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Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives ☆

Li Kexue, Peng Jigen *, Jia Junxiong

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China

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Abstract

In this paper, we are concerned with Cauchy problems of fractional differential equations with Riemann–Liouville fractional derivatives in infinite-dimensional Banach spaces. We introduce the notion of fractional resolvent, obtain some its properties, and present a generation theorem for exponentially bounded fractional resolvents. Moreover, we prove that a homogeneous α-order Cauchy problem is well posed if and only if its coefficient operator is the generator of an α-order fractional resolvent, and we give sufficient conditions to guarantee the existence and uniqueness of weak solutions and strong solutions of an inhomogeneous α-order Cauchy problem.

Keywords: Riemann–Liouville fractional derivative; Fractional resolvent; Well-posedness; Cauchy problem

1. Introduction

In this work we study the well-posedness of the following homogeneous α-order Cauchy problem

$$\begin{cases} D_t^\alpha u(t) = Au(t), & t > 0, \\ (g_{2-\alpha} * u)(0) = 0, & (g_{2-\alpha} * u)'(0) = x, \end{cases} \tag{1}$$

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* Corresponding author.
E-mail addresses: kexueli@gmail.com (K. Li), jgpeng@mail.xjtu.edu.cn (J. Peng), jiajunxiong@163.com (J. Jia).

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and the existence and uniqueness of the following inhomogeneous α-order Cauchy problem

$$\begin{cases} D_t^\alpha u(t) = Au(t) + f(t), & t > 0, \\ (g_{2-\alpha} * u)(0) = 0, & (g_{2-\alpha} * u)'(0) = x, \end{cases} \tag{2}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a closed densely defined linear operator, $(X, \|\cdot\|)$ is a Banach space, $D(A)$ is the domain of A endowed with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$, D_t^α is the α-order Riemann–Liouville fractional derivative operator, $g_{2-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$ for $t > 0$ and $g_{2-\alpha}(t) = 0$ for $t \leqslant 0$, $f : [0, T] \to X$, $x \in X$.

Fractional Cauchy problems are useful in physics to model anomalous diffusion. The observation of a universal response of electromagnetic, acoustic, and mechanical influence shows the existence of some transfer processes in a medium, which are not described by a usual diffusion equation. Nigmatullin [6] introduced a generalization diffusion equation of the non-Markovian type

$$D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \leqslant \alpha \leqslant 2,$$

where D_t^α is the Riemann–Liouville fractional derivative operator of order α, the case $\alpha = 0$ corresponds to an elliptic equation, the case $\alpha = 1$ corresponds to a parabolic equation, and $\alpha = 2$ is the case of a hyperbolic one. Zaslavsky [9] introduced the fractional kinetic equation

$$\frac{\partial^\beta g(x,t)}{\partial t^\beta} = Lg(x,t) + p_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

for Hamiltonian chaos, where $0 < \beta < 1$, L is the generator of a Feller semigroup $\{T(t)\}_{t \geqslant 0}$ and $p_0 \in C^\infty(R^1)$ is an arbitrary initial condition. Here $\partial^\beta g(x,t)/\partial t^\beta$ is the inverse Laplace transform of $s^\beta \tilde{g}(x,s)$, where $\tilde{g}(x,s) = \int_0^\infty e^{-st} g(x,t)\,dt$ is the usual Laplace transform. Eidelman and Kochubei [8] considered an evolution equation with the regularized fractional derivative of order $\alpha \in (0,1)$ with respect to the time variable, and a uniformly elliptic operator with variable coefficients acting in the spatial variables, such equations describe diffusion on inhomogeneous fractals. Meerschaert et al. [11] developed classical solutions and stochastic analogues for fractional Cauchy problems in a bounded domain $D \subset R^d$ with Dirichlet boundary conditions, stochastic solutions are constructed via an inverse stable subordinator whose scaling index corresponds to the order of the fractional time derivative. Orsingher and Beghin [10] analyzed the solutions to fractional diffusion equations of order $\alpha \in (0,2]$, and interpreted them as densities of the composition of various types of stochastic processes.

The concept of solution operators plays an important role in the theory of fractional abstract Cauchy problem. Bazhlekova [1] studied the following fractional order abstract Cauchy problem

$$\begin{cases} \mathbf{D}_t^\alpha u(t) = Au(t), & t \geqslant 0, \\ u(0) = x, & u^{(k)}(0) = 0, \quad k = 1, \dots, m-1, \end{cases} \tag{3}$$

where $m = \lceil \alpha \rceil$, the smallest integer greater than or equal to α, $\mathbf{D}_t^\alpha$ is the Caputo fractional derivative operator defined by

$$\mathbf{D}_t^\alpha u(t) = D_t^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0) \right). \tag{4}$$

The notion of solution operator is introduced in [1] as follows:

Definition 1.1. Let $\alpha > 0$. A family $\{T_\alpha(t)\}_{t \geqslant 0}$ of bounded linear operators on X is called a solution operator for (3) if the following three conditions are satisfied:

(a) $T_\alpha(t)$ is strongly continuous for $t \geqslant 0$ and $T_\alpha(0) = I$ (the identity operator);

(b) $T_\alpha(t)T_\alpha(s) = T_\alpha(s)T_\alpha(t)$ for all $t \geqslant 0$, $s \geqslant 0$;

(c) for all $x \in D(A)$, $t \geqslant 0$, $T_\alpha(t)x$ is a solution of

$$u(t) = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)\, ds.$$

Remark 1.2. The concept of solution operator for (3) is followed from Definition 1.3 in [2], solution operator is also called α-times resolvent family in some references, see for examples [12,13].

