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Dynamic boundary systems with boundary feedback and population system with unbounded birth process

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This paper is concerned with a class of dynamic boundary systems with boundary feedback. The well-posedness of the considered systems is proved under some regularity conditions. Moreover, some spectral properties are derived. As an application, the well-posedness and the asymptotic behavior of population dynamical systems with unbounded birth process '$B(t) = \int_0^\infty \beta(a)u(t-\tau,a)da, t \geq 0$' are solved. Such population dynamical systems were pointed out in [S. Piazzera, Math. Methods Appl. Sci., 27 (2004), 427-439] to be a current research topic in semigroup theory and still an open problem.

Keywords: boundary feedback; regular linear systems; age dependent population equation

1. Introduction

Dynamic boundary conditions mean that there exists dynamic on the boundary conditions, which are distinguished from static boundary conditions. Inspired by the applicability to practical problems, evolution equations with dynamic boundary conditions have been investigated intensively, via the variational method, the semigroup theory, the cosine operator function theory, or the so-called *B*-evolution theory by many authors [1–8], ever since the work of J. L. Lions came out in 1961 [9]. Among those references, we have to mention that Casarino *et al*. [3] used semigroup theory to solve linear systems with dynamical boundary conditions and obtained several sufficient and/or necessary conditions guaranteeing the system well-posedness. Recently, Mugnolo adopted such results to deal with abstract wave equations with acoustic boundary conditions [6] and damped wave equations with dynamic boundary conditions [7], respectively.

Observe that, for the boundary dynamic, maybe there exists boundary condition, too. This means that we have to consider double boundary conditions: the first layer boundary condition is dynamic boundary condition, and the second boundary condition is the usual static boundary condition for the boundary dynamic. To the best of the authors' knowledge, few references have considered such problem. Because it is shown by Salamon [10] that time delay systems can be regarded as one layer boundary system with the boundary space being the state space, initial-boundary value problem of delay equations can be regarded as double boundary problem. In this sense, population dynamical systems with delay birth process are typical double boundary problem. In [11], Piazzera considered the situation that the birth process, namely, the boundary feedback operator, is a bounded linear functional with respect to the history function. Concretely, he proved the well-posedness of such system by Desch–Schappacher perturbation theorem [12] and discussed the asymptotic behavior through spectral theory and positive semigroup theory. Simultaneously, Piazzera pointed out that population dynamical system with unbounded birth process "$B(t) = \int_0^\infty \beta(a)u(t-\tau,a)da, t \geq 0$" is a current research topic in semigroup theory and still an *open problem*, and only particular results, for example, for neutral differential equations [13] or analytic semigroups [14], are known while a general perturbation result is still missing. The main difficulty of population dynamical systems with unbounded birth process lies in that the unboundedness makes Desch–Schappacher perturbation theorem invalid. On the other hand, we observe that theory of regular linear system has been effectively used to deal with delay and boundary control system (see, for example, [15, 16]).

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Motivated by such results, in this paper, the authors shall study general results of dynamic boundary systems with boundary feedback through the tool regular linear system theory. Then, we apply the general results to population dynamical systems with unbounded birth process.

The arrangement of this paper is as follows. In Section 2, we will introduce the main tool, regular linear system theory. Section 3 is to find some regularity conditions that guarantee system (3.1) being well-posed; meanwhile, the spectrum relations are obtained. In Section 4, using the results obtained in Section 3, the open problem, population dynamical system with unbounded birth process, is considered. The problems of well-posedness and asymptotic behavior are solved.

2. Preliminaries on regular linear systems

As a preparation for the residual sections, in this section, we retrospect some concepts and results related to regular linear system developed by Salamon [10] and Weiss [17]. Throughout this section, we assume that X, U, and Y are Banach spaces, $1 < p < \infty$.

Definition 2.1
Let $\Omega = L^p(R^+, U)$ and $\Gamma = L^p(R^+, Y)$. A well-posed linear system on Ω, X, and Γ is a quadruple $\Sigma = (T, \Phi, \Psi, F)$, where

(i) $T = (T(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X.

(ii) $\Phi = (\Phi(t))_{t\geq 0}$ is a family of bounded linear operators from Ω to X such that

$$\Phi(t+\tau)u = T(t)\Phi(\tau)u + \Phi(t)u(\tau + \cdot),$$

for any $u \in \Omega$ and any $\tau, t \geq 0$.

(iii) $\Psi = (\Psi(t))_{t\geq 0}$ is a family of bounded linear operators from X to Γ such that

$$(\Psi(t+\tau)x)(\cdot) = (\Psi(t)T(\tau)x)(\cdot - \tau) \text{ on } [\tau, t+\tau], \tag{2.1}$$

for any $x \in X$ and any $\tau, t \geq 0$, and $\Psi(0) = 0$.

(iv) $F = (F(t))_{t\geq 0}$ is a family of bounded linear operators from Ω to Γ such that

$$(F(t+\tau)u)(\cdot) = (\Psi(t)\Phi(\tau)u + F(t)u(\tau + \cdot))(\cdot - \tau) \text{ on } [\tau, t+\tau], \tag{2.2}$$

for any $u \in \Omega$ and any $\tau, t \geq 0$, and $F(0) = 0$.

Here U, X, and Y are the input space, state space, and output space of Σ, respectively. The operators $\Phi(\tau)$ are called input map. The operators $\Psi(\tau)$ are called output map. The operators $F(\tau)$ are called input/output map.

