CONTROLLABILITY OF NONLOCAL FRACTIONAL DIFFERENTIAL SYSTEMS OF ORDER $\alpha \in (1, 2]$ IN BANACH SPACES*

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In this paper, we establish sufficient conditions for controllability of fractional differential systems of order $\alpha \in (1, 2]$ with nonlocal conditions in infinite dimensional Banach spaces. The results are obtained by using the Sadovskii fixed point theorem and vector-valued operator theory.

Keywords: controllability, fractional differential systems, fractional cosine family, nonlocal conditions, mild solutions.

1. Introduction

In the last decades, fractional differential systems have played an important role in physics, chemistry, engineering, biology, finance etc., due to the memory character of fractional derivative, which is a generalization of integer-order derivative and can describe many phenomena that integer derivative cannot characterize.

Controllability of infinite-dimensional integer order differential systems was studied by many authors, see the survey paper [1] and the references therein. Controllability of stochastic functional differential equations was discussed in [2, 3]. Recently, the controllability for fractional differential systems has become active. Wang and Zhou [4] studied complete controllability of fractional evolution systems without involving the compactness of characteristic solution operator. The techniques rely on fractional calculus, properties of solution operator and fixed point theorems. Sakthivel et al. [5] studied a class of dynamic control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces, they discussed the approximate controllability of nonlinear fractional stochastic control system under the assumption that the corresponding linear system is approximately controllable. Wang and Zhou [6] proved the existence and controllability results for fractional semilinear differential inclusions, the results are obtained by using fractional calculus, operator semigroup

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and Bohnenblust–Karlin's fixed point theorem. Yan [7] established sufficient conditions for the approximate controllability of control systems governed by a class of partial neutral functional differential systems of fractional order with state-dependent delay in an abstract space. The results are obtained by using the Krasnoselskii–Schaefer type fixed point theorem. Wang et al. [8] studied the solvability and optimal control of a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. Yan [9] established a sufficient condition for the controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay in Banach spaces. The approach used is analytic semigroups and fractional powers of closed operators and nonlinear alternative of Leray–Schauder type for multivalued maps. Sakthivel and Ren [10] studied the approximate controllability for a class of fractional equations with state-dependent delay and control. Sufficient conditions are established by employing semigroup theory, fixed point technique and fractional calculus. Wang and Zhou [11] discussed the nonlinear control systems of fractional order and its optimal controls in Banach spaces. Kumar and Sukavanam [12] established sufficient conditions for approximate controllability of a class of semilinear delay control systems of fractional order. The results are obtained by using contraction principle and the Schauder fixed point theorem. Rajiv Ganthi and Muthukumar [13] discussed the approximate controllability of fractional stochastic integral equation with finite delays in Hilbert spaces, the results are obtained by using the assumption that the corresponding linear integral equation is approximately controllable and a stochastic version of the Banach fixed point theorem. Wang et al. [14] studied optimal feedback controls of a system governed by semilinear fractional evolution equations via a compact semigroup in Banach spaces. Wang et al. [15] studied fractional Schrödinger equations with potential and optimal controls. The existence, uniqueness, local stability and attractivity, and data continuous dependence of mild solutions are presented. Existence and uniqueness of optimal pairs for the standard Lagrange problem are obtained.

Semilinear differential equations with nonlocal conditions have been initiated by Byszewski [16]. The nonlocal condition is a generalization of the classical initial condition, and it shows better effects in application than the classical initial condition. Wang et al. [17] established two sufficient conditions for nonlocal controllability of fractional evolution systems. Sakthivel et al. [18] studied a class of fractional neutral control systems governed by abstract nonlinear fractional neutral differential equations, they established a new set of sufficient conditions for the controllability of nonlinear fractional systems by using a fixed point analysis approach and extended the results to study the controllability of corresponding systems with nonlocal conditions.

Most of the papers above about fractional differential systems are concerned with the fractional derivative whose order is between zero and one. The purpose of this paper is to establish sufficient conditions for the controllability of fractional differential inclusions of order $\alpha \in (1, 2]$ of the form (1.1). When $\alpha = 2$, the case corresponds to the second-order differential systems, the results in this paper are generalization of classical results.