Chen and Li [5] presented a purely algebraic notion, α-resolvent operator function.

Definition 1.3. Let $\alpha > 0$. A function $\{S_\alpha(t)\}_{t \geqslant 0}$ of bounded linear operators on X is called an α-resolvent operator function if the following three conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous for $t \geqslant 0$ and $S_\alpha(0) = I$ (the identity operator);

(b) $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t)$ for all $t \geqslant 0$, $s \geqslant 0$;

(c) for all $t, s \geqslant 0$, the following equality holds

$$S_\alpha(s) J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s) S_\alpha(t) = J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s),$$

where J_t^α is the Riemann–Liouville fractional integral operator.

It is proved in [5] that a family $\{S_\alpha(t)\}_{t \geqslant 0}$ is an α-resolvent operator function if and only if it is a solution operator for some fractional Cauchy problem.

Practical problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions. Initial conditions for the Caputo fractional derivatives are expressed in terms of initials of integer order derivatives. However, on some examples from the field of viscoelasticity, Heymans and Podlubny [7] demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives or integrals, and it is possible to obtain initial values for such initial conditions by appropriate measurements.

The purpose of this paper is to develop an operator theory to study fractional abstract Cauchy problems with the Riemann–Liouville fractional derivatives. In Section 2, we present some basic definitions and preliminary facts which will be used in the next sections. In Section 3, we introduce the notion of fractional resolvent, obtain some its properties and give a generation theorem for exponentially bounded fractional resolvents. Two examples will end this section. In Section 4, we prove that (1) is well posed if and only if its coefficient operator A generates an α-order fractional resolvent. In Section 5, we obtain the existence and uniqueness of weak solutions and strong solutions of (2) under general conditions.

2. Preliminaries

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. $R_+ = [0, \infty)$. N denotes the set of natural numbers, $N_0 = N \cup \{0\}$. $\mathbb{C}$ denotes the set of complex numbers. For $z \in \mathbb{C}$, $Re\, z$ denotes the real part of z, $\arg z$ denotes the principal value of the argument of z, $\arg z \in (-\pi, \pi]$. $z^\alpha = |z|^\alpha e^{i \arg z^\alpha}$. Let $(X, \|\cdot\|)$ be a Banach space. $B(X)$ denotes the space of all bounded linear operators from X into X. If A is a linear operator in X, then $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ denotes the resolvent operator of A. For $p \geqslant 1$, $L^p((0,T); X)$ denotes the space of X-valued Bochner integrable functions $u : (0,T) \to X$ with the norm $\|u\|_{L^p((0,T);X)} = (\int_0^\infty \|u(t)\|^p\, dt)^{1/p}$, it is a Banach space. Let $J = (a, b)$, $-\infty \leqslant a \leqslant b \leqslant +\infty$. By $C(J; X)$, resp. $C^n(J; X)$, we denote the spaces of functions $u : J \to X$, which are continuous, resp. n-times continuously differentiable. Then $C([0,T]; X)$ are Banach space of continuous functions $u : [0,T] \to X$ with the norm $\|u\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|u(t)\|$. $C^\infty(R; X)$ consists of all infinitely differentiable functions. By $*$ we denote the convolution of functions

$$f(t) * g(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau, \quad t \geqslant 0.$$

Let $n \in N$, $1 \leqslant p < \infty$. Let $I = (0, T)$, or $I = [0, T]$, or $I = R_+$. The Sobolev spaces can be defined as follows (see [14, Appendix]):

$$W^{n,p}(I; X) = \left\{ u \;\middle|\; \exists \varphi \in L^p(I; X)\colon\ u(t) = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + \frac{t^{n-1}}{(n-1)!} * \varphi(t),\ t \in I \right\}.$$

Note that, $\varphi(t) = u^{(n)}(t)$, $c_k = u^{(k)}(0)$.

For $\omega \in R$, let

$$\mathrm{Lip}_\omega(R_+; X) = \left\{ G : R_+ \to X\colon\ G(0) = 0,\ \|G\|_{\mathrm{Lip}_\omega(R_+;X)} = \sup_{t > s \geqslant 0} \frac{\|G(t) - G(s)\|}{\int_s^t e^{\omega r}\, dr} < \infty \right\},$$

$$C_W^\infty\big((\omega, \infty); X\big) = \left\{ r \in C^\infty\big((\omega, \infty); X\big)\colon\ \|r\|_W = \sup_{\lambda > \omega,\, k \in N_0} \frac{(\lambda - \omega)^{k+1}}{k!} \big\| r^{(k)}(\lambda) \big\| < \infty \right\},$$

$$\begin{aligned} C_\omega^1(R_+; X) &= \left\{ f \in C^1(R_+; X)\colon\ f(0) = 0,\ \sup_{t > 0} \big\| e^{-\omega t} f'(t) \big\| < \infty \right\} \\ &= C^1(R_+; X) \cap \mathrm{Lip}_\omega(R_+; X). \end{aligned}$$

For $\beta \geqslant 0$, let

$$g_\beta(t) = \begin{cases} \frac{t^{\beta - 1}}{\Gamma(\beta)}, & t > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.1. The α-order Riemann–Liouville fractional integral is defined by

$$J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \tag{6}$$

where $u \in L^1((0,T);X)$.