By [18], there exists a unique operator $B \in L(U, X_{-1})$, called *admissible control operator* for A, such that for any $t \geq 0$ and $u \in L^p_{loc}(R^+, U)$,

$$\Phi(t)u = \int_0^t T_{-1}(t-s)Bu(s)ds \in X.$$

Here, X_{-1} is the extrapolation space corresponding to X, which is the completion of X under the norm $\|R(\lambda_0, A)\cdot\|$ with $R(\lambda_0, A)$ the resolvent of A at λ_0; $\{T_{-1}(t)\}_{t\geq 0}$ is the extrapolation semigroup of $\{T(t)\}_{t\geq 0}$ with generator A_{-1}, which is the continuous extension of $\{T(t)\}_{t\geq 0}$ on X_{-1}. In this case, we say that (A, B) generates an *abstract linear control system* on (X, U) and denote $\Phi = \Phi_{A,B}$ for brief.

It follows from [19, 20] that there exist unique continuous linear operators $\Psi(\infty) : X \to L^2_{loc}(R^+, Y)$ and $F(\infty) : L^p_{loc}(R^+, U) \to L^p_{loc}(R^+, Y)$ such that, for any $\tau \geq 0$, the operators $\Psi(\tau)$ and $F(\tau)$ are derived by truncation:

$$\Psi(\tau) = P_\tau \Psi(\infty),\ F(\tau)(t) = F(\infty)(t), \text{ for any } t \leq \tau.$$

We call $\Psi(\infty)$ the extended output map of Σ and $F(\infty)$ the extended input/output map of Σ. It is not difficult to obtain the following two equations related to the extended output map and the extended input/output map:

$$\Psi(\infty)x = (\Psi(\infty)T(\tau)x)(\cdot - \tau) \text{ on } [\tau, \infty), \forall x \in X, \forall \tau \geq 0, \tag{2.3}$$

and

$$F(\infty)u = (\Psi(\infty)\Phi(\tau)u + F(\infty)u(\tau + \cdot))(\cdot - \tau) \text{ on } [\tau, \infty), \tag{2.4}$$

for any $u \in L^p_{loc}(R^+, U)$ and $\tau \geq 0$. The space $D(A)$ with graph norm is denoted by $(D(A), \|\cdot\|_{D(A)}$. By the representation theorem in [20], there exists a unique operator $C \in L(D(A), Y)$, called *admissible observation operator* for A, such that for any $t \geq 0$ and $x \in D(A)$, $CT(t)x = (\Psi_\infty x)(t)$. In this case, we say that (A, C) generates an *abstract linear observation system* on (X, Y) and denote $\Psi = \Psi_{A,C}$ for brief.

The well-posed linear system Σ is called to be *regular*, if the limit

$$\lim_{t\to 0} \frac{1}{t}\int_0^t (F_\infty u_0)(s)ds = 0 \tag{2.5}$$

exists in Y for the constant input $u_0(t) = z, z \in U, t \geq 0$. In this case, we also say that the quadruple $\Sigma = (T, \Phi, \Psi, F)$ is a regular linear system on (X, U, Y) generated by (A, B, C), and we denote $\Sigma_{A,B,C} = \Sigma$ and $F_{A,B,C} = F$.

Definition 2.2 ([21, 22])
Let Σ be a regular linear system with family of input/output maps $\{F(t)\}_{t\geq 0}$. An operator $M \in L(Y, U)$ is called an admissible feedback for Σ if $I_Y - F(\cdot)M$ has uniformly bounded inverse.

3. Dynamic boundary systems with boundary feedback

In this section, we consider the following systems with dynamic boundary conditions and boundary feedback of the boundary dynamic

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Gx(t) = w(t), & t \geq 0, \\ \dot{w}(t) = \mathfrak{A}_m w(t), & t \geq 0, \\ Pw(t) = Kx(t), & t \geq 0, \end{cases} \tag{3.1}$$

where x and w take values in Banach spaces X and ∂X, respectively; $A_m : D(A_m) \subset X \to X$ is called *maximal operator*, $D(A_m)$ endowed with graph norm is completed; the boundary operators $G \in L(D(A_m), \partial X)$ and $P \in L(D(\mathfrak{A}_m), \partial\partial X)$ with $\partial\partial X$ being Banach space and $D(\mathfrak{A}_m)$ being endowed with graph norm, $D(\mathfrak{A}_m)$ is completed; the boundary observation operator $K \in L(D(A_m), \partial\partial X)$.

Obviously, we can convert system (3.1) to the following boundary system

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} A_m & 0 \\ 0 & \mathfrak{A}_m \end{pmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}, & t \geq 0 \\ \begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}, & t \geq 0. \end{cases}$$

We denote $\mathcal{A}_{L,K} = \begin{pmatrix} A_m & 0 \\ 0 & \mathfrak{A}_m \end{pmatrix}$ with domain $D(\mathcal{A}_{L,K}) = \left\{\begin{pmatrix} z \\ f \end{pmatrix} \in D(A_m) \times D(\mathfrak{A}_m) : Gz = f, Pf = Kz\right\}$.

Assumption 3.1
As in [23], we make the following two assumptions:

(H1) $A := A_m$ with domain $D(A) := Ker\{G\}$ generates a C_0-semigroup on X, $Im\{G\} = \partial X$;
(H2) $\mathfrak{A} := \mathfrak{A}_m$ with domain $D(\mathfrak{A}) := Ker\{P\}$ generates a C_0-semigroup on ∂X, $Im\{P\} = \partial\partial X$.

