Chapter 1. Basic Concept in Matrix Analysis

This chapter recalls basic facts about matrix, including linear vector spaces, linear transformations, norms and inner products, eigenvalues and eigenvectors.

1 Linear Vector Spaces

1.1 Linear transformation and its matrix representation

Assume that V and W are two linear vector spaces over a field $\mathbb{F}$. Here, most of the time we will have $\mathbb{F} = \mathbb{R}$ and sometimes $\mathbb{F} = \mathbb{C}$. We will also consider finite dimensional vector spaces and in such case, when $\dim V = n$, and $\dim W = m$, we have a basis in V composed of n linearly independent vectors:

$$V = \mathrm{span}\{\varphi_1, \dots \varphi_n\}.$$

Recall that a mapping $T : V \mapsto W$ is linear if

$$T(\alpha u + \beta v) = \alpha T u + \beta T v \quad \forall\, u, v \in V, \quad \alpha, \beta \in \mathbb{R} \text{ (or } \quad \alpha, \beta \in \mathbb{C})$$

The set of all linear operators from V to W will be denoted by $\mathcal{L}(V, W)$ and $\mathcal{L}(V) = \mathcal{L}(V, V)$. It is clear that $\mathcal{L}(V, W)$ is also a linear vector space.

When the bases in V and W ($W = \mathrm{span}\{\psi_1, \dots \psi_n\}$) are fixed, the linear operator T has a **matrix representation** $\boldsymbol{A} \in \mathbb{F}^{n\times n}$, which is obtained as follows. Consider $T : V \mapsto W$ and obviously then, the image of $T\varphi_j$, $j = 1, \dots, n$ is an element of W. Using the basis $\{\psi_k\}$ in W we can write

$$T\varphi_j = \sum_{k=1}^{m} \alpha_{kj} \psi_k.$$

Therefore, for $v \in V$, $v = \sum_{j=1}^{n} a_j \varphi_j$ we have:

$$\begin{aligned} Tv &= T \sum_{j=1}^{n} a_j \varphi_j = \sum_{j=1}^{n} a_j T\varphi_j \\ &= \sum_{j=1}^{n} a_j \sum_{k=1}^{m} \alpha_{kj} \psi_k = \sum_{k=1}^{m} \left(\overbrace{\sum_{j=1}^{n} \alpha_{kj} a_j}^{b_k} \right) \psi_k \\ &= \sum_{k=1}^{m} b_k \psi_k. \end{aligned}$$

The matrix representation of T then is $\boldsymbol{A} = (\alpha_{kj})$, $k = 1, \dots, m$, $j = 1, \dots, n$ and we have the following relation between the coefficients of v and Tv:

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{a}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

It is often convenient to identify the linear transformations with their matrix representations, when the bases are fixed. When there is no ambiguity, we shall use the same notation A, B, etc. (without boldface) to denote both the operator and its matrix representation.

A linear mapping $f : V \mapsto \mathbb{R}$ or $f : V \mapsto \mathbb{C}$ is called linear functional. The set of all linear functionals is denoted by V' and it becomes a vector space by setting

$$(f + g)(v) = f(v) + g(v), \quad (\alpha f)(v) = \alpha f(v), \quad \alpha \in \mathbb{F}.$$

The dimension of V' is the same as the dimension of V; and for fixed basis in V, $\{\varphi_1, \dots, \varphi_n\}$, a basis in V', $\{\varphi'_1, \dots, \varphi'_n\}$ is defined by

$$\varphi'_k(v) = a_k, \quad \text{if} \quad v = \sum_{j=1}^{n} a_j \varphi_j.$$

or

$$\varphi'_k(\varphi_j) = \delta_{kj}.$$

1.2 Norms and inner products

We introduce the notions of norm and inner product which generalize the length and dot product for vectors in $\mathbb{R}^3$.

Definition 1.1 (Norm) *A norm on the vector space V is a function $\|\cdot\| : V \mapsto \mathbb{R}$ that satisfies, for any $u, v \in V$ and $\alpha \in \mathbb{R}$, the following three properties*

1. $\|v\| = 0$, *iff* $v = 0$;
2. $\|\alpha v\| = |\alpha| \|v\|$;
3. $\|u + v\| \leq \|u\| + \|v\|$.

Note that these three properties imply that $\|v\| \geq 0$. Indeed, by property 3. and 2. above we have

$$0 = \|v - v\| \leq \|v\| + |(-1)| \, \|v\| = 2\|v\|.$$

Important examples of norms are the ℓ^p-norms on $\mathbb{R}^n$. Let $V = \mathbb{R}^n$ and $p \geq 1$. Define

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}. \tag{1.1}$$

It can be verified that $\|\cdot\|_p$ satisfies all the three conditions in the above definition and hence it defines a norm in $\mathbb{R}^n$. The three most important cases are $p = 1$, $p = 2$ and $p = \infty$:

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}, \quad \text{and } \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Another example of a norm on a vector space V with a fixed basis $\{\varphi_j\}_{j=1}^n$ is as follows:

$$q_p(v) = \|x\|_p, \quad \text{where} \quad v = \sum_{j=1}^n x_j \varphi_j, \quad p \geq 1. \tag{1.2}$$

We recommend that the reader verifies that all properties of the norm are satisfied by $q_p(\cdot)$.

The equivalence of different norms on a vector space V is an important tool in the analysis when we would like to obtain quantitative results on the rate of convergence of sequences. We introduce this concept next.

Definition 1.2 *We say that two norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent norms on V if there exist constants c and C such that the following inequalities hold for all $v \in V$:*

$$c\|v\| \leq |\!|\!|v|\!|\!| \leq C\|v\|. \tag{1.3}$$

Next theorem shows the important fact that in finite dimensional space all norms are equivalent.

