1 \le i \le N}\{d_i\}$. By (8), for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$ such that, whenever $t \ge T_\varepsilon$,

$$\|F(x(t)) - F(x^*)\| \le \frac{\varepsilon \cdot d_{min}}{\|W\|}.$$

Therefore, we conclude from (9) that $\|y(t) - y^*\| < e^{-(t-t_0)d_{min}} \|y_0 - y^*\| + \varepsilon$, when $t > t_0 \ge T_\varepsilon$. Letting $t \to +\infty$ in the above inequality yields $\lim_{t \to +\infty} \|y(t) - y^*\| < \varepsilon$, which then implies $\lim_{t \to +\infty} y(t) = y^*$

since ε is arbitrary. That is, when $t \to +\infty$, any trajectory of (1) converges to y^*.

Now to complete the proof. We need to verify that there exists a subsequence $\{t_n\}$ such that $\lim_{n\to+\infty}\|F(x(t_n))-F(x^*)\|=0$. Obviously, if for all $i \in \{1,2,\ldots,N\}$,

$$\liminf_{t\to+\infty}|f_i(x_i(t))-f_i(x_i^*)|=0, \tag{10}$$

then the above proposition holds clearly. In what follows, we will show the correctness of (10).

If (10) were not true for some $i_0 \in \{1,2,\ldots,N\}$, then there would be two positive constants t_0 and $\wp$, such that

$$|f_{i_0}(x_{i_0}(t))-f_{i_0}(x_{i_0}^*)| \ge \wp, \tag{11}$$

whenever $t \ge t_0$. This, combined with the definition of minimum Lipschitz constant of f_{i_0} and the monotonically decreasing property of $\beta_{x_{i_0}^*}$, leads to the fact that for any $t \ge t_0$, $|x_{i_0}(t)-x_{i_0}^*| \ge \wp\cdot L_{i_0}^{-1}$, and further, $\beta_{x_{i_0}^*}(|x_{i_0}(t)-x_{i_0}^*|) \le \beta_{x_{i_0}^*}(\wp\cdot L_{i_0}^{-1})$. Thus, by (7), we have

$$\begin{aligned}\frac{dE(y(t))}{dt} &\le -\lambda_{i_0}d_{i_0}(\beta_{x_{i_0}^*}^{-1}(|x_{i_0}(t)-x_{i_0}^*|)-L_{i_0}^{-1})|f_{i_0}(x_{i_0}(t))-f_{i_0}(x_{i_0}^*)|^2\\ &\le -\lambda_{i_0}d_{i_0}(\beta_{x_{i_0}^*}^{-1}(\wp\cdot L_{i_0}^{-1})-L_{i_0}^{-1})|f_{i_0}(x_{i_0}(t))-f_{i_0}(x_{i_0}^*)|^2.\end{aligned}$$

Since $\lim_{t\to+\infty}E(y(t))$ exists and $\beta_{x_{i_0}^*}^{-1}(\wp\cdot L_{i_0}^{-1})>L_{i_0}^{-1}$, when integrate both side of the above inequality from t_0 to $+\infty$, it can be deduced that

$$\begin{aligned}\lim_{t\to+\infty}E(y(t))-E(y(t_0)) &\le -\lambda_{i_0}d_{i_0}(\beta_{x_{i_0}^*}^{-1}(\wp\cdot L_{i_0}^{-1})-L_{i_0}^{-1})\int_{t_0}^{+\infty}|f_{i_0}(x_{i_0}(t))-f_{i_0}(x_{i_0}^*)|^2dt\\ &\le -\lambda_{i_0}d_{i_0}(\beta_{x_{i_0}^*}^{-1}(\wp\cdot L_{i_0}^{-1})-L_{i_0}^{-1})\wp^2\int_{t_0}^{+\infty}dt.\end{aligned} \tag{12}$$

The left side of (12) is a constant, while the right side of it approaches to $-\infty$, which is a contradiction. The result of (10) is thus proved. This completes the proof of the lemma. □

Based on Lemma 1, we further give the globally exponential stability conclusion of network (1). In what follows, when the nonlinear activation mapping $F(x)$ is locally continuously differentiable at x^*, then $F'(x)|_{x=x^*}$, the derivative of $F(x)$ at x^*, is denoted by $F'(x^*)$.