By [23], under the Assumption 3.1, the direct sums

$$D(A_m) = D(A) \oplus Ker\{\lambda - A_m\}, \lambda \in \rho(A)$$

and

$$D(\mathfrak{A}_m) = D(A) \oplus Ker\{\lambda - \mathfrak{A}_m\}, \lambda \in \rho(\mathfrak{A})$$

hold. Moreover, the Dirichlet operators,

$$D_{\lambda,A} := (G|_{Ker\{\lambda - A_m\}})^{-1} : \partial X \to Ker\{\lambda - A_m\}$$

and

$$D_{\lambda,\mathfrak{A}} := (P|_{Ker\{\lambda - \mathfrak{A}_m\}})^{-1} : \partial\partial X \to Ker\{\lambda - \mathfrak{A}_m\}$$

exist and are bounded.

According to [10, 24], the system operator, control operator, and observation operator of boundary system described by (A_m, G, K) are A,

$$B := (\lambda - A_{-1})D_{\lambda,A},$$

and $C = K|_{D(A)}$, respectively. Similarly, the system operator and control operator of boundary system described by $(\mathfrak{A}_m, P)$ are $\mathfrak{A}$ and

$$\mathfrak{B} := (\lambda - \mathfrak{A}_{-1})D_{\lambda,\mathfrak{A}},$$

respectively.

Definition 3.2
A pair of continuous functions $\begin{pmatrix} x \\ w \end{pmatrix} : [0, \infty) \to X \times \partial X$ is called classical solution of (3.1) with initial value $\begin{pmatrix} z \\ f \end{pmatrix} \in D(\mathcal{A}_{L,K})$ if it is continuously differentiable on $[0, \infty)$, $x(t) \in D(A_m)$, $w(t) \in D(\mathfrak{A}_m)$, $Gx(t) = w(t)$, and $Pw(t) = Kx(t)$.

Theorem 3.3
Assume that operator $\mathcal{A}_{L,K}$ generates a C_0-semigroup on the space $X \times \partial X$. Then, for each initial value $\begin{pmatrix} z \\ f \end{pmatrix} \in D(\mathcal{A}_{L,K})$, there is a unique classical solution of (3.1). Moreover, the classical solution depends continuously on the initial data.

Proof

Assume that $\begin{pmatrix} z \\ f \end{pmatrix} \in D(\mathcal{A}_{L,K})$. Let $\{\mathcal{T}_{L,K}(t)\}_{t\geq 0}$ be the semigroup generated by $\mathcal{A}_{L,K}$. Then, we can obtain that $\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} := \mathcal{T}_{L,K}(t)\begin{pmatrix} z \\ f \end{pmatrix}$ is the classical solution of (3.1). In fact, by the properties of C_0-semigroup,

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \in D(\mathcal{A}_{L,K})$$

and

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \mathcal{A}_{L,K}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} A_m x(t) \\ \mathfrak{A}_m x(t) \end{pmatrix}.$$

The continuous dependence on the initial data is obtained directly from the properties of C_0-semigroup. The proof is completed. □

Theorem 3.3 means that if $\mathcal{A}_{L,K}$ generates a C_0-semigroup, then system (3.1) is well-posed, namely, classical solution exists and depends continuously on the initial data.

Theorem 3.4

Assume that (A_m, G, K) generates a regular linear system and $(\mathfrak{A}_m, P)$ generates an abstract linear control system. Then, system (3.1) is well-posed.

Proof

By assumption, it is obtained that (A, B, C) generates a regular linear system and $(\mathfrak{A}, \mathfrak{B})$ generates an abstract linear control system. Because both G and P are surjections, operator matrix $\begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}$ is a surjection. The restriction of $\begin{pmatrix} A_m & 0 \\ 0 & \mathfrak{A}_m \end{pmatrix}$ to $Ker\left\{\begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}\right\}$ equals to $\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}$. We compute

$$\begin{aligned}\begin{pmatrix} A_m & 0 \\ 0 & \mathfrak{A}_m \end{pmatrix} - \begin{pmatrix} A_{-1} & 0 \\ 0 & \mathfrak{A}_{-1} \end{pmatrix} &= \begin{pmatrix} A_m - A_{-1} & 0 \\ 0 & \mathfrak{A}_m - \mathfrak{A}_{-1} \end{pmatrix} \\ &= \begin{pmatrix} BG & 0 \\ 0 & \mathfrak{B}P \end{pmatrix} \\ &= \begin{pmatrix} B & 0 \\ 0 & \mathfrak{B} \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}.\end{aligned}$$

Because $\begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}$ is surjection, we obtain by [24, Theorem 2.3] that the corresponding control operator of system $\left(\begin{pmatrix} A_m & 0 \\ 0 & \mathfrak{A}_m \end{pmatrix}, \begin{pmatrix} G & 0 \\ 0 & P \end{pmatrix}\right)$ is $\begin{pmatrix} B & 0 \\ 0 & \mathfrak{B} \end{pmatrix}$. The combination of (A_m, G, K) generating regular linear system and $(\mathfrak{A}_m, P)$ generating abstract linear control system implies that $\left(\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \mathfrak{B} \end{pmatrix}\right)$ generates abstract linear control system. Moreover,

$$\Phi_{\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \mathfrak{B} \end{pmatrix}} = \begin{pmatrix} \Phi_{A,B} & 0 \\ 0 & \Phi_{\mathfrak{A},\mathfrak{B}} \end{pmatrix}.$$