Set $J_t^0 u(t) = u(t)$. The fractional integral operators $\{J_t^\alpha\}_{\alpha \geqslant 0}$ satisfy the semigroup property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \quad \alpha, \beta \geqslant 0. \tag{7}$$

Definition 2.2. The α-order Riemann–Liouville fractional derivative is defined by

$$D_t^\alpha u(t) = \frac{d^m}{dt^m}(g_{m-\alpha} * u)(t) = \frac{d^m}{dt^m} J_t^{m-\alpha} u(t), \tag{8}$$

where $u \in L^1((0,T);X)$, $g_{m-\alpha} * u \in W^{m,1}((0,T);X)$. When $\alpha = m$, $m \in N$, define $D_t^m = \frac{d^m}{dt^m}$.

The Riemann–Liouville derivative operator D_t^α is a left inverse of the integral operator J_t^α but in general not a right inverse, that is,

$$D_t^\alpha J_t^\alpha u = u, \quad u \in L^1\big((0,T);X\big), \tag{9}$$

and

$$\big(J_t^\alpha D_t^\alpha u\big)(t) = u(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * u)^{(k)}(0) g_{\alpha+k+1-m}(t), \tag{10}$$

where $u \in L^1((0,T);X)$, $g_{m-\alpha} * u \in W^{m,1}((0,T);X)$.

The Laplace transform formula for the Riemann–Liouville fractional integral is defined by

$$L\big\{J_t^\alpha u(t)\big\} = \frac{1}{\lambda^\alpha} \hat{u}(\lambda), \tag{11}$$

where $\hat{u}(\lambda)$ is the Laplace of u defined by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt, \quad Re\,\lambda > \omega. \tag{12}$$

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C}.$$

The Mittag-Leffler function is related to the Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}\big(\omega t^\alpha\big)\, dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\,\lambda > |\omega|^{1/\alpha}, \tag{13}$$

and to the following asymptotic formulas as $z \to \infty$. If $0 < \alpha < 2$, $\beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\big(z^{1/\alpha}\big) + \varepsilon_{\alpha,\beta}(z), \qquad |\arg z| \leqslant \frac{1}{2}\alpha\pi, \tag{14}$$

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \qquad \big|\arg(-z)\big| < \left(1 - \frac{1}{2}\alpha\right)\pi, \tag{15}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + O\big(|z|^{-N}\big), \quad z \to \infty.$$

3. Fractional resolvent

In this section, we introduce the notion of fractional resolvent, and obtain some its properties, give a generation theorem for exponentially bounded fractional resolvents and present two examples illustrating the abstract theory.

Definition 3.1. Let $1 \leqslant \alpha \leqslant 2$. A family $\{T_\alpha(t)\}_{t\geqslant 0} \subset B(X)$ is called a strongly continuous fractional resolvent of order α (or α-order fractional resolvent, for short) if it satisfies the following assumptions:

(a) $T_\alpha(t)$ is strongly continuous on R_+, and

$$\lim_{t\to 0+} \frac{T_\alpha(t)}{t^{\alpha-1}} x = \frac{1}{\Gamma(\alpha)} x \quad \text{for all } x \in X; \tag{16}$$

(b)

$$T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s) \quad \text{for all } t, s \geqslant 0;$$

(c) the equality holds

$$T_\alpha(s) J_t^\alpha T_\alpha(t) - J_s^\alpha T_\alpha(s) T_\alpha(t) = \frac{s^{\alpha-1}}{\Gamma(\alpha)} J_t^\alpha T_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^\alpha T_\alpha(s) \tag{17}$$

for all $t, s \geqslant 0$.

The linear operator A defined by

$$D(A) = \left\{ x \in X\colon \lim_{t\to 0+} \frac{T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)} x}{t^{2\alpha-1}} \text{ exists} \right\}$$

and

$$Ax = \Gamma(2\alpha) \lim_{t\to 0+} \frac{T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x}{t^{2\alpha-1}} \quad \text{for } x \in D(A)$$

is the generator of the α-order fractional resolvent $T_\alpha(t)$, $D(A)$ is the domain of A.

Remark 3.2. The integrals in (17) are understood strongly in the sense of Bochner.

Definition 3.3. The α-order fractional resolvent $T_\alpha(t)$ is called exponentially bounded if there exist constants $M \geqslant 1$, $\omega \geqslant 0$ such that

$$\|T_\alpha(t)\| \leqslant Me^{\omega t}, \quad t \geqslant 0. \tag{18}$$

An operator A is said to belong to $\mathcal{C}^\alpha(M, \omega)$ if A generates an α-order fractional resolvent $T_\alpha(t)$ satisfying (18). Denote $\mathcal{C}^\alpha(\omega) = \bigcup\{\mathcal{C}^\alpha(M, \omega); M \geqslant 1\}$.