Theorem 1.3 *Any two norms on a finite dimensional linear vector space V are equivalent.*

Proof. We fix a basis in V, $\{\varphi_j\}$, $j = 1, \ldots, n$ and let $\|\cdot\|$ be any norm on V. We will show that $\|\cdot\|$ is equivalent norm to the norm $q_\infty(\cdot)$ defined in (1.2). This will prove the result, because norm-equivalence is a transitive relation. Clearly, the set S defined as

$$S = \left\{ y \in \mathbb{R}^n \;\middle|\; \|y\|_\infty = 1 \right\}$$

is closed and bounded as a subset of $\mathbb{R}^n$. Moreover, we have

$$S = \left\{ y \in \mathbb{R}^n \;\middle|\; q_\infty(v) = 1, \quad v = \sum_{j=1}^n y_j \varphi_j \right\}$$

Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}_+$ defined by

$$x = (x_1, \dots x_n)^T, \quad f(x) := \left\| \sum_{j=1}^{n} x_j \varphi_j \right\|.$$

It is clear that f is continuous, because by the triangle inequality we have that

$$\begin{aligned} |f(x) - f(y)| &\leq \left\| \sum_{j=1}^{n} (x_j - y_j)\varphi_j \right\| \leq \|x - y\|_\infty \sum_{j=1}^{n} \|\varphi_j\| \\ &\leq K\|x - y\|_\infty \end{aligned}$$

where, $K = \sum_{j=1}^{n} \|\varphi_j\|$, which is a constant because the basis is fixed. We have that $f(x)$ is continuous on $\mathbb{R}^n$, and, in particular, on S. By the extreme value theorem from calculus, because S is a compact set in $\mathbb{R}^n$, it follows that $f(x)$ attains its maximum and minimum values at some points $x_{\min} \in S$ and $x_{\max} \in S$, that is,

$$0 < f(x_{\min}) = \min_{x \in S} f(x), \quad f(x_{\max}) = \max_{x \in S} f(x) < \infty.$$

Therefore,

$$c = f(x_{\min}) \leq \|v\| \leq f(x_{\max}) = C.$$

Applying the above inequality with $\frac{1}{q_\infty(v)} v$ for general $v \in V$ completes the proof. ■

Operator norms Notice that $\mathcal{L}(V)$ is a linear vector space, hence a norm of an operator $A \in \mathcal{L}(V)$, in general, could be any function which satisfies the conditions stated in Definition 1.1. Several examples are

Example 1.4 *For a matrix $A \in R^{n \times m}$, we can map A to a vector in $\mathbb{R}^{nm}$ and define the following "entrywise" norm:*

$$\|A\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^p \right)^{1/p}. \tag{1.4}$$

For $p = 2$, this is called the Frobenius norm:

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{1/2}. \tag{1.5}$$

For $p = \infty$, this is called the max-norm:

$$\|A\|_\infty = \max\{|a_{ij}|\}. \tag{1.6}$$

Operator norm may be defined as a **sub-ordinate norm** to a vector norm. We have the following proposition which defines such a norm and proves two important properties.

Proposition 1.5 *Let U, V, and W be linear vector spaces with norms $\|\cdot\|_U$, $\|\cdot\|_V$, and $\|\cdot\|_W$, respectively. Define, for $A \in \mathcal{L}(V,W)$,*

$$\|A\|_{\mathcal{L}(\mathcal{V},\mathcal{W})} = \sup_{v\in V} \frac{\|Av\|_W}{\|v\|_V}, \quad \textit{for all } A \in \mathcal{L}(V,W) \tag{1.7}$$

Then, $\|A\|_{\mathcal{L}(\mathcal{V},\mathcal{W})}$ is a norm in $\mathcal{L}(V,W)$ which satisfies, for any $v \in V$,

$$\|Av\|_W \leq \|A\|_{\mathcal{L}(V,W)} \|v\|_V. \tag{1.8}$$

In addition, for any $B \in \mathcal{L}(U,V)$, we have

$$\|AB\|_{\mathcal{L}(U,W)} \leq \|A\|_{\mathcal{L}(V,W)} \|B\|_{\mathcal{L}(U,V)}. \tag{1.9}$$

Proof. The first inequality (1.8) follows from the definition of $\|A\|_{\mathcal{L}(V,W)}$:

$$\|Av\|_W = \|v\|_V \frac{\|Av\|_W}{\|v\|_V} \leq \|v\|_V \sup_{y\in V} \frac{\|Ay\|_W}{\|y\|_V} = \|A\|_{\mathcal{L}(V,W)} \|v\|_V.$$

The second inequality follows almost immediately. From (1.8) we get

$$\|ABu\|_W \leq \|A\|_{\mathcal{L}(V,W)} \|Bu\|_V \leq \|A\|_{\mathcal{L}(V,W)} \|B\|_{\mathcal{L}(U,V)} \|u\|_U.$$

Dividing both sides by $\|u\|_U$ and taking the sup over all $u \in U$ completes the proof. ■

An operator norm $\|\cdot\|$ is called **sub-multiplicative** if it satisfies (1.9). A vector norm $\|\cdot\|_\alpha$ and an operator norm $\|\cdot\|_\beta$ are called **consistent** if

$$\|Av\|_\alpha \leq \|A\|_\beta \|v\|_\alpha, \quad \text{for all } A \in \mathcal{L}(V), v \in V.$$

Since the linear functionals are mappings from V to $\mathbb{F}$ their norms are also defined as in (1.7) which relates the norms on V and V':

$$\|f\|_{V'} = \sup_{v\in V} \frac{|f(v)|}{\|v\|_V}. \tag{1.10}$$

We shall now give the definition of inner (scalar) product.