In this paper we are concerned with a class of fractional differential systems

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t, x(t)) + Bu(t), & t \in [0, b], \\ x(0) + g(x) = x_0, \qquad x'(0) = y_0, \end{cases} \tag{1.1}$$

where $\alpha \in (1, 2]$, ${}^C D_t^\alpha$ is the Caputo fractional derivative, A is the infinitesimal generator of a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on a Banach space X, the state $x(\cdot)$ takes values in X, the control function $u(\cdot)$ is given in $L^2([0, b]; U)$, a Banach space of admissible control functions, with U a Banach space, B is a bounded linear operator from U into X, $F : [0, b] \times X \to X$, $g : C([0, b]; X) \to X$, $x_0, y_0 \in X$.

This paper is organized as follows. In Section 2, we give the basic notation and preliminary facts. In Section 3, we give the controllability result of the system (1.1). At last, an example is presented to illustrate the main results.

2. Preliminaries

In this section, we introduce some basic definitions and notation which are used throughout this paper. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$. Let X be a Banach space with norm $\|\cdot\|$. By $\mathcal{B}(X)$ we denote the space of all bounded linear operators on X. Let $1 \leq p < \infty$. By $L^p([0, b]; X)$ we denote the space of X-valued Bochner integrable functions $f : [0, b] \to X$ with the norm

$$\|f\|_{L^p([0,b];X)} = \left(\int_0^b \|f(t)\|^p dt \right)^{1/p}. \tag{2.1}$$

By $C([0, b]; X)$, resp. $C^1([0, b]; X)$, we denote the spaces of functions $f : [0, b] \to X$, which are continuous, resp. 1-times continuously differentiable. $C([0, b]; X)$ and $C^1([0, b]; X)$ are Banach spaces endowed with the norms

$$\|f\|_C = \sup_{t\in[0,b]} \|f(t)\|_X, \qquad |f\|_{C^1} = \sup_{t\in[0,b]} \sum_{k=0}^{1} \|f^{(k)}(t)\|_X. \tag{2.2}$$

Let I be the identity operator on X. If A is a linear operator on X, then $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. For the sake of simplicity, we use the notation for $\beta > 0$,

$$k_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \qquad t > 0, \tag{2.3}$$

where $\Gamma(\beta)$ is the Gamma function. If $\beta = 0$, we set $k_0(t) = \delta(t)$, the delta distribution.

DEFINITION 2.1 ([19]). The *Riemann–Liouville fractional integral of order $\alpha > 0$* is defined by

$$J_t^\alpha x(t) = \int_0^t k_\alpha(t - s)x(s)ds, \tag{2.4}$$

where $x(t) \in L^1([0, b]; X)$.

DEFINITION 2.2 ([19]). The *Riemann–Liouville fractional derivative of order* $\alpha \in (1, 2]$ is defined by

$$D_t^\alpha x(t) = \frac{d^2}{dt^2} J_t^{2-\alpha} x(t), \tag{2.5}$$

where $x(t) \in L^1([0, b]; X)$, $D_t^\alpha x(t) \in L^1([0, b]; X)$.

DEFINITION 2.3 ([19]). The *Caputo fractional derivative of order* $\alpha \in (1, 2]$ is defined by

$${}^C D_t^\alpha x(t) = D_t^\alpha (x(t) - x(0) - x'(0)t), \tag{2.6}$$

where $x(t) \in L^1([0, b]; X) \cap C^1([0, b]; X)$, $D_t^\alpha x(t) \in L^1([0, b]; X)$.

The Laplace transform for the Riemann–Liouville fractional integral is given by

$$L\{J_t^\alpha x(t)\} = \frac{1}{\lambda^\alpha} \widehat{x}(\lambda), \tag{2.7}$$

where $\widehat{x}(\lambda)$ is the Laplace transform of x given by

$$\widehat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t) dt, \qquad \mathrm{Re}\lambda > \omega. \tag{2.8}$$

The Laplace transform for the Caputo derivative is given by

$$L\{{}^C D_t^\alpha x(t)\} = \lambda^\alpha \widehat{x}(\lambda) - x(0)\lambda^{\alpha-1} - x'(0)\lambda^{\alpha-2}. \tag{2.9}$$

Consider the following problem,

$${}^C D_t^\alpha x(t) = Ax(t), \qquad x(0) = \eta, \qquad x'(0) = 0, \tag{2.10}$$

where $\alpha \in (1, 2]$, $A : D(A) \subset X \to X$ is a closed densely defined linear operator in Banach space X.