Proposition 3.4. *Let A be the generator of an α-order fractional resolvent $T_\alpha(t)$ on X. Then $\sup_{t\in[0,T]} \|T_\alpha(t)\| < \infty$ for every $T > 0$.*

Proof. Any given $T > 0$, defined a mapping $S : X \to C([0, T]; X)$ by $(Sx)t = T_\alpha(t)x$, $t \in [0, T]$. It is easy to show that S is linear and closed, hence by the closed graph theorem S is bounded, there exists a constant $M > 0$ such that $\sup_{t\in[0,T]} \|T_\alpha(t)x\| \leqslant M\|x\|$ for all $x \in X$. Therefore, by the uniform boundedness theorem it follows that $\sup_{t\in[0,T]} \|T_\alpha(t)\| < \infty$. □

Proposition 3.5. *Let $T_\alpha(t)$ be an α-order fractional resolvent such that $\|T_\alpha(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for some $M \geqslant 1$, $\omega \geqslant 0$. Let $R(\lambda) = \int_0^\infty e^{-\lambda t} T_\alpha(t)\,dt$, $\lambda > \omega$. Then for $\lambda, \mu > \omega$,*

$$R(\lambda) - R(\mu) = \left(\mu^\alpha - \lambda^\alpha\right) R(\mu)R(\lambda). \tag{19}$$

Proof. Since $J_t^\alpha T_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * T_\alpha(t)$ and $R(\lambda) = \int_0^\infty e^{-\lambda t} T_\alpha(t)\,dt$, $\lambda > \omega$, it follows from the convolution theorem of Laplace transforms that

$$\int_0^\infty e^{-\lambda t} J_t^\alpha T_\alpha(t)\,dt = \frac{R(\lambda)}{\lambda^\alpha}. \tag{20}$$

Taking Laplace transform with respect to t in both sides of (17), we have

$$T_\alpha(s)\frac{R(\lambda)}{\lambda^\alpha} - J_s^\alpha T_\alpha(s)R(\lambda) = \frac{s^{\alpha-1}R(\lambda)}{\Gamma(\alpha)\lambda^\alpha} - \frac{1}{\lambda^\alpha} J_s^\alpha T_\alpha(s). \tag{21}$$

For $\mu > \omega$, $R(\mu) = \int_0^\infty e^{-\mu s} T_\alpha(s)\,ds$. Then

$$\int_0^\infty e^{-\mu s} J_s^\alpha T_\alpha(s)\,ds = \frac{R(\mu)}{\mu^\alpha}. \tag{22}$$

Taking Laplace transform with respect to s in both sides of (21), we have

$$\frac{R(\mu)R(\lambda)}{\lambda^\alpha}-\frac{R(\mu)R(\lambda)}{\mu^\alpha}=\frac{R(\lambda)}{\mu^\alpha\lambda^\alpha}-\frac{R(\mu)}{\lambda^\alpha\mu^\alpha}.$$

The proof is complete. □

Remark 3.6. When $\alpha=1$, (19) reduces to $R(\lambda)-R(\mu)=(\mu-\lambda)R(\mu)R(\lambda)$. This is equivalent to $T_1(s+t)=T_1(s)T_1(t)$, by Proposition 2.2 in [4], then $T_1(t)$ is a C_0-semigroup.

Proposition 3.7. *Let $\{T_\alpha(t)\}_{t\geqslant 0}$ be an α-order fractional resolvent on X, and A its generator. Then*

(a) *$T_\alpha(t)D(A)\subset D(A)$ and $AT_\alpha(t)x=T_\alpha(t)Ax$ for each $x\in D(A)$.*

(b) *For each $x\in X$, $t\geqslant 0$,*

$$T_\alpha(t)x=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x+AJ_t^\alpha T_\alpha(t)x. \tag{23}$$

(c) *For each $x\in D(A)$, $t\geqslant 0$,*

$$T_\alpha(t)x=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x+J_t^\alpha T_\alpha(t)Ax, \tag{24}$$

and

$$T_\alpha(\cdot)x\in C^1\big((0,\infty);X\big). \tag{25}$$

(d) *A is closed and densely defined.*

Proof. (a) Let $x\in D(A)$, for $t\geqslant 0$, $s\geqslant 0$, from $T_\alpha(t)T_\alpha(s)=T_\alpha(s)T_\alpha(t)$ it follows that

$$T_\alpha(s)T_\alpha(t)x-\frac{s^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(t)x=T_\alpha(t)\left(T_\alpha(s)x-\frac{s^{\alpha-1}}{\Gamma(\alpha)}x\right),$$

since $T_\alpha(t)$ is bounded, then

$$\Gamma(2\alpha)\lim_{s\to 0+}\frac{T_\alpha(s)T_\alpha(t)x-\frac{s^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(t)x}{s^{2\alpha-1}}=T_\alpha(t)Ax.$$

That is, $T_\alpha(t)x\in D(A)$ and $AT_\alpha(t)x=T_\alpha(t)Ax$.