Definition 1.6 (Inner (or Scalar) Product) *An inner product of V is a sesquilinear form $(\cdot,\cdot) : V \times V \mapsto \mathbb{C}$ such that for any $u \in V$, $v \in V$, $w \in V$ and $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$,*

1. $(u,v) = \overline{(v,u)}$;
2. $(v,v) \geq 0$ *and* $(v,v) = 0$, *iff* $v = 0$;

3. $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$

If V is a vector space over the real numbers, the second item in the definition says that the inner product is symmetric. Note that (v, v) is always real by 1.

Example 1.7 *The Euclidean inner product when $V = \mathbb{C}^n$ is defined by*

$$(x, y)_{\ell^2} = \sum_{i=1}^{n} x_i \overline{y}_i,$$

and analogously also for $V = \mathbb{R}^n$ (actually the same expression as $\bar{y}_i = y_i$ in this case).

Lemma 1.8 (Cauchy-Schwarz inequality) *Assume that $(\cdot, \cdot)$ is an inner product on V, then*

$$|(u, v)|^2 \le (u, u)(v, v), \quad \textit{for all } u \in V,\ v \in V,$$

Proof. By the definition of inner product, we have, for any $u, v \in V$ and $\alpha = \frac{(u,v)}{(v,v)} \in \mathbb{C}$ we have

$$\begin{aligned} 0 &\le (u - \alpha v, u - \alpha v) = (u, u) - \alpha\overline{(u, v)} - \overline{\alpha}(u, v) + |\alpha|^2 (v, v) \\ &= (u, u) - \frac{|(u, v)|^2}{(v, v)}, \end{aligned}$$

which completes the proof. ■

The Cauchy Schwarz inequality shows that $\|v\| = (v, v)^{1/2}$ is a norm. This norm is sometimes said to be a norm induced by the scalar product. It is immediate to see that $(v, v)^{1/2}$ satisfies conditions 1. and 2. in the definition of a norm (Definition 1.1). The triangle inequality (condition 3.) follows from the Cauchy Schwarz inequality by

$$\begin{aligned} \|u + v\|^2 &= (u + v, u + v) = \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}((u, v)) \\ &\le \|u\|^2 + \|v\|^2 + 2|(u, v)| \le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2 . \end{aligned}$$

Example 1.9 *Let $V = \mathbb{R}^n$. For the inner product given by Example 1.7, the Cauchy-Schwarz inequality gives*

$$\left| \sum_{i=1}^{n} x_i y_i \right|^2 \le \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2.$$

Example 1.10 *$V = L^2(a,b)$, the vector space consisting of square integrable functions over interval (a,b). Define an inner product by*

$$(f,g)_{L^2} = \int_a^b f(x)g(x)dx.$$

The corresponding Cauchy-Schwarz inequality is

$$\left| \int_a^b f(x)g(x)dx \right|^2 \leq \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx.$$

2 Eigenvalues, eigenvectors of matrices

In this section, we briefly review some facts related to eigen-decomposition of square matrices.

2.1 Eigenvalues: algebraic and geometric multiplicity

Given $A \in \mathbb{C}^{n\times n}$, if $\lambda \in \mathbb{C}$ and $x \neq 0$ are such that

$$Ax = \lambda x,$$

then we call λ an eigenvalue of A and x an eigenvector of A corresponding to the eigenvector x. Note that

$$Ax = \lambda x \iff (\lambda I - A)x = 0. \tag{2.11}$$

Given $A \in \mathcal{L}(V)$ we use the notation $\sigma(A)$ to denote the set of all eigenvalues of A, or, equivalently the <u>spectrum</u> of A, namely,

$$\sigma(A) = \{\lambda \in \mathbb{C} \bigm| \det(A - \lambda I) = 0\} \tag{2.12}$$

The characteristic polynomial of A is defined to be $P_A(z) = \det(A - zI)$. Any polynomial of degree n with complex coefficients

$$q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

is a characteristic polynomial of a matrix. Indeed, $q(z) = P_{A_q}(z)$ where

$$A_q = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}.$$

To find the eigenvalues of A is equivalent to solving $P_A(z) = 0$.

If $n \geq 5$, by Abel's theorem (a.k.a Abel-Ruffini theorem), the eigenvalues of A cannot be expressed in terms of the algebraic operations $+$, $-$, $\div$, $\times$, $\sqrt[n]{\ }$. Thus, on a computer, we can only find eigenvalues iteratively using approximate methods.

"In algebra, the Abel–Ruffini theorem (also known as Abel's impossibility theorem) states that there is no algebraic solution, that is, solution in radicals, to the general polynomial equations of degree five or higher with arbitrary coefficients. The theorem is named after Paolo Ruffini, who made an incomplete proof in 1799, and Niels Henrik Abel, who provided a proof in 1824." – Wiki

Definition 2.1 *The algebraic multiplicity $M_a(\lambda_i)$ of an eigenvalue λ_i, is the number of times that λ_i appears in the factorization of $P_A(z)$:*

$$P_A(z) = (z - \lambda_1)^{\gamma_1}(z - \lambda_2)^{\gamma_2} \cdots (z - \lambda_r)^{\gamma_r}, \text{ where } \gamma_1 + \gamma_2 + \cdots + \gamma_r = n.$$

Definition 2.2 *The geometric multiplicity of an eigenvalue λ_i is*

$$M_g(\lambda_i) = \dim E_i, \text{ where } E_i = \ker(\lambda_i I - A).$$

It is immediate to see that $M_g(\lambda) \leq M_a(\lambda)$. In addition, if $M_g(\lambda) < M_a(\lambda)$, then the eigenvalue λ is called <u>defective eigenvalue</u>; if $\sum_{\lambda\in\sigma(A)} M_g(\lambda) < \sum_{\lambda\in\sigma(A)} M_a(\lambda)$, then the matrix A is called <u>defective matrix</u>.