Theorem 1. *Assume that $A=diag\{a_1,a_2,\ldots,a_N\}$, $\mathbf{R}(F)$ is bounded and $y^*\in\Omega$. Let $L=diag\{L_1,L_2,\ldots,L_N\}$ with each L_i being the minimum Lipschitz constant of f_i. If F is diagonal L-DAM at $x^*(:=Ay^*+b)$ and locally continuously differentiable at x^*, each f_i is continuous and strictly monotonically increasing, then y^* is the unique equilibrium state of network (1) and it is globally exponentially stable if there exist two positive definite diagonal matrices Γ and Λ, such that $S(\Gamma,L)=\Gamma DL^{-1}-\Gamma AW+(\Gamma AW)^T/2$ is nonnegative definite and $K(\Lambda,F'(x^*))=\Lambda D(F'(x^*))^{-1}-\Lambda WA+(\Lambda WA)^T/2$ is positive definite.*

Proof. Note first that $F'(x^*)$ is a positive definite diagonal matrix since F is diagonal nonlinear and each f_i is strictly monotonically increasing.

By Lemma 1, we already known that network (1) has a unique equilibrium state y^*, and it is globally asymptotically stable on $\mathcal{R}^N$ when $S(\Gamma,L)\ge 0$, F is diagonal L-DAM at x^* with $\mathbf{R}(F)$ being bounded, and each f_i is continuous and strictly monotonically increasing. Thus, we only need to show y^* is ultimately globally exponentially stable under the assumption that $K(\Lambda,F'(x^*))$ is positive definite and F is locally continuously differentiable at x^*. This is equivalent to justify the unique equilibrium state of the following network:

$$\Lambda\frac{dy(t)}{dt}=-\Lambda Dy(t)+\Lambda WF(Ay(t)+b)+\Lambda q,\quad y_0\in\mathcal{R}^N, \tag{13}$$

i.e., y^*, is globally exponentially stable, where Λ is the chosen diagonal matrix with which matrix $K(\Lambda,F'(x^*))$ is positive definite. Define $T:\mathcal{R}^N\to\mathcal{R}^N$ by

$$T(y)=-\Lambda Dy+\Lambda WF(Ay+b)+\Lambda q.$$

Obviously, T is locally continuously differentiable at y^*. Let $T'(y^*)$ be the derivative of $T(y)$ at y^*. According to the theory developed recently in [21], we only need to justify $\alpha(T'(y^*))<0$, and then, the globally exponential stability of network (1) can be verified. Since A and $F'(x^*)$ all are diagonal matrices, a direct calculation gives that

$$\alpha(T'(y^*))=\alpha(-\Lambda D+\Lambda WF'(x^*)A)=\alpha(-\Lambda D+\Lambda WAF'(x^*)).$$

Next, we will show that $\alpha(-\Lambda D+\Lambda WAF'(x^*))<0$ when $K(\Lambda,F'(x^*))>0$. Let $\tilde{K}(\Lambda,F'(x^*))=-K(\Lambda,F'(x^*))$. By the positive definition assumption of $K(\Lambda,F'(x^*))$, it follows that $\lambda(\tilde{K}(\Lambda,F'(x^*)))$, the eigenvalues of $\tilde{K}(\Lambda,F'(x^*))$, all are negative, namely,

$$\begin{aligned}\lambda(\tilde{K}(\Lambda,F'(x^*)))&=\lambda\left(-\Lambda D(F'(x^*))^{-1}+\frac{\Lambda WA+(\Lambda WA)^T}{2}\right)\\ &=\lambda\left(\frac{(-\Lambda D(F'(x^*))^{-1}+\Lambda WA)+(-\Lambda D(F'(x^*))^{-1}+\Lambda WA)^T}{2}\right).\\ &<0\,.\end{aligned} \tag{14}$$

For any real matrix $M_{N\times N}$, $\mu_2(M)$, the matrix measure induced by the 2-norm $\|x\|_2:=(\sum_{i=1}^N|x_i|^2)^{1/2}$, is defined as

$$\mu_2(M)=\lim_{\rho\to0^+}\frac{\|I+\rho M\|_2-1}{\rho},$$

and it has the property $\mu_2(M)=\max_i\lambda_i((M+M^T)/2)$. When all the eigenvalues of $(M+M^T)/2$ are negative, we know $\mu_2(M)<0$. Thus, by (14), we have $\mu_2(-\Lambda D(F'(x^*))^{-1}+\Lambda WA)<0$, i.e.,