(b) For each $x\in X$,

$$\begin{aligned}\left\|\Gamma(2\alpha)\frac{J_s^\alpha T_\alpha(s)x}{s^{2\alpha-1}}-x\right\|&=\left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^s s^{1-2\alpha}(s-\tau)^{\alpha-1}T_\alpha(\tau)x\,d\tau-x\right\|\\&=\left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^1 s^{1-\alpha}(1-\tau)^{\alpha-1}T_\alpha(s\tau)x\,d\tau-x\right\|\end{aligned}$$

$$= \left\| \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 s^{1-\alpha}\Gamma(\alpha)(1-\tau)^{\alpha-1}T_\alpha(s\tau)x\,d\tau - x \right\|$$

$$= \left\| \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}\Gamma(\alpha)\frac{T_\alpha(s\tau)}{(s\tau)^{\alpha-1}}x\,d\tau \right.$$

$$\left. - \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}x\,d\tau \right\|$$

$$\leqslant \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}\,d\tau \sup_{\tau\in[0,1]} \left\| \Gamma(\alpha)\frac{T_\alpha(s\tau)}{(s\tau)^{\alpha-1}}x - x \right\|$$

$$\leqslant \Gamma(\alpha) \sup_{\tau\in[0,1]} \left\| \frac{T_\alpha(s\tau)}{(s\tau)^{\alpha-1}}x - \frac{1}{\Gamma(\alpha)}x \right\|. \tag{26}$$

By (a) of Definition 3.1, we have

$$\sup_{\tau\in[0,1]} \left\| \frac{T_\alpha(s\tau)}{(s\tau)^{\alpha-1}}x - \frac{1}{\Gamma(\alpha)}x \right\| \to 0 \quad \text{as } s \to 0+.$$

Then

$$\left\| \Gamma(2\alpha)\frac{J_s^\alpha T_\alpha(s)x}{s^{2\alpha-1}} - x \right\| \to 0 \quad \text{as } s \to 0+. \tag{27}$$

By (c) of Definition 3.1, we have

$$\left(T_\alpha(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} \right) J_t^\alpha T_\alpha(t)x = J_s^\alpha T_\alpha(s)\left(T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x \right), \tag{28}$$

thus,

$$A J_t^\alpha T_\alpha(t)x = \Gamma(2\alpha)\lim_{s\to 0+} \frac{(T_\alpha(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)})J_t^\alpha T_\alpha(t)x}{s^{2\alpha-1}}$$

$$= \Gamma(2\alpha)\lim_{s\to 0+} \frac{J_s^\alpha T_\alpha(s)(T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x)}{s^{2\alpha-1}}$$

$$= T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x, \tag{29}$$

therefore (b) holds.

(c) For $x \in D(A)$, the limit

$$\lim_{s\to 0+} \frac{T_\alpha(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}x}{s^{2\alpha-1}}$$

exists, then the function

$$f(s) = \frac{T_\alpha(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}x}{s^{2\alpha-1}}$$

is bounded for sufficiently small $s > 0$. For $t > 0$, by the dominated convergence theorem, we obtain

$$\begin{aligned} T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x &= A J_t^\alpha T_\alpha(t)x \\ &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \lim_{s\to 0+} \frac{T_\alpha(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)}}{s^{2\alpha-1}} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)x\,d\tau \\ &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \lim_{s\to 0+} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau) \frac{T_\alpha(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}x}{s^{2\alpha-1}}\,d\tau \\ &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau) \lim_{s\to 0+} \frac{T_\alpha(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}x}{s^{2\alpha-1}}\,d\tau \\ &= J_t^\alpha T_\alpha(t)Ax. \end{aligned} \tag{30}$$

From (30) it follows that $T_\alpha(t)x$ is differentiable on R_+ for all $x \in D(A)$ and

$$\begin{aligned} \frac{d}{dt}T_\alpha(t)x &= \frac{d}{dt}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}x + J_t^\alpha T_\alpha(t)Ax\right) \\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{d}{dt}J_t^\alpha T_\alpha(t)Ax \\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^{\alpha-1} T_\alpha(t)Ax \\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} T_\alpha(\tau)Ax\,d\tau \\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} T_\alpha(t\tau)Ax\,d\tau. \end{aligned} \tag{31}$$

Now use the dominated convergence theorem to (31) to conclude that

$$T_\alpha(\cdot)x \in C^1\big((0,\infty); X\big), \quad x \in D(A).$$

(d) Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From part (c), we have

$$\begin{aligned} T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x &= \lim_{n\to\infty} T(t)x_n - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x_n \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)Ax_n\, d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)y\, d\tau \\ &= J_t^\alpha T_\alpha(t)y. \end{aligned} \tag{32}$$

Using (27), we have

$$\begin{aligned} Ax &= \Gamma(2\alpha) \lim_{t\to 0+} \frac{T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x}{t^{2\alpha-1}} \\ &= \Gamma(2\alpha) \lim_{t\to 0+} \frac{J_t^\alpha T_\alpha(t)y}{t^{2\alpha-1}} \\ &= y. \end{aligned} \tag{33}$$

The closeness of A is proved.