DEFINITION 2.4 ([19]). Let $\alpha \in (1, 2]$. A family $\{C_\alpha(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called a *solution operator* (or a strongly continuous α-order fractional cosine family) for (2.10) if the following conditions are satisfied:

(a) $C_\alpha(t)$ is strongly continuous for $t \geq 0$ and $C_\alpha(0) = I$;
(b) $C_\alpha(t)D(A) \subset D(A)$ and $AC_\alpha(t)\eta = C_\alpha(t)A\eta$ for all $\eta \in D(A)$, $t \geq 0$;
(c) $C_\alpha(t)\eta$ is a solution of $x(t) = \eta + \int_0^t k_\alpha(t-s)Ax(s)ds$ for all $\eta \in D(A)$, $t \geq 0$.

A is called the *infinitesimal generator* of $C_\alpha(t)$. The strongly continuous α-order fractional cosine family is also called *α-order cosine family* for short.

DEFINITION 2.5. The *fractional sine family* $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ associated with C_α is defined by

$$S_\alpha(t) = \int_0^t C_\alpha(s) ds, \qquad t \geq 0. \tag{2.11}$$

DEFINITION 2.6. The *fractional Riemann–Liouville family* $P_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ associated with C_α is defined by

$$P_\alpha(t) = J_t^{\alpha-1} C_\alpha(t). \tag{2.12}$$

DEFINITION 2.7. The α-order cosine family $C_\alpha(t)$ is called *exponentially bounded* if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C_\alpha(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{2.13}$$

An operator A is said to belong to $\mathcal{C}^\alpha(M,\omega)$, if the problem (2.10) has an α-order cosine family $C_\alpha(t)$ satisfying (2.13).

In the following, we will derive the appropriate definition of mild solutions of (1.1). Assume $A \in \mathcal{C}^\alpha(M,\omega)$ and let $C_\alpha(t)$ be the corresponding α-order cosine family. Then we have (see [19], (2.6))

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)\eta = \int_0^\infty e^{-\lambda t}C_\alpha(t)\eta dt, \qquad \mathrm{Re}\lambda > \omega, \qquad \eta \in X. \tag{2.14}$$

By (2.11), (2.14), we have

$$\lambda^{\alpha-2}R(\lambda^\alpha, A)\eta = \int_0^\infty e^{-\lambda t}S_\alpha(t)\eta dt, \qquad \mathrm{Re}\lambda > \omega, \qquad \eta \in X. \tag{2.15}$$

By (2.12), (2.14), we have

$$R(\lambda^\alpha, A)\eta = \int_0^\infty e^{-\lambda t}P_\alpha(t)\eta dt, \qquad \mathrm{Re}\lambda > \omega, \qquad \eta \in X. \tag{2.16}$$

Assume that the Laplace transform of $F(t,x(t))$ with respect to t exists, by $\widehat{F}(\lambda)$ we denote the Laplace transform of $F(t,x(t))$. By $\widehat{u}(\lambda)$, we denote the Laplace transform of $u(t)$. Taking the Laplace transform to (1.1), by (2.9), we obtain

$$\lambda^\alpha\widehat{x}(\lambda) - (x_0 - g(x))\lambda^{\alpha-1} - y_0\lambda^{\alpha-2} = A\widehat{x}(\lambda) + \widehat{F}(\lambda) + B\widehat{u}(\lambda).$$

Then

$$\widehat{x}(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha, A)(x_0 - g(x)) + \lambda^{\alpha-2}R(\lambda^\alpha, A)y_0 + R(\lambda^\alpha, A)\widehat{F}(\lambda) + R(\lambda^\alpha, A)B\widehat{u}(\lambda).$$

By the property of Laplace transforms and (2.14), (2.15), (2.16),

$$x(t) = C_\alpha(t)(x_0 - g(x)) + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)(F(s,x(s)) + Bu(s))ds.$$

DEFINITION 2.8. A function $x \in C([0,b];X)$ is called a *mild solution* of (1.1) if x satisfies

$$x(t) = C_\alpha(t)(x_0 - g(x)) + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)(F(s,x(s)) + Bu(s))ds, \quad t \in [0,b]. \tag{2.17}$$

DEFINITION 2.9. The system (1.1) is said to be *controllable on* $[0,b]$, if for every $x_0, y_0, x_1 \in X$, there exists a control $u \in L^2([0,b];U)$ such that a mild solution x of the system (1.1) satisfies $x(b) + g(x) = x_1$.