$$\mu_2((-\Lambda D+\Lambda WAF'(x^*))F'(x^*)^{-1})<0.$$

Let $H=(h_{ij})_{N\times N}:=-\Lambda D+\Lambda WAF'(x^*)$ and $\Upsilon=diag\{\gamma_1,\gamma_2,\ldots,\gamma_N\}:=F'(x^*)^{-1}$, then $\mu_2(H\Upsilon)<0$. Define $\Upsilon^{1/2}=diag\{\gamma_1^{1/2},\gamma_2^{1/2},\ldots,\gamma_N^{1/2}\}$. Denote by μ_Υ the matrix measure induced by the vector norm $\|x\|_\Upsilon=\|\Upsilon^{-1/2}x\|_2$, one can get that

$$\begin{aligned}\mu_\Upsilon(H)&=\lim_{\rho\to0^+}\frac{\|I+\rho H\|_\Upsilon-1}{\rho}\\ &=\lim_{\rho\to0^+}\frac{\sup_{\|x\|_\Upsilon=1}\|(I+\rho H)x\|_\Upsilon-1}{\rho}\\ &=\lim_{\rho\to0^+}\frac{\sup_{\|\Upsilon^{-1/2}x\|_2=1}\|\Upsilon^{-1/2}(I+\rho H)x\|_2-1}{\rho}\\ &=\lim_{\rho\to0^+}\frac{\sup_{\|\Upsilon^{-1/2}x\|_2=1}\|\Upsilon^{-1/2}(I+\rho H\Upsilon^{1/2}\Upsilon^{-1/2})x\|_2-1}{\rho}\\ &=\lim_{\rho\to0^+}\frac{\sup_{\|\Upsilon^{-1/2}x\|_2=1}\|(I+\rho\Upsilon^{-1/2}H\Upsilon^{1/2})(\Upsilon^{-1/2}x)\|_2-1}{\rho}\\ &\overset{y=\Upsilon^{-\frac{1}{2}}x}{=}\lim_{\rho\to0^+}\frac{\sup_{\|y\|_2=1}\|(I+\rho\Upsilon^{-1/2}H\Upsilon^{1/2})y\|_2-1}{\rho}\\ &=\lim_{\rho\to0^+}\frac{\|I+\rho\Upsilon^{-1/2}H\Upsilon^{1/2}\|_2-1}{\rho}=\mu_2(\Upsilon^{-1/2}H\Upsilon^{1/2})\\ &=\tfrac{1}{2}\lambda_{max}(\Upsilon^{-1/2}H\Upsilon^{1/2}+\Upsilon^{1/2}H^T\Upsilon^{-1/2})\\ &=\tfrac{1}{2}\lambda_{max}(\Upsilon^{1/2}(\Upsilon^{-1}H\Upsilon+H^T)\Upsilon^{-1/2}).\end{aligned} \tag{15}$$

In accordance with the fact that for any $N\times N$ matrix M and any invertible matrix $P_{N\times N}$, the eigenvalues of PAP^{-1} are equal to that of A, it can be deduced from (15) that

$$\mu_\Upsilon(H)=\tfrac{1}{2}\lambda_{max}(\Upsilon^{-1}H\Upsilon+H^T)=\tfrac{1}{2}\lambda_{max}(\Upsilon^{-1}(H\Upsilon+(H\Upsilon)^T)). \tag{16}$$

We now prove that when $M_{N\times N}$ is a Hermite matrix and $B_{N\times N}$ is a positive definite diagonal matrix, then

$$\lambda_{max}(B^{-1}M)=\max_{x\neq 0}\frac{x^TMx}{x^TBx}.$$

Since for any invertible matrix Q, $P(QP)P^{-1}=PQ$, thus $\lambda(QP)=\lambda(PQ)$. By the definition of Rayleigh quotient, it follows that

$$\begin{aligned}\lambda_{max}(B^{-1}M)&=\lambda_{max}(B^{-1/2}B^{-1/2}M)=\lambda_{max}(B^{-1/2}(B^{-1/2}M))\\ &=\lambda_{max}(B^{-1/2}MB^{-1/2})=\max_{y\neq 0}\frac{y^T(B^{-1/2}MB^{-1/2})y}{y^Ty}\\ &\overset{x\triangleq B^{-1/2}y}{=}\max_{x\neq 0}\frac{x^TMx}{x^TBx}.\end{aligned} \tag{17}$$