Example 2.3 $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *Then $P_A(z) = z^3$. $\lambda = 0$ is the only eigenvalue with $M_a(0) = 3$ and $M_g(0) = 1$.*

Definition 2.4 *A matrix $A \in \mathbb{C}^{n\times n}$ is* **diagonalizable** *if there exists an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.*

Theorem 2.5 *A is non-defective if and only if it is diagonalizable.*

Similar matrices represent the same linear operator under two different bases, with P being the change of basis matrix. For example, consider a linear operator

$$T : V \mapsto V.$$

Under the basis $\{\phi_i\}$, we have $v = \sum_{i=1}^n a_i\phi_i$, $Tv = \sum_{i=1}^n b_i\phi_i$, and the relation $b = Aa$. If we consider another basis $\{\psi_i\}$, we have $v = \sum_{i=1}^n \tilde{a}_i\psi_i$, $Tv = \sum_{i=1}^n \tilde{b}_i\psi_i$, and the relation $\tilde{b} = B\tilde{a}$. It is easy to check that

$$B = P^{-1}AP$$

where

$$\psi_i = \sum_{i=1}^n P_{ij}\phi_j.$$

2.2 Schur decomposition

The following lemma is used in the proof of the Schur decomposition Theorem 2.7 and also in constructing QR decompositions later in this course.

Lemma 2.6 *Let $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$, be such that $\|x\| = \|y\|$ and $(x, y) = y^*x$ is real. Then, there exists a unitary transformation Q such that $Qx = y$.*

Proof. We set $Q = 1-\alpha(x-y)(x-y)^*$. Choosing $\alpha = \frac{2}{\|x-y\|^2}$ a straightforward computation shows that $Q^*Q = QQ^* = I$, and it follows that Q is unitary. Next we check whether $Qx = y$. Note that from the conditions on the norms of x and y, and the fact that their inner product is real, we have

$$\|x - y\|^2 = \|x\|^2 - y^*x - x^*y + \|y\|^2 = 2(x - y)^*x.$$

Therefore,

$$Qx = x - \alpha(x - y)[(x - y)^*x] = x - \frac{2(x - y)^*x}{\|x - y\|^2}(x - y) = y.$$

■

The Schur decomposition theorem, which we prove next, shows that every matrix is unitarily similar to an upper triangular matrix.

Theorem 2.7 *Let $A \in \mathbb{C}^{n\times n}$. Then there exists a unitary Q and an upper triangular U such that $A = Q^*UQ$.*

Proof. The proof is by induction with respect to n. The result is clear for $n = 1$ with $Q = 1$ and $U = a_{11}$. Assume that the Schur decomposition exists for matrices from $\mathbb{C}^{(n-1)\times(n-1)}$ and let $x \in \mathbb{C}^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, with $(x, x) = 1$ is an eigenvector of A, corresponding to eigenvalue

λ. Next, define $\beta \in \mathbb{C}$ as follows:

$$\beta = \begin{cases} 1, & \text{if} \quad x_1 = 0 \\ \frac{x_1}{|x_1|}, & \text{if} \quad x_1 \neq 0. \end{cases} \tag{2.13}$$

Consider now the vector $y = \beta(1, 0, \ldots, 0)^* = \beta e_1$ and note that $\|y\| = \|x\|$ and that $y^* x = |x_1| \in \mathbb{R}$. Therefore, x and y satisfy all conditions of Lemma 2.6 and we can construct a unitary matrix Q_1 such that $Q_1 x = y = \beta e_1$. We then have that $\beta Q_1^* e_1 = x$ and hence,

$$Q_1 A Q_1^* e_1 = \frac{1}{\beta} Q_1 A x = \frac{\lambda}{\beta} Q_1 x = \lambda e_1.$$

Thus, we obtain

$$Q_1 A Q_1^* = \begin{pmatrix} \lambda & w_2^T \\ 0 & A_2 \end{pmatrix}$$

where $w_2 \in \mathbb{C}^{n-1}$ and $A_2 \in \mathbb{C}^{(n-1)\times(n-1)}$. By the induction assumption, there exists a unitary matrix $\widetilde{Q}_2 \in \mathbb{C}^{(n-1)\times(n-1)}$ such that $U_2 = \widetilde{Q}_2 A_2 \widetilde{Q}_2^*$ is upper triangular matrix. We then set

$$Q_2 Q_1 A Q_1^* Q_2^* = \begin{pmatrix} \lambda & w_2^* \\ 0 & U_2 \end{pmatrix} = U, \quad \text{with} \quad Q_2 = \begin{pmatrix} 1 & 0^T \\ 0 & \widetilde{Q}_2 \end{pmatrix}.$$

It is also clear that the eigenvalues of U and A coincide (with their multiplicities), and the proof is complete. ■

Theorem 2.8 *A matrix $A \in \mathbb{C}^{n\times n}$ can be written as $A = Q^* \Lambda Q$, where Q is unitary and Λ is diagonal, if and only if A is normal, namely $A^* A = A A^*$.*

Proof. By Schur decomposition above, $A = Q^* U Q$. Obviously

$$A^* A = A A^* \Leftrightarrow U^* U = U U^* \Leftrightarrow U \text{ is diagonal.}$$

■

2.3 Gershgorin Circle Theorem

Theorem 2.9 *If $A \in \mathbb{C}^{n\times n}$, then the spectrum of A, $\sigma(A)$, is contained in the union of the following disks in the complex plane.*

$$D_k = \big\{ z \in \mathbb{C} \,\big|\, |z - a_{kk}| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}| \big\}.$$