For every $x \in X$, set $x_t = J_t^\alpha T_\alpha(t)x$, from (29) it follows that $x_t \in D(A)$, and by (27) we have $\Gamma(2\alpha)t^{1-2\alpha}x_t \to x$ as $t \to 0+$. Thus $\overline{D(A)} = X$. □

Proposition 3.8. *Let $T_\alpha(t)$ and $S_\alpha(t)$ be α-order fractional resolvents with generators A and B respectively. If $A = B$ then $T_\alpha(t) = S_\alpha(t)$ for $t \geqslant 0$.*

Proof. For $x \in D(A)$, by properties (c), (a) of Proposition 3.7, we obtain

$$\begin{aligned} \frac{t^{\alpha-1}}{\Gamma(\alpha)} * S_\alpha(t)x &= \big(T_\alpha(t) - J_t^\alpha AT_\alpha(t)\big) * S_\alpha(t)x \\ &= T_\alpha(t) * S_\alpha(t)x - g_\alpha(t) * AT_\alpha(t) * S_\alpha(t)x \\ &= T_\alpha(t) * S_\alpha(t)x - g_\alpha(t) * T_\alpha(t) * AS_\alpha(t)x \\ &= T_\alpha(t) * \big(S_\alpha(t)x - g_\alpha(t) * AS_\alpha(t)x\big) \\ &= T_\alpha(t) * \frac{t^{\alpha-1}}{\Gamma(\alpha)}x \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} * T_\alpha(t)x, \end{aligned}$$

therefore $T_\alpha(t)x = S_\alpha(t)x$ for every $x \in D(A)$, $t \geqslant 0$, from density of $D(A)$, it follows that $T_\alpha(t) = S_\alpha(t)$. □

Lemma 3.9. *(See [3, Theorem 2.4.1].) Let $\omega \in R$. The Laplace–Stieltjes transform is an isometric isomorphism of* $\mathrm{Lip}_\omega(R_+; X)$ *onto* $C_W^\infty((\omega,\infty); X)$. *In particular, for $M > 0$ and $r \in C_W^\infty((\omega,\infty); X)$, the following are equivalent*:

(i) $\|(\lambda-\omega)^{k+1}\frac{1}{k!}r^{(k)}(\lambda)\| \leqslant M$ $(\lambda > \omega,\ k \in N_0)$.

(ii) *There exists $G : R_+ \to X$ satisfying $G(0) = 0$ and $\|G(t+h) - G(t)\| \leqslant M\int_t^{t+h} e^{\omega r}\,dr$ $(t, h \geqslant 0)$, such that $r(\lambda) = \int_0^\infty e^{-\lambda t}\,dG(t)$ for all $\lambda > \omega$.*

Lemma 3.10. *(See [3, Lemma 3.3.3].) For $\omega \in R$, the space $C_\omega^1(R_+; X)$ is a closed subspace of* $\mathrm{Lip}_\omega(R_+; X)$. *In particular, if $S \in \mathrm{Lip}_\omega(R_+; B(X))$, then $\{x \in X\colon\ S(\cdot)x \in C^1(R_+; X)\}$ is a closed subspace of X.*

It is well known that both strongly continuous semigroups and strongly continuous cosine functions are necessarily exponentially bounded. However, whether an α-order fractional resolvent $T_\alpha(t)$ is exponentially bounded is unknown in general. In fact, if $T_\alpha(t)$ is exponentially bounded, we have the following generation theorem.

Theorem 3.11. *Let $1 \leqslant \alpha \leqslant 2$. Let A be a closed linear operator with domain $D(A)$ dense in a Banach space X. Then $A \in \mathcal{C}^\alpha(M,\omega)$ if and only if $(\omega^\alpha,\infty) \subset \rho(A)$ and*

$$\left\|\frac{d^k}{d\lambda^k}\big(R\big(\lambda^\alpha, A\big)\big)\right\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}, \quad \lambda > \omega,\ k \in N_0. \tag{34}$$

Proof. (Necessity) Suppose $A \in \mathcal{C}^\alpha(M,\omega)$ and $T_\alpha(t)$ is the fractional resolvent generated by A. For $x \in X$, $\lambda > \omega$, we define

$$R(\lambda)x = \int_0^t e^{-\lambda t} T_\alpha(t)x\,dt. \tag{35}$$

Since $\|T_\alpha(t)\| \leqslant Me^{\omega t}$, $R(\lambda)$ is well defined for every λ satisfying $\lambda > \omega$. By properties (b), (c), (d) of Proposition 3.7 and the equality (11), it follows that

$$\lambda^\alpha R(\lambda)x - x = AR(\lambda)x, \quad x \in X,$$
$$\lambda^\alpha R(\lambda)x - x = R(\lambda)Ax, \quad x \in D(A).$$

Thus, $\lambda^\alpha I - A$ is invertible and $R(\lambda) = R(\lambda^\alpha, A)$, that is

$$\big\{\lambda^\alpha\colon\ \lambda > \omega\big\} \subset \rho(A) \tag{36}$$

and

$$R\big(\lambda^\alpha, A\big)x = \int_0^\infty e^{-\lambda t} T_\alpha(t)x\,dt, \quad \lambda > \omega,\ x \in X. \tag{37}$$

From (37), we have

$$\frac{d}{d\lambda}\big(R\big(\lambda^\alpha, A\big)\big)x = \frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}T_\alpha(t)x\,dt = -\int_0^\infty te^{-\lambda t}T_\alpha(t)x\,dt.$$

Proceeding by induction we get

$$\frac{d^k}{d\lambda^k}\big(R\big(\lambda^\alpha, A\big)\big)x = (-1)^k\int_0^\infty t^k e^{-\lambda t}T_\alpha(t)x\,dt.$$