LEMMA 2.1 (Sadovskii fixed point theorem). *Let N be a condensing operator on a Banach space X, i.e. N is continuous and takes bounded sets into bounded sets, and $\gamma(N(D)) < \gamma(D)$ for every bounded set D of X with $\gamma(D) > 0$. If $N(S) \subset S$ for a convex, closed and bounded set S of X, then N has a fixed point in S (where $\gamma(\cdot)$ denotes the Kuratowski measure of noncompactness).*

3. Main results

To prove the main results, we give the following hypotheses:

(H_1) A is the infinitesimal generator of an α-order cosine family $C_\alpha(t)$ on X, and there exists a constant $M \geq 1$ such that

$$\|C_\alpha(t)\| \leq M. \tag{3.1}$$

(H_2) The linear operator $W : L^2([0,b];U) \to X$ defined by

$$Wu = \int_0^b P_\alpha(b-s)Bu(s)ds \tag{3.2}$$

has an induced inverse operator W^{-1} which takes values in $L^2([0,b];U)/\ker W$, and there exist constants M_1, M_2 such that

$$\|B\| \leq M_1, \qquad \|W^{-1}\| \leq M_2. \tag{3.3}$$

(H_3) $F : [0,b] \times X \to X$ satisfied the Carathéodory condition, i.e. $F(\cdot,x)$ is measurable for all $x \in X$, and $F(t,\cdot)$ is continuous for a.e. $t \in [0,b]$.

(H_4) There exists a function $F : [0,b] \times X \to X$, where X is compact.

(H_5) There exists a function $L_F(\cdot) \in L^1([0,b];\mathbb{R}_+)$ such that

$$\|F(t,x) - F(t,z)\| \leq L_F(t)\|x - z\|, \qquad x, z \in X. \tag{3.4}$$

(H_6) There exists a constant L_g such that

$$\|g(\phi) - g(\varphi)\| \leq L_g\|\phi - \varphi\|, \qquad \phi, \varphi \in C([0,b];X). \tag{3.5}$$

THEOREM 3.1. *Suppose* (H_1)–(H_6) *hold. Then the fractional differential control system* (1.1) *is controllable on* $[0,b]$ *provided that*

$$\left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\left(L_g + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\|L_F\|_{L^1}\right) < 1. \tag{3.6}$$

Proof: Using hypothesis (H_2), for an arbitrary function $x(\cdot) \in C([0,b];X)$, we define the control

$$\begin{aligned} u_x(t) = W^{-1}\{&x_1 - g(x) - C_\alpha(b)(x_0 - g(x)) - S_\alpha(b)y_0 \\ &- \int_0^b P_\alpha(b-s)F(s,x(s))ds\}(t), \qquad t \in [0,b]. \end{aligned} \tag{3.7}$$

Using this control, we shall show that the operator G defined by

$$(Gx)(t) = C_\alpha(t)(x_0 - g(x)) + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)(F(s,x(s)) + Bu_x(s))ds, \qquad t \in [0,b], \tag{3.8}$$

has a fixed point. For any $\delta > 0$, consider

$$N_\delta = \{x \in C([0,b];X) : \|x\| \le \delta\}. \tag{3.9}$$

It is obvious that N_δ is a bounded closed convex set in $C([0,b];X)$. We shall show that there exists a $\delta > 0$ such that $G(N_\delta) \subset N_\delta$. If this is not true, then for each $\delta > 0$, there exists a function $x^\delta \in N_\delta$, but $G(N_\delta)$ does not belong to N_δ, i.e. $\|G(x^\delta)(t)\| > \delta$ for some $t \in [0,b]$. From (H_1), (H_2), (H_3), (H_5), (H_6), (2.11), (2.12), we have

$$\begin{aligned}
\delta &< \|(Gx^\delta)(t)\| \\
&\le M\|x_0\| + M\|g(x^\delta)\| + Mb\|y_0\| + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \|F(s,x^\delta(s))\|ds \\
&\quad + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\Bigg(\|x_1\| + g(x^\delta) + M\|x_0\| + M\|g(x^\delta)\| \\
&\quad + Mb\|y_0\| + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \|F(s,x^\delta(s))\|ds\Bigg) \\
&\le \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\|x_1\| + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\|x_0\| + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)Mb\|y_0\| \\
&\quad + M(L_g\|x^\delta\| + \|g(0)\|) + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b (L_1(s)\|x^\delta(s)\| + \|F(s,0)\|)ds \\
&\quad + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\Bigg((M+1)(L_g\|x^\delta\| + \|g(0)\|) \\
&\quad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b (L_F(s)\|x^\delta(s)\| + \|F(s,0)\|)ds\Bigg) \\
&\le \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\|x_1\| + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\|x_0\| + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)Mb\|y_0\| \\
&\quad + \left(1 + \frac{(M+1)M_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\|g(0)\| + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)ML_g\|x^\delta\| \\
&\quad + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|L_F\|_{L^1}\|x^\delta\| \\
&\quad + \left(1 + \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \|F(s,0)\|ds.
\end{aligned} \tag{3.10}$$