By (16) and (17) and the fact that $\max_{x\neq 0}x^T(H\Upsilon+(H\Upsilon)^T)x<0$ (which is deduced by $\mu_2(H\Upsilon)<0$), we have

$$\begin{aligned}\mu_{\Upsilon}(H)&=\frac{1}{2}\max_{x\neq 0}\frac{x^T(H\Upsilon+(H\Upsilon)^T)x}{x^T\Upsilon x}\\ &\leq\frac{1}{2}\max_{x\neq 0}x^T(H\Upsilon+(H\Upsilon)^T)x\cdot\min_{x\neq 0}x^T\Upsilon^{-1}x<0.\end{aligned} \tag{18}$$

On noting that for any real matrix B,

$$\alpha(B)=\inf_{\|\cdot\|\in\Psi}\mu(B),$$

where Ψ denotes the set of all equivalent norm of $\|\cdot\|$ [19,20], then

$$\alpha(H)=\alpha(-\Lambda D+\Lambda WAF'(x^*))<0.$$

This completes the proof of Theorem 1. □

The following corollary replaces verifying the positive definiteness of matrix $K(\Lambda,F'(x^*))$ with certifying the relationship of two matrices defined by the network in the sense of componentwise comparison, and it is more available for applications.

Corollary 1. *Consider network (1) with $A=I$. Assume that $\mathbf{R}(F)$ is bounded and $y^*\in\Omega$. Let $L=diag\{L_1,L_2,\ldots,L_N\}$ with each L_i being the minimum Lipschitz constant of f_i. If F is diagonal L-DAM at $x^*(:=Ay^*+b)$ and locally continuously differentiable at x^*, every f_i is continuous and strictly monotonically increasing, then y^* is the unique equilibrium state of network (1), and it is globally exponentially stable if $L>F'(x^*)$ in the sense of componentwise comparison and there exists a positive definite diagonal matrix Γ, such that $S(\Gamma,L)=\Gamma DL^{-1}-(\Gamma AW+(\Gamma AW)^T)/2$ is nonnegative definite.*

Proof. It is obvious that $S(\Gamma,L)=K(\Gamma,L)$ when $A=I$, and if $L>F'(x^*)$ in the sense of componentwise comparison, then the nonnegative definiteness of $S(\Gamma,L)$ implies the positive definiteness of $K(\Gamma,F'(x^*))$. Thus, Corollary 1 follows from Theorem 1 directly. □

Remark 2. Lemma 1, Theorem 1 and Corollary 1 give some definite answers to the problem that what kinds of dynamics behavior will be for network (1) under the critical conditions. In addition, as the special cases of network (1), the two basic RNN models: local field neural network model and static neural network model, are extensively applied in learning, pattern recognition, associative memory, solving optimization problems, etc. The applicability and efficiency of such applications crucially hinge upon their dynamics, and therefore the analysis of dynamical behaviors of such two networks becomes imperative (actually is a first step) for any practical design and application of the networks. Due to their intrinsic difficulty, these two RNN models have been fallen short of a generic, in-depth theoretical analysis under the critical conditions. While, by Lemma 1, Theorem 1 and Corollary 1, we can easily provide the corresponding dynamic results for them and the results are original. Either Lemma 1, Theorem 1 and Corollary 1, or the results achieved for local field neural network model and static neural network model can not only sharpen and extend most of the existing non-critical conclusions, but also generalize, the latest critical results for RNNs, see, e.g., [1–8,10–14,17,18,22–24].