Therefore,

$$\begin{aligned}\left\|\frac{d^k}{d\lambda^k}\big(R\big(\lambda^\alpha, A\big)\big)x\right\| &= \left\|\int_0^\infty t^k e^{-\lambda t}T_\alpha(t)x\,dt\right\| \\ &\leqslant M\int_0^\infty t^k e^{(\omega-\lambda)t}\|x\|\,dt \\ &\leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}\|x\|. \end{aligned} \tag{38}$$

(Sufficiency) By Lemma 3.9, for $M \geqslant 1$, $\omega \geqslant 0$, there exists a function $S : [0,\infty) \to B(X)$ satisfying $S(0) = 0$, and

$$\big\|S(t+h) - S(t)\big\| \leqslant M\int_t^{t+h} e^{\omega s}\,ds, \quad t, h \geqslant 0, \tag{39}$$

such that

$$\frac{R(\lambda^\alpha, A)}{\lambda} = \int_0^\infty e^{-\lambda t}S(t)\,dt, \quad \lambda > \omega. \tag{40}$$

From (40), we see that $S(t)$ commutes with A and

$$\int_0^\infty e^{-\lambda t}S(t)\,dt = \frac{1}{\lambda^{\alpha+1}} + \frac{1}{\lambda^\alpha}\int_0^\infty e^{-\lambda t}S(t)A\,dt. \tag{41}$$

For $x \in D(A)$, taking inverse Laplace transform to (41), we obtain that

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x + J_t^\alpha S(t)Ax, \quad t \geqslant 0. \tag{42}$$

Since $\alpha \in [1,2]$, it follows from (42) that $\frac{d}{dt}S(t)x$ exists for all $x \in D(A)$ and $t \geqslant 0$. For $\alpha = 1$,

$$\frac{d}{dt}S(t)x = x + S(t)Ax. \tag{43}$$

For $\alpha \in (1,2]$,

$$\begin{aligned}\frac{d}{dt}S(t)x &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{\alpha-1}S(\tau)Ax\,d\tau \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}S(\tau)Ax\,d\tau.\end{aligned} \tag{44}$$

Let $U(t)x = \int_0^t \tau^{\alpha-2}S(t-\tau)Ax\,d\tau$, $\alpha \in (1,2]$, $x \in D(A)$, $t \geqslant 0$. By (44), we have

$$\frac{d}{dt}S(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha-1)}U(t)x. \tag{45}$$

We shall show that $U(t)x$ is continuous on R_+ for each $x \in D(A)$. In fact, for $\Delta t > 0$ and $x \in D(A)$, we have

$$\begin{aligned}\|U(t+\Delta t)x - U(t)x\| &= \left\| \int_0^{t+\Delta t} \tau^{\alpha-2}S(t+\Delta t-\tau)Ax\,d\tau - \int_0^t \tau^{\alpha-2}S(t-\tau)Ax\,d\tau \right\| \\ &\leqslant \int_0^t \tau^{\alpha-2}\|S(t+\Delta t-\tau) - S(t-\tau)\|\|Ax\|\,d\tau \\ &\quad + \int_t^{t+\Delta t} \tau^{\alpha-2}\|S(t+\Delta t-\tau)\|\|Ax\|\,d\tau.\end{aligned} \tag{46}$$

For $\tau \in [0,t]$, by (39),

$$\|S(t+\Delta t-\tau) - S(t-\tau)\| \leqslant M\int_{t-\tau}^{t+\Delta t-\tau} e^{\omega s}\,ds = \frac{M}{\omega}\left(e^{\omega(t+\Delta t-\tau)} - e^{\omega(t-\tau)}\right).$$

Then, we have

$$\int_0^t \tau^{\alpha-2}\|S(t+\Delta t-\tau) - S(t-\tau)\|\|Ax\|\,d\tau \to 0 \quad \text{as } \Delta t \to 0+. \tag{47}$$

Note that $S(0)=0$, for $\tau \in [t, t+\Delta t]$, by (39),

$$\left\|S(t+\Delta t-\tau)\right\| \leqslant M \int_0^{t+\Delta t-\tau} e^{\omega s}\, ds \leqslant M \int_0^{\Delta t} e^{\omega s}\, ds. \tag{48}$$

By (48),

$$\int_t^{t+\Delta t} \tau^{\alpha-2}\left\|S(t+\Delta t-\tau)\right\|\|Ax\|\, d\tau \leqslant M\|Ax\| \int_0^{\Delta t} e^{\omega s}\, ds \int_t^{t+\Delta t} \tau^{\alpha-2}\, d\tau \to 0 \tag{49}$$

as $\Delta t \to 0+$. Put (47) and (49) into (46) to conclude that $U(t+\Delta t)x - U(t)x \to 0$ as $\Delta t \to 0+$. Similarly, we can prove that $U(t+\Delta t)x - U(t)x \to 0$ as $\Delta t \to 0-$. Hence $U(t)x$ is continuous on R_+ for each $x \in D(A)$. Therefore, from (43), (44) and (45), it follows that $S(\cdot)x \in C^1(R_+; X)$ for each $x \in D(A)$. Since $D(A)$ is dense in X, it follows from (39) and Lemma 3.10 that $S(\cdot)x \in C^1(R_+; X)$ for all $x \in X$. Let $T_\alpha(t)x = \frac{d}{dt}S(t)x$, $x \in X$, $t \geqslant 0$. Then $T_\alpha(t)$ is strongly continuous on R_+. It follows that $T_\alpha(t)$ is linear. For every $\varepsilon > 0$, there exists a sufficiently small $\delta > 0$, such that for $h \in (0, \delta)$,

$$\left\|T_\alpha(t)x - \frac{S(t+h)x - S(t)x}{h}\right\| < \varepsilon\|x\|, \quad x \in X. \tag{50}$$