Proof. For $\lambda \in \sigma(A)$, take x such that $Ax = \lambda x$ and $\|x\|_\infty = 1$. Let k be such that $|x_k| = 1$. Then $(Ax)_k = \lambda x_k$, thus

$$\lambda x_k = \sum_{j=1}^{n} a_{kj} x_j = \sum_{j \neq k} a_{kj} x_j + a_{kk} x_k \implies (\lambda - a_{kk}) x_k = \sum_{j \neq k} a_{kj} x_j.$$

Hence

$$|\lambda - a_{kk}| = |\lambda - a_{kk}|\,|x_k| = \left| \sum_{j \neq k} a_{kj} x_j \right| \leq \sum_{j \neq k} |a_{kj}||x_j| \leq \sum_{j \neq k} |a_{kj}|.$$

■

2.4 Min-Max theorem (Courant-Fischer)

Given $A \in \mathbb{R}^{n \times n}$, $A = A^T$. We know that

$$\lambda_{\min} = \min_{x \in \mathbb{R}^n} \frac{(Ax, x)}{(x, x)} \leq \max_{x \in \mathbb{R}^n} \frac{(Ax, x)}{(x, x)} = \lambda_{\max}.$$

Theorem 2.10 *(Courant-Fisher min-max theorem) Suppose $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A \in \mathbb{R}^{n \times n}$, $A = A^T$. Then*

$$\lambda_k = \min_{\{S \,|\, \dim S = k\}} \max_{x \in S} \frac{(Ax, x)}{(x, x)}.$$

Proof. Let $u_1, u_2, \cdots, u_k$ be the corresponding eigenvectors and $S_k = \mathrm{Span}\{u_1, u_2, \ldots, u_k\}$. Clearly $\dim S_k = k$. So

$$\lambda_k = \max_{x \in S_k} \frac{(Ax, x)}{(x, x)} \geq \min_{\{S \,|\, \dim S = k\}} \max_{x \in S} \frac{(Ax, x)}{(x, x)}.$$

Consider $S_k' = \mathrm{Span}\{u_k, u_{k+1}, \cdots, u_n\}$. We have $\dim S_k' = n - k + 1$ and

$$\frac{(Ax, x)}{(x, x)} \geq \lambda_k, \;\; \forall x \in S_k'.$$

Now for any S such that $\dim S = k$, $\dim S + \dim S_k' = n + 1$ and thus $S \cap S_k' \neq \{0\}$. Let y be in $S \cap S_k'$ and we have

$$\max_{x \in S} \frac{(Ax, x)}{(x, x)} \geq \frac{(Ay, y)}{(y, y)} \geq \lambda_k.$$

Hence

$$\min_{\{S \,|\, \dim S = k\}} \max_{x \in S} \frac{(Ax, x)}{(x, x)} \geq \lambda_k.$$

The proof is complete. ■

Corollary 2.11 (Max-Min Theorem) *Using the same setup as above, we have*

$$\lambda_k = \max_{\{S \mid \dim S = n-k+1\}} \min_{x \in S} \frac{(Ax,x)}{(x,x)}.$$

The proof is similar.

Theorem 2.12 *(Interlacing Theorem) Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric, $P \in \mathbb{R}^{n\times m}$ is full rank and $P^*P = I$, where $m < n$. Let $B = P^*AP$. Let $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$ and $\lambda_1(B) \le \lambda_2(B) \le \cdots \le \lambda_m(B)$ be the eigenvalues of A and B, respectively. Then*

$$\lambda_j(A) \le \lambda_j(B) \le \lambda_{n-m+j}(A), \quad j = 1, 2, \cdots, m.$$

Proof. Let $v_1, v_2, \cdots, v_m$ be the eigenvectors of B. Denote again $S_j = \mathrm{Span}\{v_1, v_2, \cdots, v_j\}$. Then

$$\begin{aligned}
\lambda_j(B) &= \max_{x \in S_j} \frac{(Bx,x)}{(x,x)} = \max_{x \in S_j} \frac{(P^*APx,x)}{(x,x)} \\
&= \max_{x \in S_j} \frac{(P^*APx,x)}{(P^*Px,x)} && \text{Because } P^*P = I \\
&= \max_{x \in S_j} \frac{(APx,Px)}{(Px,Px)} = \max_{y \in P(S_j)} \frac{(Ay,y)}{(y,y)}.
\end{aligned}$$

Since P is full rank, $\dim P(S_j) = \dim S_j = j$. So

$$\lambda_j(B) = \max_{y \in P(S_j)} \frac{(Ay,y)}{(y,y)} \ge \min_{\{S \mid \dim S = j\}} \max_{x \in S} \frac{(Ax,x)}{(x,x)} = \lambda_j(A).$$

In the last equality we used the min-max theorem.

We shall now prove $\lambda_j(B) \le \lambda_{n-m+j}(A)$. Let $\widetilde{S}_{m-j+1} = \mathrm{Span}\{v_j, v_{j+1}, \cdots, v_m\}$, then

$$\begin{aligned}
\lambda_j(B) &= \min_{x \in \widetilde{S}_{m-j+1}} \frac{(Bx,x)}{(x,x)} = \min_{y \in P(\widetilde{S}_{m-j+1})} \frac{(Ay,y)}{(y,y)} \\
&\le \max_{\{S \mid \dim S = n-(n-m+j)+1\}} \min_{x \in S} \frac{(Ax,x)}{(x,x)} = \lambda_{n-m+j}(A),
\end{aligned}$$

where the last equality is due to the corollary of the min-max theorem. ■

2.5 Relations between spectral radius and norms

For a matrix $A : \mathbb{R}^n \mapsto \mathbb{R}^n$, the spectral radius of A, $\rho(A)$ is defined as

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

The relations between spectral radius and the norm of a matrix are interesting because they show connections between algebraic quantity (spectral radius) and geometric quantity (such as length and norm).