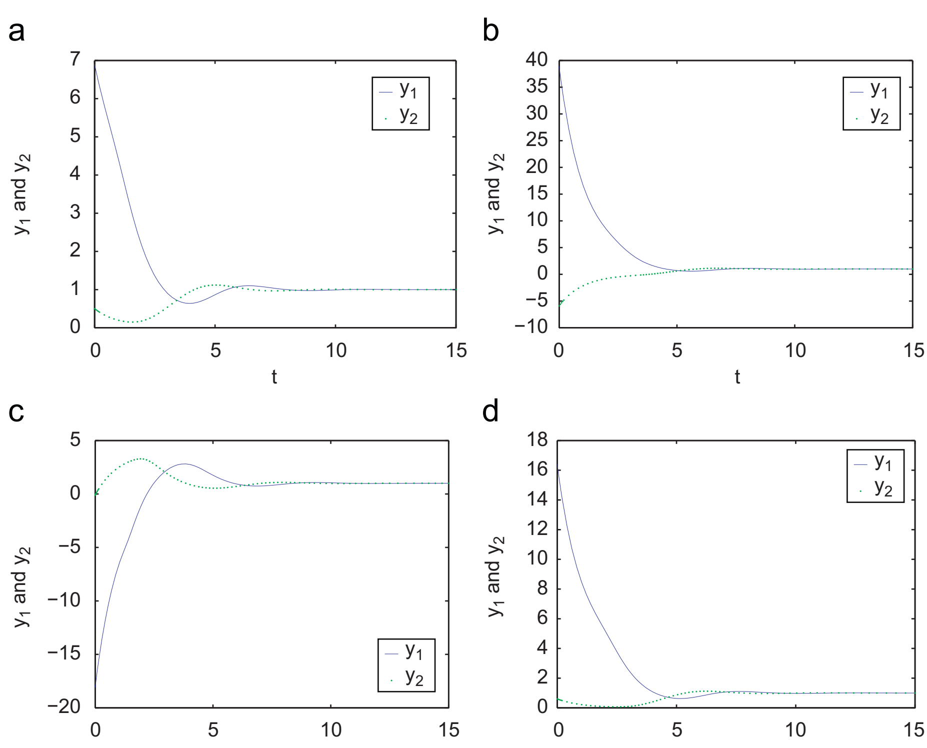


Fig. 1. Transient behaviors of RNN in network (19) with random initial points.

4. Illustrative example

In what follows, we provide an illustrative example to demonstrate the validity of the critical dynamics results formulated in Section 3. Consider the following Hopfield-type RNN:

$$\begin{cases} \frac{dy_1(t)}{dt} = -y_1(t) + f_1(y_1(t)) + 3f_2(y_2(t)) - (4 + \ln 3), \\ \frac{dy_2(t)}{dt} = -y_2(t) - f_1(y_1(t)) + 0.2f_2(y_2(t)) + 0.8 - \ln 3, \end{cases} \tag{19}$$

where $f_i(s) = (4/(1+e^{-s}))$ $(i=1,2)$.

In this example, $D = A = I$, $W = \begin{pmatrix} 1 & 3 \\ -1 & 0.2 \end{pmatrix}$, $b = \vec{0}$, $q = \begin{pmatrix} -4 & -\ln 3 \\ 0.8 & -\ln 3 \end{pmatrix}$, each $L_i = 1$ and the equilibrium state set is $\Omega_e = \{(-\ln 3, -\ln 3)^T\}$.

For any positive definite diagonal matrix Γ, it is easy to verify that $\Gamma L^{-1} - (\Gamma W + W^T\Gamma)/2$ is not positive definite, so almost all of the exponential stability conclusions of such kind of RNN are not suitable here, see, e.g., [9,18,23]. Meanwhile, by the results established in [14], one can only get the globally attractive conclusions of network (19), and, it is easy to verify that the critically exponentially stable conditions used in [3] do not satisfied for this example. We will show Theorem 1 established in Section 3 can be applied here to achieve the globally exponential stability for network (19).

By choosing $\Gamma = diag\{1,3\}$, we have

$$\Gamma L^{-1} - \frac{\Gamma W + W^T\Gamma}{2} \geq 0.$$

On the other hand, for $y^* = (-\ln 3, -\ln 3)^T \in \Omega_e$, we know $x^* = Ay^* + b = (-\ln 3, -\ln 3)^T$ and further, $F'(x^*) = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$. It follows that

$$\Gamma F'(x^*)^{-1} - \frac{\Gamma W + W^T\Gamma}{2} > 0.$$

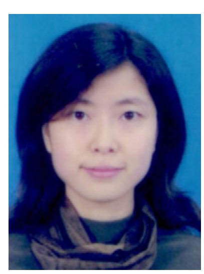
Meanwhile, by the definition of f_i, it is clear that each f_i is continuous and strictly monotonically increasing, F is locally continuously differentiable at y^*. Since f_i is sigmoidal, then by Remark 1, it is L_i-DAM. That is, the conditions in Theorem 1 all hold. According to Corollary 1, network (19) is globally exponentially stable. The following Fig. 1 depicts the time responses of state variables of the network with random initial points, which can confirm that the proposed conditions in Theorem 1 ensure the globally exponential stability of RNNs.

5. Conclusion

In this work, we have developed the critical stability theory of RNNs with general forms. Based on exploring some intrinsic properties of the networks, and by using the energy function method and matrix measure theory, it is shown that a RNN with decreasing anti-monotone activation mapping has a unique equilibrium state and which is globally asymptotically stable under the critical conditions. Further, the RNN is globally exponentially stable under the conditions that a discriminant matrix determined by the network is positive definite at the unique equilibrium state. The obtained critical dynamics results extend directly the existing non-critical conclusions without adding any further requirements, and at the same time, they generalize almost all of the critical conclusions. The achieved conclusions can not only provide a wider application range for RNNs, but also can be applied directly to many concrete RNN models.