Similarly, it is not hard to obtain that $\left(\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}\right)$ generates an abstract linear observation system with

$$\Psi_{\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}} = \begin{pmatrix} 0 & \Psi_{\mathfrak{A},I} \\ \Psi_{A,K} & 0 \end{pmatrix}.$$

We denote $F = \begin{pmatrix} 0 & F_{\mathfrak{A},\mathfrak{B},I} \\ F_{A,B,C} & 0 \end{pmatrix}$. Then, by definition, it is not hard to test that $\left(T, \Phi_{\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}}, \Psi_{\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}}\right)$ is a regular linear system generated by $\left(\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A} \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \mathfrak{B} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}\right)$. We compute

$$I - F = \begin{pmatrix} I & -F_{\mathfrak{A},\mathfrak{B},I} \\ -F_{A,B,C} & I. \end{pmatrix}$$

By the proof of [25, Lemma 3.2], it follows that $\|F_{\mathfrak{A},\mathfrak{B},I}(t)\| \to 0$ as $t \to 0$. Observe that $\|F_{A,B,C}(t)\|$ is bounded. Thus,

$$\lim_{t\to 0} \|F_{A,B,C}(t)F_{\mathfrak{A},\mathfrak{B},I}(t)\| = 0.$$

Hence, $I - F$ has well-posed inverse. Thus, I is an admissible feedback operator. By [16, Theorem 9], it follows that $\mathcal{A}_{L,K}$ generates a C_0-semigroup. Hence, system (3.1) is well-posed. This completes the proof. □

Theorem 3.5

Assume that (A_m, G, K) generates a regular linear system and $(\mathfrak{A}_m, P)$ generates an abstract linear control system. Let $\lambda \in \rho(A) \cap \rho(\mathfrak{A})$. Then,

$$\begin{aligned}\lambda \in \sigma_P(\mathcal{A}_{L,K}) &\Longleftrightarrow 1 \in \sigma_P(R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B))\\ &\Longleftrightarrow 1 \in \sigma_P(KR(\lambda, A_{-1})BR(\lambda, \mathfrak{A}_{-1})\mathfrak{B}).\end{aligned}$$

Proof

By [16, Proposition 1], it follows that for $\lambda \in \sigma_P\left(\begin{pmatrix} A & 0 \\ 0 & \mathfrak{A}\end{pmatrix}\right)$, $\lambda \in \sigma_P(\mathcal{A}_{L,K})$ if and only if

$$1 \in \sigma_P\left(\begin{pmatrix} 0 & I \\ K & 0\end{pmatrix}\begin{pmatrix} \lambda - A_{-1} & 0 \\ 0 & \lambda - \mathfrak{A}_{-1}\end{pmatrix}^{-1}\begin{pmatrix} B & 0 \\ 0 & \mathfrak{B}\end{pmatrix}\right).$$

The equivalence relations of this theorem are obtained by [26, Theorem II.2.8]. The proof is therefore completed. □

Theorem 3.6

Assume that (A_m, G, K) generates a regular linear system and $(\mathfrak{A}_m, P)$ generates an abstract linear control system. Let $\lambda \in \rho(A) \cap \rho(\mathfrak{A})$. Then,

$$\begin{aligned}\lambda \in \rho(\mathcal{A}_{L,K}) &\Longleftarrow 1 \in \rho(R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)\\ &\Longleftrightarrow 1 \in \rho(KR(\lambda, A_{-1})BR(\lambda, \mathfrak{A}_{-1})\mathfrak{B}).\end{aligned}$$

Moreover, in the case $1 \in \rho(R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B))$, denote

$$M_{inv}(\lambda) = (I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)^{-1}R(\lambda, \mathfrak{A}_{-1}),$$

then,

$$R(\lambda, \mathcal{A}_{L,K}) = \begin{pmatrix} R(\lambda, A) + R(\lambda, A_{-1})BM_{inv}(\lambda)\mathfrak{B}KR(\lambda, A) & R(\lambda, A_{-1})BM_{inv}(\lambda) \\ M_{inv}(\lambda)\mathfrak{B}KR(\lambda, A) & M_{inv}(\lambda)\end{pmatrix}. \tag{3.2}$$

Proof

By [16, Theorem 9], it follows that

$$\mathcal{A}_{L,K} = \begin{pmatrix} A_{-1} & 0 \\ 0 & \mathfrak{A}_{-1}\end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & \mathfrak{B}\end{pmatrix}\begin{pmatrix} 0 & I \\ K & 0\end{pmatrix} = \begin{pmatrix} A_{-1} & B \\ \mathfrak{B}K & \mathfrak{A}_{-1}\end{pmatrix}$$

with domain

$$D(\mathcal{A}_{L,K}) = \left\{\begin{pmatrix} x \\ y\end{pmatrix} \in D(A_m) \times \partial X : A_{-1}x + By \in X, \mathfrak{B}Kx + \mathfrak{A}_{-1}y \in \partial X\right\}.$$

In order to obtain the condition such that $\lambda \in \rho(\mathcal{A}_{L,K})$. We consider the following operator equation

$$(\lambda - \mathcal{A}_{L,K})V(\lambda) = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$$

with $V(\lambda) \in L(X \times \partial X)$. Remark that we can partition the operator $V(\lambda)$ into a 2×2 matrix of operators corresponding to the two components of its domain space and its range space:

$$V(\lambda) = \begin{pmatrix} V_1(\lambda) & V_2(\lambda) \\ V_3(\lambda) & V_4(\lambda)\end{pmatrix}.$$