In this subsection, using the Schur decomposition theorem we prove such relations between spectral radius of a linear transformation and a its sub-ordinate norm. Recall that a norm is sub-ordinate if it is defined via a vector norm. For a linear transformation $A : V \mapsto V$, where V is a vector space equipped with a norm $\|\cdot\|$:

$$\|A\| = \sup_{v \in V} \frac{\|Av\|}{\|v\|}.$$

$$\rho(A) = \|A\|_2 \quad \text{if } A \text{ is normal.}$$

$$\|A\|_2^2 = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \in \mathbb{R}^n} \frac{x^* A^* A x}{x^* x} = \rho(A^* A) = \rho(A)^2$$

In the proof of the main result, Theorem 2.14 we need the following lemma.

Lemma 2.13 *Let $S \in \mathcal{L}(V)$ be an invertible operator and $\|\cdot\|$ be a sub-ordinate norm. Then $N_S(A) = \|SAS^{-1}\|$ is also a sub-ordinate norm.*

Proof. It is easy to verify that $M_S(v) = \|Sv\|$ is a vector norm on V. Further, by the definition of a sub-ordinate norm we have:

$$N_S(A) = \sup_{v \in V} \frac{\|SAS^{-1}v\|}{\|v\|} = \sup_{w \in V} \frac{\|SAw\|}{\|Sw\|} = \sup_{w \in V} \frac{M_S(Aw)}{M_S(w)},$$

Here we have changed the variables $v = Sw$, and the second identity above follows from the invertibility of S. This concludes the proof. ■

The following theorem states several important relations between spectral radius and norms of a linear transformation.

Theorem 2.14 *Assume that $A \in \mathcal{L}(V)$, V is a vector space,* $\dim V = n$. *Then the following equalities hold:*

1. $\rho(A) = \inf_{\|\cdot\|} \|A\|$, *where the infimum is taken over all sub-ordinate norms.*

2. $\lim_{k\to\infty} A^k = 0$ *if and only if* $\rho(A) < 1$.

3. $\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$;

Proof. To prove 1. we first show that $\rho(A) \leq \|A\|$ for any sub-ordinate norm $\|\cdot\|$. Let $x \in V$ be such that $Ax = \lambda x$ with $|\lambda| = \rho(A)$. We then have

$$\|A\| = \sup_{v\in V} \frac{\|Av\|}{\|v\|} \geq \frac{\|Ax\|}{\|x\|} = \rho(A).$$

Next we show that for any $\delta > 0$ there exists a sub-ordinate norm $\|\cdot\|_\delta$ such that $\|A\|_\delta < \rho(A) + \delta$. This will conclude the proof of 1. We fix the basis in V and below we will talk about the matrix representation of A in this fixed basis. From the Schur decomposition theorem we know that there exists unitary transformation Q such that

$$Q^*AQ = \Lambda + U,$$

where Λ is a diagonal matrix with the eigenvalues of A on the diagonal, and U is a strictly upper triangular matrix. Note that $U^n = 0$. For $0 < \varepsilon < 1$, let us introduce a diagonal matrix D, $D = \mathrm{diag}(\varepsilon, \varepsilon^2, \ldots, \varepsilon^n)$. As $U_{ij} = 0$ for all $i \geq j$, it is straightforward to verify that $(D^{-1}UD)_{i,j} = \varepsilon^{j-i}U_{ij}$ for $i < j$, and $(D^{-1}UD)_{i,j} = 0$ for $i \geq j$. Choosing $\varepsilon = \min\{\frac{\delta}{2\|U\|_\infty}, 1\}$ and we have

$$\|D^{-1}UD\|_\infty \leq \varepsilon\|U\|_\infty < \delta.$$

Let us set $S = D^{-1}Q^*$ and from Lemma 2.13 we have that $N_S(A) = \|SAS^{-1}\|_\infty$ is a sub-ordinate norm. Hence,

$$N_S(A) = \|\Lambda + D^{-1}UD\|_\infty \leq \rho(A) + \|D^{-1}UD\|_\infty < \rho(A) + \delta. \tag{2.14}$$

Here we have used that $D^{-1}\Lambda D = \Lambda$ and $\|\Lambda\|_\infty = \rho(A)$. This completes the proof of 1.

To show 2.: If $\rho(A) < 1$, say $\rho(A) = 1 - \varepsilon$, for some $0 < \varepsilon \leq 1$ and we choose $\delta = \frac{\varepsilon}{2}$ and by 1. we have that there exists a norm $\|\cdot\|_\delta$ such that

$$\|A\|_\delta < \rho(A) + \delta = 1 - \delta < 1.$$

As the sub-ordinate norms are continuous functions, we have

$$0 \leq \|\lim_{k\to\infty} A^k\|_\delta = \lim_{k\to\infty} \|A^k\|_\delta < \lim_{k\to\infty} (1-\delta)^k = 0,$$

which shows one of the implications in 2. To show the other direction, we assume that $\lim_{k\to\infty} A^k = 0$, and let x be an eigenvector of A, such that $Ax = \lambda x$ with $|\lambda| = \rho(A)$. As in the proof of 1.

$$\|A^k\| = \sup_{v\in V} \frac{\|A^k v\|}{\|v\|} \geq \frac{\|A^k x\|}{\|x\|} = \rho(A)^k \geq 0.$$

Taking the limit on both sides shows that $\lim_{k\to\infty} \rho(A)^k = 0$, and, hence $\rho(A) < 1$. This concludes the proof of 2.