References

[1] S. Arik, V. Tavsanoglu, A sufficient condition for absolute stability of a large class of dynamical neural networks, IEEE Trans. Circuits Syst. 47 (2000) 758–760.
[2] J. Cao, J. Wang, Global asymptotic and robust stability of recurrent neural networks with time delays, IEEE Trans. Circuits Syst. 52 (2005) 417–426.
[3] T.P. Chen, S.I. Amari, Stability of asymmetric Hopfield networks, IEEE Trans. Neural Networks 12 (2001) 159–163.
[4] T.P. Chen, S.I. Amari, New theorems on global convengence of some dynamical systems, Neural Networks 14 (2001) 251–255.
[5] T.P. Chen, Global convergence of delayed dynamical systems, IEEE Trans. Neural Networks 12 (2001) 1532–1536.
[6] T.P. Chen, L.B. Rong, Robust global exponential stability of Cohen–Grossberg neural networks with time delays, IEEE Trans. Neural Networks 15 (2004) 203–206.
[7] M. Forti, A. Tesi, New conditions for global stability of neural networks with applications to linear and quadratic programming problems, IEEE Trans. Circuits Syst. 42 (1995) 354–366.
[8] Z.H. Guan, G. Chen, Y. Qin, On equilibria, stability and instability of Hopfield neural networks, IEEE Trans. Neural Networks 11 (2000) 534–540.
[9] T.W. Huang, J.D. Cao, C.D. Li, Necessary and sufficient condition for the absolute exponential stability of a class of neural networks with finite delay, Phys. Lett. A 352 (2006) 94–98.
[10] X.B. Liang, J. Wang, Absolute exponential stability of neural networks with a general class of activation functions, IEEE Trans. Circuits Syst. 47 (2000) 1258–1263.
[11] X.B. Liang, J. Si, Global exponential stability of neural networks with globally Lipschitz continuous activation and its application to linear variational inequality problem, IEEE Trans. Neural Networks 12 (2001) 349–359.
[12] X.W. Liu, T.P. Chen, A new result on the global convergence of Hopfield neural networks, IEEE Trans. Circuits Syst. 49 (2002) 1514–1516.
[13] W.L. Lu, T.P. Chen, R^n_+-global stability of a Cohen–Grossberg neural network system with nonnegative equilibria, Neural Networks 20 (2007) 714–722.
[14] J.G. Peng, Z.B. Xu, H. Qiao, B. Zhang, A critical analysis on global convergence of Hopfield-type neural networks, IEEE Trans. Circuits Syst. 52 (2005) 804–814.
[15] C. Qiao, Z.B. Xu, A critical global convergence analysis of recurrent neural networks with general projection mappings, Neurocomputing 72 (2009) 1878–1886.
[16] C. Qiao, Z.B. Xu, On the P-critical dynamics analysis of projection recurrent neural networks, Neurocomputing 73 (2010) 2783–2788.
[17] H. Qiao, J.G. Peng, Z.B. Xu, Nonlinear measures: a new approach to exponential stability analysis for Hopfield-type neural networks, IEEE Trans. Neural Networks 12 (2001) 360–370.
[18] H. Qiao, J.G. Peng, Z.B. Xu, B. Zhang, A reference model approach to stability analysis of neural networks, IEEE Trans. Syst. Man Cybern. 33 (2003) 925–936.
[19] T. Storm, On logarithmic norms, SIAM J. Numer. Anal. 2 (1975) 741–753.
[20] G. Soderlind, Bounds on nonlinear operators in finite dimensional Banach space, Numer. Math. 50 (1986) 27–44.
[21] L.S. Wang, R. Zhang, Z.B. Xu, J.G. Peng, Some characterizations of global exponential stability of a generic class of continuous-time recurrent neural networks, IEEE Trans. Syst. Man Cybern. 39 (2009) 763–772.
[22] Y.S. Xia, J. Wang, On the stability of globally projected dynamical systems, J. Optim. Theory Appl. 106 (2000) 129–150.
[23] Z.B. Xu, H. Qiao, J.G. Peng, B. Zhang, A comparative study of two modeling approaches in neural networks, Neural Networks 17 (2004) 73–85.
[24] Y.Q. Yang, J.D. Cao, Solving quadratic programming problems by delayed projection neural network, IEEE Trans. Neural Networks 17 (2006) 1630–1634.



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