Then, if $1 \in \rho(R(\lambda, \mathfrak{A}_{-1}\mathfrak{B}KR(\lambda, A_{-1})B))$, we can obtain that

$$\begin{aligned}V_2(\lambda) &= R(\lambda, A_{-1})B(I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)^{-1}R(\lambda, \mathfrak{A})\\ V_3(\lambda) &= (I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)^{-1}R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A)\\ V_1(\lambda) &= R(\lambda, A) + R(\lambda, A_{-1})BV_3(\lambda)\\ V_4(\lambda) &= (I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)^{-1}R(\lambda, \mathfrak{A}),\end{aligned}$$

which implies that $\lambda \in \rho(\mathcal{A}_{L,K})$ and $R(\lambda, \mathcal{A}_{L,K}) = V(\lambda)$. The relation

$$1 \in \rho(R(\lambda, \mathfrak{A}_{-1}\mathfrak{B}KR(\lambda, A_{-1})B)) \Longleftrightarrow 1 \in \rho(KR(\lambda, A_{-1})BR(\lambda, \mathfrak{A}_{-1}\mathfrak{B}))$$

is obvious. This completes the proof. □

4. Population dynamical systems with unbounded birth process

In this section, we consider the *open problem* stated in [11, Remark 6], population dynamical system with unbounded birth process described as follows.

$$\begin{cases} \frac{\partial w(t,a)}{\partial t} = -\frac{\partial w(t,a)}{\partial a} - \mu(a)w(t,a) \\ w(t,0) = \int_0^\infty \beta(a)w(t-r,a)da, t \geq 0 \\ w(s,a) = \phi(s,a), s \in [-r,0] \text{ and } a \geq 0. \end{cases} \tag{4.1}$$

Here, $w(t;a)$ represents the density of the population of age a at time t, $\mu \in L^\infty_{loc}(R^+)$ is the death rate, and $\beta \in L^\infty(R^+)$ is the birth rate. Denote

$$\lim_{a\to\infty} \mu(a) =: \mu_\infty > 0.$$

We denote $X = L^p([-r,0], L^1(R^+))$, $\partial X = L^1(R^+)$, and $\partial\partial X = \mathbb{C}$. Then, system (4.1) can be transformed to the form of (3.1) with the operators:

- $A_m := \frac{d}{d\theta}$ with domain $D(A_m) := W^{1,p}([-r,0], L^1(R^+))$;
- $GF = F(0), \forall F \in L^p([-r,0], L^1(R^+))$;
- $\mathfrak{A}_m := -\frac{d}{d\sigma} - \mu(\cdot)$ with domain $D(\mathfrak{A}_m) = W^{1,1}(R^+)$;
- $Pf = f(0), \forall f \in L^1(R^+)$;
- $Kg = \int_0^\infty \beta(a)g(-r,a)da, \forall g \in L^p([-r,0], L^1(R^+))$.

Thus, $G \in L(W^{1,p}([-r,0], L^1(R^+)), L^1(R^+))$ and $P \in L(W^{1,1}(R^+), \mathbb{C})$. The equation

$$w(t;0) =: B(t) = Kw_t, t \geq 0,$$

is called the birth process, where $w_t := w(t+\cdot)$ is the history function.

It is well known that A generates a left-shift semigroup on X and $\mathfrak{A}$ generates a right-shift semigroup on ∂X. By [27], (A,B) is an abstract linear control system satisfying

$$(\lambda - A_{-1})^{-1}B = e^{\lambda}.$$

It has been shown in [28, Proposition 2.1] that the spectrum $\sigma(\mathfrak{A})$ is

$$\sigma(\mathfrak{A}) = \{\lambda \in \mathbb{C} : Re\lambda \leq -\mu_\infty\}.$$

Moreover, by [28, (24)], we have

$$Ker(\lambda - \mathfrak{A}_m) = \begin{cases} <\phi_\lambda> := <e^{-\int_0^\cdot(\lambda+\mu(s))ds}>, & Re > -\mu_\infty, \\ 0, & otherwise. \end{cases}$$

Lemma 4.1
The boundary control system

$$\begin{cases} \frac{\partial}{\partial t}w(t,\sigma) = -\frac{\partial}{\partial\sigma}w(t,\sigma) - \mu(\sigma)w(t,\sigma), & t \in R^+, \sigma \in R^+, \\ w(t,0) = u(t), & t \in R^+. \end{cases} \tag{4.2}$$

with $w(t,\sigma) \in \mathbb{C}, \forall t \in R^+, \sigma \in R^+$, is well-posed, that is, the corresponding control operator $\beta_{r,\mu}$ is admissible for $\mathfrak{A}$.