Finally to prove 3. Let $\varepsilon > 0$ be arbitrary. We set $B = (\rho(A) + \varepsilon)^{-1}A$. Note that $\rho(B) < 1$ and by 2. $\lim_{k\to\infty} B^k = 0$. By the continuity of the norm, this implies that $\lim_{k\to\infty} \|B^k\| = 0$. By the definition of a limit, this means that there exists an N, such that $\|B^k\| < 1$ for all $k \geq N$. Note that $\|B^k\| < 1$ implies that $\|A^k\| < (\rho(A) + \varepsilon)^k$, or, equivalently, $\|A^k\|^{\frac{1}{k}} < (\rho(A) + \varepsilon)$, for $k \geq N$. Therefore, for any $\varepsilon > 0$ there exists an N, such that

$$\rho(A) \leq \|A^k\|^{\frac{1}{k}} \leq \rho(A) + \varepsilon, \quad k \geq N.$$

This concludes the proof of 3. and the proof of the Theorem. ■

We also state a theorem and a corollary, which are often useful in analysis.

Theorem 2.15 *Let $A \in \mathbb{C}^{n\times n}$ be a given matrix. If $\rho(A) < 1$, then $(I - A)$ is invertible and*

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j.$$

Proof. Let $\rho(A) = (1 - \gamma)$, where $0 < \gamma \leq 1$. Let $\delta = \frac{\gamma}{2}$. From the previous theorem we know that there exists a sub-ordinate norm $\|\cdot\|_\delta$ such that

$$\|A\|_\delta \leq \rho(A) + \delta = 1 - \frac{\gamma}{2} < 1.$$

Let us denote

$$S_N = \sum_{j=0}^{N} A^j.$$

By the triangle inequality ond the sub-ordinate property of the norm we obtain that

$$\begin{aligned} \|S_N\| &= \|\sum_{j=0}^{N} A^j\| \leq \sum_{j=0}^{N} \|A^j\| \leq \sum_{j=0}^{N} \|A\|^j \\ &\leq \sum_{j=0}^{N} \left(1 - \frac{\gamma}{2}\right)^j \leq \sum_{j=0}^{\infty} \left(1 - \frac{\gamma}{2}\right)^j = \frac{2}{\gamma} \end{aligned}$$

This tells us that $\lim_{N\to\infty} S_N$ exists and is bounded and we denote $S = \lim_{N\to\infty} S_N$. Next, we observe that

$$\begin{aligned} (I - A)S_N &= (I - A)(I + A + \ldots + A^N) \\ &= I + A + \ldots + A^N - A - \ldots - A^{N+1} = I - A^{N+1}. \end{aligned}$$

Taking the limit on both sides and also considering $S_N(I-A)$ shows that

$$S(I-A) = (I-A)S = I,$$

which is another way to say that $(I-A)$ is invertible and its inverse is S. ■

The following corollary is immediate and its proof amounts to some small changes in the proof of Theorem 2.15.

Corollary 2.16 *If $A \in \mathbb{C}^{n\times n}$ and $\|A\| < 1$, where $\|\cdot\|$ is a subordinate norm, then $(I-A)$ is invertible and the following bound holds:*

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

2.6 Special Cases of Eigenvalue Problems

We have the following special cases of eigenvalue problems which are often encountered in applications.

1. A is diagonalizable if A has n distinct eigenvalues.

2. A is unitarily diagonalizable (there exists unitary Q and diagonal matrix Λ such that $A = Q\Lambda Q^*$) if and only if $AA^* = A^*A$, that is, if and only if A is normal.

3. A is Hermitian (or symmetric, when $\mathbb{F} = \mathbb{R}$) matrix $A = A^*$. Any Hermitian matrix is unitarily diagonalizable.

4. For any $A, B \in \mathcal{L}(V)$, $\sigma(AB) = \sigma(BA)\backslash\{0\}$ and $\sigma(A) = \sigma(T^{-1}AT)$ for any invertible $T \in \mathcal{L}(V)$. A special case is that if A, $B \in \mathbb{R}^{n\times n}$ and B is invertible, we have $AB = B^{-1}(BA)B$. It is obvious that $\sigma(AB) = \sigma(BA)$.

3 SVD: Singular Value Decomposition

The singular value decomposition is one of the important and practical tools in numerical linear algebra. To draw a simple analogy, let us consider a selfadjoint matrix $A \in \mathbb{C}^{n\times n}$, i.e., $A^* = A$. It is well known that we can write A as a sum of rank 1 projections on its eigenvectors. Let $A\varphi_j = \lambda_j\varphi_j$, $j = 1 : n$ and $\|\varphi_j\|_2 = 1$. From Schur Decomposition Theorem (and many other theorems) we have the following identity

$$A = \sum_{j=1}^{n} \lambda_j \varphi_j \varphi_j^*.$$

The theorem which we prove next, provides such a decomposition for general matrix $A \in \mathbb{R}^{m\times n}$.

Theorem 3.1 *Any matrix $A \in R^{m\times n}$ $(m \geq n)$ can be written as*

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^* = \sum_{j=1}^{r} \sigma_j u_j v_j^* \tag{3.15}$$

where $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ are orthogonal matricies, and $\Sigma = \operatorname{diag}(\sigma_i)$ is diagonal such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots \sigma_n = 0.$$

There are different ways to prove this important theorem. Here we give two constructive proofs. Let us first recall a classical result from linear algebra.