From (3.10) and $\|x^\delta\| \le \delta$, it follows that

$$\begin{aligned}
\delta &< \|(Gx^\delta)(t)\| \\
&\le \frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\|x_1\| + \left(1+\frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\|x_0\| + \left(1+\frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)Mb\|y_0\| \\
&\quad + \left(1+\frac{(M+1)M_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\|g(0)\| \\
&\quad + \left(1+\frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \|F(s,0)\|ds \\
&\quad + \left(1+\frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\left(L_g + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\|L_F\|_{L^1}\right)\delta. \qquad (3.11)
\end{aligned}$$

Dividing both sides of (3.11) by δ and taking the limit as $\delta \to \infty$, we obtain

$$\left(1+\frac{MM_1M_2b^\alpha}{\Gamma(\alpha)}\right)M\left(L_g + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\|L_F\|_{L^1}\right) \ge 1. \qquad (3.12)$$

This contradicts with (3.6). Therefore for some $\delta > 0$, $G(N_\delta) \subset N_\delta$.

Now we decompose the operator G as two operators G_1 and G_2 ($G = G_1 + G_2$), where

$$(G_1x)(t) = C_\alpha(t)(x_0 - g(x)) + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)Bu_x(s)ds, \qquad t \in [0,b],$$

$$(G_2x)(t) = \int_0^t P_\alpha(t-s)F(s,x(s))ds, \qquad t \in [0,b].$$

We shall show that G_1 is a contraction operator, and G_2 is a completely continuous operator. Let $x, y \in N_\delta$. From (H_1)–(H_6), it follows that

$$\begin{aligned}
&\|(G_1x)(t) - (G_1y)(t)\| \\
&\qquad \le ML_g\|x-y\| + \frac{M^2M_1M_2b^{\alpha-1}L_g}{\Gamma(\alpha)}\|x-y\| \\
&\qquad\quad + \frac{M^2M_1M_2b^{2\alpha-1}}{(\Gamma(\alpha))^2}\|L_F\|_{L^1}\|x-y\| \\
&\qquad \le \left(\left(1+\frac{MM_1M_2b^{\alpha-1}}{\Gamma(\alpha)}\right)ML_g + \frac{M^2M_1M_2b^{2\alpha-1}}{(\Gamma(\alpha))^2}\|L_F\|_{L^1}\right)\|x-y\|.
\end{aligned}$$

Hence, by (3.6), it is clear that G_1 is a contraction operator.

Next we prove that G_2 is completely continuous. Let $x_n \in N_\delta$ with $x_n \to x$ in N_δ. By (H_3), (H_5), it follows that for $s \in [0,b]$,

$$\begin{gathered}
F(s,x_n(s)) \to F(s,x(s)), \qquad n \to \infty, \\
\|F(s,x_n(s) - F(s,x(s))\| \le 2\delta L_F(s)
\end{gathered}$$

From the dominated convergence theorem, it is easy to show that G_2 is continuous on N_δ. To obtain the compactness of G_2, by the Ascoli–Arzela theorem, we need to prove that $G_2(N_\delta) \subset C([0,b];X)$ is equicontinuous and $\{G_2(N_\delta)(t)\}_{t\in[0,b]}$ is precompact. For any $x \in N_\delta$ and $h > 0$ we have

$$\begin{aligned}
&\|G_2x(t+h) - G_2x(t)\| \\
&\quad = \left\| \int_0^{t+h} P_\alpha(t+h-s)F(s,x(s))ds - \int_0^t P_\alpha(t-s)F(s,x(s))ds \right\| \\
&\quad \le \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_t^{t+h} \|F(s,x(s))\|ds \\
&\qquad + \int_0^t \|P_\alpha(t+h-s)F(s,x(s)) - P_\alpha(t-s)F(s,x(s))\|ds \\
&\quad \le \frac{Mb^{\alpha-1}\delta}{\Gamma(\alpha)} \int_t^{t+h} L_F(s)ds + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_t^{t+h} \|F(s,0)\|ds \\
&\qquad + \int_0^t \|P_\alpha(t+h-s)F(s,x(s)) - P_\alpha(t-s)F(s,x(s))\|ds. \qquad (3.13)
\end{aligned}$$