Proof
As in Section 3, the Dirichlet operators corresponding to system (4.2) and system

$$\begin{cases} \frac{\partial}{\partial t}w(t,\sigma) = -\frac{\partial}{\partial\sigma}w(t,\sigma), & t \in R^+, \sigma \in R^+, \\ w(t,0) = u(t), & t \in R^+, \end{cases} \tag{4.3}$$

are denoted by $D_{\lambda,\mathfrak{A}}$ and $D_{\lambda,\mathfrak{A}+\mu(\cdot)}$, respectively. The control operator corresponding to system (4.2) and system (4.3) are denoted by $\beta_{r,\mu}$ and β_r, respectively. By [23], for all $x \in \mathbb{C}$, the Dirichlet operator $D_{\lambda,\mathfrak{A}+\mu(\cdot)}$ satisfies the following equations

$$\begin{cases} \left(\lambda + \frac{d}{d\sigma}\right) D_{\lambda,\mathfrak{A}+\mu(\cdot)}x = 0, \ \forall t \geq 0, \\ (D_{\lambda,\mathfrak{A}+\mu(\cdot)}x)(0) = x. \end{cases} \tag{4.4}$$

This implies

$$D_{\lambda,\mathfrak{A}+\mu(\cdot)}x = e^{-\lambda}x.$$

Thus, $D_{\lambda,\mathfrak{A}+\mu(\cdot)} = e^{-\lambda}$. Similarly, we can obtain $D_{\lambda,\mathfrak{A}} = e^{-\lambda-\int_0^{\cdot}\mu(\theta)d\theta}$. We compute

$$\begin{aligned}&R(\lambda,\mathfrak{A}+\mu(\cdot))\left(\mu(\cdot)e^{-\lambda-\int_0^{\cdot}\mu(\theta)d\theta}\right)\\&=e^{-\lambda}\int_0^{\cdot}\mu(s)e^{-\int_0^s\mu(\theta)d\theta}ds\\&=e^{-\lambda}-e^{-\lambda-\int_0^{\cdot}\mu(\theta)d\theta}.\end{aligned}$$

This implies that

$$\beta_{r,\mu} = (\lambda-\mathfrak{A}_{-1})D_{\lambda,\mathfrak{A}} = (\lambda-\mathfrak{A}_{-1}-\mu(\cdot))D_{\lambda,\mathfrak{A}+\mu(\cdot)} = \beta_r.$$

The semigroup generated by $\mathfrak{A}+\mu(\cdot)$ is denoted by $\{S_r(t)\}_{t\geq 0}$. For any $u\in L^p([0,\tau),\mathbb{C})$, we obtain $u\in L^1([0,\tau),\mathbb{C})$ and

$$\int_0^{\tau} S_r(\tau-s)D_{\lambda,\mathfrak{A}+\mu(\cdot)}u(s)ds = \int_{\tau-\cdot}^{\tau} e^{-\lambda(\cdot-\tau+s)}u(s)ds.$$

By simple computation, it is obtained that

$$\left(\int_{\tau-\cdot}^{\tau} e^{-\lambda(\cdot-\tau+s)}u(s)ds\right)(0)=0$$

and

$$\int_{\tau-\cdot}^{\tau} e^{-\lambda(\cdot-\tau+s)}u(s)ds\in W^{1,1}(R^+,\mathbb{C}).$$

Thus,

$$\int_0^{\tau} S_r(t-s)D_{\lambda,\mathfrak{A}+\mu(\cdot)})u(s)ds\in D(\mathfrak{A}).$$

Hence, $\beta_r = (\lambda - A_{-1}-\mu(\cdot))D_{\lambda,\mathfrak{A}+\mu(\cdot)}$ is admissible for $\mathfrak{A}+\mu(\cdot)$. Because operator $\mu(\cdot)$ is bounded, $\beta_{r,\mu}=\beta_r$ is admissible for $\mathfrak{A}$. Here, $(\mathfrak{A},\beta_{r,\mu})$ corresponds with system (4.2). The proof is completed. □

Theorem 4.2
The triple (A,B,C) generates a regular linear system.

Proof
For any $f\in W^{1,p}([-r,0],L^1(R^+))$, we compute

$$\begin{aligned}\|Kf\| &= \left\|\int_0^{\infty}\beta(a)f(-r,a)da\right\|\\&\leq \|\beta\|_{\infty}\|\delta_{-r}f\|\\&\leq \|\beta\|_{\infty}\|\delta_{-r}\|\|f\|_{W^{1,p}([-r,0],X)}.\end{aligned}$$

Thus, $K\in L(W^{1,p}([-r,0],L^1(R^+)),\mathbb{C})$. The semigroup generated by A is denoted by $\{S(t)\}_{t\geq 0}$. Then, we can obtain that

$$\begin{aligned}\int_0^r\|KS(t)f\|^p dt &= \int_0^r\left\|\int_0^{\infty}\beta(a)f(t-r,a)da\right\|^p dt\\&\leq\int_{-r}^0\left\|\int_0^{\infty}\beta(a)f(\sigma,a)da\right\|^p d\sigma\\&\leq\int_{-r}^0\|\beta\|_{\infty}^p\left\|\int_0^{\infty}f(\sigma,a)da\right\|^p d\sigma\\&=\|\beta\|_{\infty}^p\int_{-r}^0\|f(\sigma,\cdot)\|^p d\sigma\\&=\|\beta\|_{\infty}^p\|f\|^p,\end{aligned}$$

which implies that K is admissible for A. By [15], $\Phi_{A,B}: L^p(R^+,\partial X)\to X$ is defined by

$$(\Phi_{A,B}(t)g)(\theta)=\begin{cases}g(t+\theta), & t+\theta\geq 0,\\0, & \text{if not.}\end{cases}$$

Therefore, for any $u \in W^{1,p}_{loc}\left(R^+, L^1\left(R^+\right)\right)$, we obtain

$$\begin{aligned}\int_0^r \|K\Phi_{A,B}(t)u\|^p dt &\le \int_0^r \left\|\int_0^\infty \beta(a)u(t-r,a)da\right\|^p dt\\ &\le \int_{-r}^0 \left\|\int_0^\infty \beta(a)u(\sigma,a)da\right\|^p d\sigma\\ &\le \int_{-r}^0 \|\beta\|_\infty^p \left\|\int_0^\infty u(\sigma,a)da\right\|^p d\sigma\\ &= \|\beta\|_\infty^p \int_0^\infty \|u(\sigma,\cdot)\|^p d\sigma\\ &= \|\beta\|_\infty^p \|u\|^p.\end{aligned}$$

Hence, (A, B, C) generates a regular linear system. This completes the proof. □

With Theorem 3.4, Lemma 4.1, and Theorem 4.2, we obtain the following theorem.