Lemma 3.2 *Let $A \in \mathbb{R}^{m\times n}$ and $r = \operatorname{rank}(A) = \dim(\mathrm{Range(A)})$. Then*

$$\dim \operatorname{Ker}(A) = n - r, \quad \dim \operatorname{Ker}(A^*) = m - r.$$

Recall the concept of **rank** of a matrix.

The **column rank** of A is the dimension of the column space of A (number of linearly independent column vectors), while the **row rank** of A is the dimension of the row space of A (number of linearly independent row vectors).

column rank = row rank

A matrix is said to have full rank if its rank equals the largest possible for a matrix of the same dimensions, which is the lesser of the number of rows and columns. A matrix is said to be rank deficient if it does not have full rank.

The rank is also the dimension of the image of the linear transformation that is given by multiplication by A. More generally, if a linear operator on a vector space (possibly infinite-dimensional) has finite-dimensional image (e.g., a finite-rank operator), then the rank of the operator is defined as the dimension of the image.

3.1 Low-rank approximation

A matrix $A \in R^{m\times n}$ $(m \geq n)$ has SVD

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^* = \sum_{j=1}^{n} \sigma_j u_j v_j^* \quad \text{with } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0. \tag{3.16}$$

The best rank k approximation to A in the spectral norm is

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^*,$$

and

$$\|A - A_k\|_2 = \sigma_{k+1}.$$

3.2 Proof I of Theorem 3.1

Since $A^*A \in \mathbb{R}^{n\times n}$ is selfadjoint and positive (semi)-definite we can write

$$A^*A = V\Lambda V^*$$

where $V = (v_1, \ldots, v_n)$ is orthogonal and Λ is diagonal with nonnegative entries λ_i. Without loss of generality, we assume that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots \lambda_n = 0. \tag{3.17}$$

It follows that

$$(AV)^*(AV) = V^*A^*AV = \Lambda. \tag{3.18}$$

This means that $AV \in \mathbb{R}^{m\times n}$ has orthogonal columns. Define

$$\sigma_i = \|Av_i\| = \sqrt{\lambda_i}, \quad i = 1 : n. \tag{3.19}$$

and let

$$u_i = Av_i/\sigma_i, \quad 1 \leq i \leq r.$$

Note that

$$Av_i = 0, \quad r+1 \leq i \leq n.$$

Thus, with $U_r = (u_1, \ldots, u_r)$, we have that

$$AV = (U_r\Sigma_r, 0), \quad \text{and, hence} \quad A = (U_r\Sigma_r, 0)V^* = \sum_{j=1}^{r} \sigma_j u_j v_j^*.$$

Next, we complete the set of vectors $(u_1, \ldots, u_r)$ to an orthogonal basis in $\mathbb{R}^m$. We set

$$U = (U_r, \widetilde{U}_{m-r}), \quad \text{where } \widetilde{U}_{m-r} = (u_{r+1}, \ldots u_m) \in \mathbb{R}^{m,m-r}.$$

Without loss of generality we may assume that the columns of $\widetilde{U}_{m-r}$ are orthogonal

$$\widetilde{U}_{m-r}^*\widetilde{U}_{m-r} = I_{(m-r)\times(m-r)}, \text{and, hence,} \quad U^*U = I.$$

It is then immediate to see that

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^*, \quad \Sigma = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

as desired.

The proof given above contains a useful information, summarized as follows:

1. The singular values of A are the square roots of eigenvalues of AA^*.

2. Rank(A) is the number of the nonzero singular values.

Exercise 3.3 *Two matrices $A, B \in \mathbb{C}^{n\times n}$ are* ***unitarily equivalent*** *if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{n\times n}$. Proves that that A and B are unitarily equivalent if and only if they have the same singular values.*

4 Problems

Exercise. Let V be a real finite dimensional vector space with basis $(\varphi_1, \dots, \varphi_n)$. Show that the functionals $(\varphi_1', \dots, \varphi_n')$ defined as

$$\varphi_k'(v) = a_k, \quad \text{if} \quad v = \sum_{j-1}^{n} a_j \varphi_j.$$

are linearly independent and that any $f \in V'$ can be represented as $f = \sum_{j=1}^n \gamma_j \varphi_j'$. This proves that $\dim V' = n$.

Exercise. If $A = A^T$ and $E = E^T$, show that

$$\lambda_k(A) + \lambda_{\min}(E) \le \lambda_k(A+E) \le \lambda_k(A) + \lambda_{\max}(E).$$

Hint: rewrite $\lambda_k(A+E)$ using min-max theorem, then split the terms.

Exercise. If $S : V \mapsto V$ is invertible, show that $M_S(v) = \|Sv\|_V$ is a vector norm on V.

Exercise. Prove that $A \in \mathbb{R}^{n\times n}$ is self-adjoint if and only if A has all real eigenvalues and a complete set of eigenvectors (namely these eigenvector form a basis of $\mathbb{R}^n$).

Exercise. Prove that the eigenvalues of the symmetric tri-diagonal matrix $A = \text{diag}(b, a, b)$, for any two real numbers a, b, are given by

$$\lambda_k = a + 2b\cos\frac{k\pi}{N+1}, \quad 1 \le k \le N$$

and the eigenvectors $v_k, 1 \le k \le N$, are given by

$$v_{k,j} = \sin\left(\frac{jk\pi}{N+1}\right), \quad 1 \le j \le N$$

where $v_{k,j}$ is the jth component of the kth vector w_k.

Exercise 4.1 *If $A = U\Sigma V^*$, then*

$$\begin{pmatrix} 0 & 2A^* \\ 2A & 0 \end{pmatrix} = \begin{pmatrix} V & V \\ U & -U \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \begin{pmatrix} V & V \\ U & -U \end{pmatrix}^*$$