Since $P_\alpha(t)$ is strongly continuous for $t \ge 0$ and F is compact, then from (3.13), it follows that $G_2(N_\delta) \subset C([0,b];X)$ is equicontinuous. Since F is compact, then the set $P_\alpha(t-s)F(s,x(s)) : t,s \in [0,b],\ x \in N_\delta$ is precompact. This fact together with

$$G_2(N_\delta)(t) \subset t\overline{\mathrm{conv}}\{P_\alpha(t-s)F(s,x(s)) : s,t \in [0,b],\ \ x \in N_\delta\},\quad t \in [0,b]$$

yield that $G_2(N_\delta)(t) \subset X$ is precompact. Therefore, $G = G_1 + G_2$ is a condensing operator on N_δ. By Lemma 2.1, G has a fixed point x on N_δ. It is easy to prove that x is a mild solution of the system (1.1) satisfying $x(b) + g(x) = x_1$. The proof is complete. □

REMARK 3.1. In order to describe various real world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive fractional differential equations have been used to the system model. The controllability for fractional differential equations with nonlocal conditions and impulses will be a problem.

4. An example

Consider the following fractional differential system

$$\begin{cases} {}^C D_t^\alpha x(t,z) = \dfrac{\partial^2 x}{\partial z^2}(t,z) + h(t,x(t,z)) + Bu(t,z), \quad z \in (0,\pi), \quad t \in [0,b], \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,b], \\ x(0,z) + g(x) = x_0(z), \quad \dfrac{\partial x}{\partial t}(0,z) = y_0(z), \quad z \in (0,\pi), \end{cases} \qquad (4.1)$$

where $\alpha \in (1,2]$.

Let $X = L^2(0, \pi)$ and let $A : D(A) \subset X \to X$ be defined by

$$A\mu = \mu''$$

where $D(A) = \{\mu : \mu$ and μ' are absolutely continuous, $\mu'' \in X,\ \mu(0) = \mu(\pi) = 0\}$. Then

$$A\mu = \sum_{n=1}^{\infty} -n^2(\mu, \mu_n), \qquad \mu \in D(A),$$

where $\mu_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1, 2, \ldots,$ is the orthonormal set of eigenvalues of A. It is easy to show that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, and

$$C(t)\mu = \sum_{n=1}^{\infty} \cos nt(\mu, \mu_n)\mu_n, \qquad \mu \in X, \qquad t \in \mathbb{R}.$$

For $\alpha = 2$, set $x(t)(z) = x(t, z)$, $F(t, x(t))(z) = h(t, x(t, z))$, $(Bu)(t)(z) = Bu(t, z)$. Then the problem (4.1) can be rewritten as

$$\begin{cases} x''(t) = Ax(t) + F(t, x(t)) + Bu(t), & t \in [0, b], \\ x(0) + g(x) = x_0, \qquad x'(0) = y_0, \end{cases} \tag{4.2}$$

Therefore, by Theorem 3.1, if the hypotheses (H_2)–(H_6) are satisfied, the differential system (1.1) is controllable on $[0, b]$.

For $\alpha \in (1, 2)$, since A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, from the subordinate principle (see Theorem 3.1, [19]), it follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_\alpha(t)$ such that $C_\alpha(0) = I$, and

$$C_\alpha(t) = \int_0^\infty \varphi_{t,\alpha/2}(s)C(s)ds, \qquad t > 0,$$

where $\varphi_{t,\alpha/2}(s) = t^{-\alpha/2}\phi_{\alpha/2}(st^{-\alpha/2})$, and

$$\phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \qquad 0 < \gamma < 1.$$

Set $x(t)(z) = x(t, z)$, $F(t, x(t))(z) = h(t, x(t, z))$, $(Bu)(t)(z) = Bu(t, z)$. Then the problem (4.1) can be rewritten as

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + F(t, x(t)) + Bu(t), & t \in [0, b], \\ x(0) + g(x) = x_0, \qquad x'(0) = y_0, \end{cases} \tag{4.3}$$

Therefore, by Theorem 3.1, if the hypotheses (H_2)–(H_6) are satisfied, the fractional differential system (1.1) is controllable on $[0, b]$.

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