Theorem 4.3
The population dynamical system (4.1) is well-posed.

Observe that the operator $KR(\lambda, A_{-1})B$ has one-dimensional range; hence, it is compact. This implies that the operators

$$R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B$$

and

$$KR(\lambda, A_{-1})BR(\lambda, \mathfrak{A}_{-1})\mathfrak{B}$$

are both compact. Thus, in Theorem 3.6, '$\Longleftarrow$' can be replaced by '$\Longleftrightarrow$'. On the other hand, observe that $Re\lambda > -\mu_\infty$ implies $\lambda \in \rho(A) \cap \rho(\mathfrak{A})$. Therefore, we can obtain the following theorem.

Theorem 4.4
Let $Re\lambda > -\mu_\infty$. Then,

$$\begin{aligned}\lambda \in \sigma(\mathcal{A}_{L,K}) &\Longleftrightarrow 1 \in \sigma(R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)\\ &\Longleftrightarrow 1 \in \sigma(KR(\lambda, A_{-1})BR(\lambda, \mathfrak{A}_{-1})\mathfrak{B})).\end{aligned}$$

Corollary 4.5
Let $Re\lambda > -\mu_\infty$. Then,

$$\lambda \in \sigma(\mathcal{A}_{L,K}) \Longleftrightarrow \lambda \in \sigma_P(\mathcal{A}_{L,K}).$$

Proof
The result is obtained directly from the combination of Theorems 3.5, 3.6, and 4.4. □

Theorem 4.6
Let $Re\lambda > -\mu_\infty$. Then,

$$\lambda \in \sigma(\mathcal{A}_{L,K}) \Longleftrightarrow \xi(\lambda) = 0$$

where

$$\xi(\lambda) = -1 + \int_0^\infty \beta(a)e^{-\int_0^a(\lambda+\mu(s))ds}e^{-\lambda r}da$$

Proof
Similar to the proof of [11, Theorem 12], we obtain that $\lambda \in \sigma_P(\mathcal{A}_{L,K})$ if and only if $1 = K(\phi_\lambda e_\lambda)$, that is, $1 = \int_0^\infty \beta(a)e^{-\int_0^a(\lambda+\mu(s))ds}e^{-\lambda r}da$. □

Theorem 4.7
The semigroup generated by $(\mathcal{A}_{L,K}, D(\mathcal{A}_{L,K}))$ is positive and the following statements hold:

i) $w_0(\mathcal{A}_{L,K}) < 0 \Leftrightarrow \xi(0) < 0$,
ii) $w_0(\mathcal{A}_{L,K}) = 0 \Leftrightarrow \xi(0) = 0$,
iii) $w_0(\mathcal{A}_{L,K}) > 0 \Leftrightarrow \xi(0) > 0$.

Proof
The combination of $(\mathfrak{A}, \mathfrak{B})$ generating an abstract linear control system and (A, B, C) generating a regular linear system implies that

$$\|R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B\| \to 0$$

as $\lambda \to \infty$. It follows that

$$\|R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B\| < 1$$

for $Re\lambda$ big enough. Therefore, the operator $(I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)$ is invertible and its inverse is given by Neumann series. Observe that

$$KR(\lambda, A_{-1})Bf = Ke^{\lambda}f = \int_0^{\infty} \beta(a)e^{-\lambda r}f(s)da, \forall f \in L^1(R^+),$$

and

$$R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}x = e^{-\lambda}x, \forall x \in \mathbb{C}.$$

Therefore, $(I - R(\lambda, \mathfrak{A}_{-1})\mathfrak{B}KR(\lambda, A_{-1})B)$ is positive. Obviously, $R(\lambda, A)$, $R(\lambda, A)B = e^{\lambda}$ and $R(\lambda, \mathfrak{A})$ are positive. Moreover, we can compute

$$\begin{aligned} KR(\lambda, A)w &= K\int_{\cdot}^{0} e^{-(s-\cdot)}w(s)ds \\ &= \int_0^{\infty} \beta(a)\int_{-r}^{0} e^{-(s+r)}w(s,a)dsda, \forall w \in L^1([-r,0], L^1(R^+))), \end{aligned}$$

which implies that $KR(\lambda, A)$ is positive. Thus, we can obtain from (3.2) that $R(\lambda, \mathcal{A}_{L,B})$ is positive. By [26, Therem VI.1.15], it follows that operator $\mathcal{A}_{L,K}$ generates a positive C_0-semigroup on Banach space $X \times \partial X$. Because this is an AL-space, it follows from [26, Theorem VI.1.15] that $w_0(\mathcal{A}_{L,K}) = s(\mathcal{A}_{L,K})$ (spectrum boundness is qual to growth boundness). Note that the function ξ is continuous and strictly decreasing with $\lim_{\lambda \to -\infty} \xi(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} \xi(\lambda) = 1$. The rest of the proof is the same as the proof of [11, Theorem 13]. □

The following result is directly obtained from Theorems 4.6 and 4.7.

Corollary 4.8
If

$$\|\beta\|_{\infty} \int_0^{\infty} e^{-\int_0^a \mu(s)ds}da < 1,$$

then the growth bound of the semigroup satisfies $w_0(\mathcal{A}_{L,K}) < 0$. In particular, all solutions (classical or mild) of (4.1) are uniformly exponentially stable.

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References

1. Andrews KT, Kuttler KL, Shillor M. Second order evolution equations with dynamic boundary conditions. *Journal of Mathematical Analysis and Applications* 1996; **197**:781–795.
2. Carroll RW, Showalter RE. *Singular and Degenerate Cauchy Problems*. Academic Press: New York, 1976.
3. Casarino V, Engel K-J, Nagel R, Nickel G. A semigroup approach to boundary feedback systems. *Integral Equations Operator Theory* 2003; **47**:289–306.
4. Escher J. Quasilinear parabolic systems with dynamical boundary conditions. *Communications in Partial Differential Equations* 1993; **18**:1309–1364.
5. Guo B-Z, Xu G-Q. On basis property of a hyperbolic system with dynamic boundary condition. *Differential and Integral Equations* 2005; **18**(1):35–60.
6. Mugnolo D. Abstract wave equations with acoustic boundary conditions. *Mathematische Nachrichten* 2006; **279**(3):299–318.
7. Mugnolo D. Damped wave equations with dynamic boundary condtions. *Journal of Applied Analysis* 2011; **17**:241–275.
8. Vázquez JL. Heat equation with dynamical boundary conditions of reactive-diffusive type. *Journal of Differential Equations* 2011; **250**:2143–2161.
9. Lions JL. *Equations Differentielles Operationelles et Problemes aux Limites*. Springer: Berlin, 1961.
10. Salamon D. Infinite-dimensional linear system with unbounded control and observation: a functional analytic approach. *Transactions on American Mathematical Society* 1987; **300**:383–431.
11. Piazzera S. An age dependent population equation with delayed birth process. *Mathematical Methods in the Applied Sciences* 2004; **27**:427–439.
12. Desch W, Schappacher W. Some generation results for perturbed semigroups. In *Semigroup Theory and Applications (Proceedings Trieste 1987)*. Lecture Notes in Pure and Applied Mathematics, Vol. 116, Clément P, Invernizzi S, Mitidieri E, Vrabie II (eds). Marcel Dekker: New York, 1989; 125–152.
13. Nagel R, Huy NT. Linear neutral partial differential equations: a semigroup approach. *International Journal of Mathematics and Mathematical Sciences* 2003; **23**:1433–1445.
14. Greiner G, Kuhn KG. Linear and semilinear boundary conditions: the analytic case. In *Semigroup Theory and Evolution Equations (Delft, 1989)*. Lecture Notes in Pure and Applied Mathematics, Vol. 135, Clément P, Mitidieri E, de Pagter B (eds). Dekker: New York, 1991; 193–211.
15. Hadd S, Idrissi A, Rhandi A. The regular linear systems associated to the shift semigroups and application to control delay systems. *Mathematics of Control Signals and Systems* 2006; **18**:272–291.

16. Hadd S, Zhong Q-C. Perturbation theory of boundary value problems and approximate controllability of perturbed boundary control problems. *Proceedings of the 47th IEEE Conference on Decision and Control Cancun*, Mexico, December 9–11, 2008, 1465–1470.
17. Weiss G. Transfer functions of regular linear systems Part I: characterizations of regularity. *Transactions of the American Mathematical Society* 1994; **342**(2):827–854.
18. Weiss G. Admissibility of unbounded control operators. *SIAM Journal of Control Optimization* 1989; **27**:527–545.
19. Weiss G. The representation of regular linear systems on Hilbert spaces. In *Control and Estimation of Distributed Parameter Systems (Proceedings Vorau 1988)*, Kappel F, Kunisch K, Schappacher W (eds). Birkhäuser: Basel, Switzerland, 1989; 401–416.
20. Weiss G. Admissible observation operators for linear semigroups. *Israel Journal of Mathematics* 1989; **65**:17–43.
21. Staffans OJ. *Well-Posed Linear Systems*. Cambridge University Press: Cambridge, U.K., 2005.
22. Weiss G. Regular linear systems with feedback. *Mathematics of Control Signals and Systems* 1994; **7**:23–57.
23. Greiner G. Perturbing the boundary conditions of a generator. *Houston Journal of Mathematics* 1987; **13**:213–229.
24. Malinen J, Staffans OJ. Conservative boundary control systems. *Journal of Differential Equations* 2006; **231**:290–312.
25. Hadd S. Exact controllability of infinite dimensional systems persists under small perturbations. *Journal of Evolution Equations* 2005; **5**:545–555.
26. Engel K-J, Nagel R. *One Parameter Semigroups for Linear Evolutional Equations*. Springer-Verlag: New York, 2000.
27. Hadd S, Idrissi A. On the admissibility of observation for perturbed C_0-semigroups on Banach spaces. *Systems Control Letter* 2006; **55**:1–7.
28. Greiner G. A typical Perron-Frobenius theoremwith applications to an age-dependent population equation. In *Infinite-Dimensional Systems (Retzhof, 1983)*. Lecture Notes in Math., Vol. 1076, Kappel F, Schappacher W (eds). Springer: Berlin, 1984; 86–100.