By (39),

$$\left\|\frac{S(t+h)x - S(t)x}{h}\right\| \leqslant \frac{M\|x\|}{h} \int_t^{t+h} e^{\omega s}\, ds, \quad x \in X. \tag{51}$$

From (50) and (51), it follows that

$$\left\|T_\alpha(t)x\right\| \leqslant \left(\varepsilon + \frac{M}{h}\int_t^{t+h} e^{\omega s}\, ds\right)\|x\|, \tag{52}$$

letting $\varepsilon \to 0+,\ h \to 0+$, we have

$$\left\|T_\alpha(t)\right\| \leqslant M e^{\omega t}, \quad t \geqslant 0. \tag{53}$$

For $x \in D(A)$, since $S(\cdot)x \in C^1(R_+; X)$, $S(0)=0$, by (42) we have

$$\begin{aligned} T_\alpha(t)x &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \tau^{\alpha-1} S(t-\tau)Ax\, d\tau \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)}\int_0^t \tau^{\alpha-1}\frac{d}{dt}S(t-\tau)Ax\, d\tau \end{aligned}$$

$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)}\int_0^t \tau^{\alpha-1}T_\alpha(t-\tau)Ax\,d\tau. \tag{54}$$

From the fact that $S(t)$ commutes with A and A is closed, it follows that $T_\alpha(t)$ commutes with A. Therefore, the closedness of A and the density of $D(A)$ imply that

$$T_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AJ_t^\alpha T_\alpha(t)x, \quad x \in X. \tag{55}$$

Since $T_\alpha(t)$ commutes with A and A is closed, for all $t, s \geqslant 0$, it follows from (55) that

$$\begin{aligned} J_s^\alpha T_\alpha(s)T_\alpha(t)x &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T_\alpha(s)x + J_s^\alpha T_\alpha(s)AJ_t^\alpha T_\alpha(t)x \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T_\alpha(s)x + AJ_s^\alpha T_\alpha(s)J_t^\alpha T_\alpha(t)x \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T_\alpha(s)x + T_\alpha(s)J_t^\alpha T_\alpha(t)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}J_t^\alpha T_\alpha(t)x, \quad x \in X. \end{aligned} \tag{56}$$

By (53), we have

$$\left\|\frac{J_t^\alpha T_\alpha(t)x}{t^{\alpha-1}}\right\| = \frac{1}{\Gamma(\alpha)}\left\|\frac{1}{t^{\alpha-1}}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)x\,d\tau\right\| \leqslant \frac{Mte^{\omega t}\|x\|}{\alpha\Gamma(\alpha)} \to 0 \tag{57}$$

as $t \to 0+$.

By (56), (57), we have

$$J_s^\alpha T_\alpha(s)\lim_{t\to 0+}\left(\frac{T_\alpha(t)}{t^{\alpha-1}}x - \frac{1}{\Gamma(\alpha)}x\right) = 0. \tag{58}$$

By (9), (58), we obtain

$$T_\alpha(s)\lim_{t\to 0+}\left(\frac{T_\alpha(t)}{t^{\alpha-1}}x - \frac{1}{\Gamma(\alpha)}x\right) = 0. \tag{59}$$

From (55), we see that if $T_\alpha(t)x = 0$, $t > 0$, then $x = 0$. This together with (59) implies

$$\lim_{t\to 0+}\frac{T_\alpha(t)}{t^{\alpha-1}}x = \frac{1}{\Gamma(\alpha)}x, \quad x \in X. \tag{60}$$

We next show that $T_\alpha(t)$ commutes with $T_\alpha(s)$. For all $x \in D(A)$ and $t, s \geqslant 0$, by (54) we have

$$\begin{aligned} T_\alpha(t)T_\alpha(s)x &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + g_\alpha(t) * T_\alpha(t)AT_\alpha(s)x \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + g_\alpha(t) * AT_\alpha(t)T_\alpha(s)x, \end{aligned} \tag{61}$$

and

$$T_\alpha(s)T_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + g_\alpha(t) * T_\alpha(s)AT_\alpha(t)x$$
$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + g_\alpha(t) * AT_\alpha(s)T_\alpha(t)x. \tag{62}$$

From (61) and (62), we see that both $w_1(t) = T_\alpha(t)T_\alpha(s)x$ and $w_2(t) = T_\alpha(s)T_\alpha(t)x$ are solutions of

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + g_\alpha(t) * Au(t). \tag{63}$$

Hence for $x \in D(A)$ we have

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x * w_1(t) = \bigl(w_2(t) - g_\alpha(t) * Aw_2(t)\bigr) * w_1(t)$$
$$= w_2(t) * w_1(t) - g_\alpha(t) * Aw_2(t) * w_1(t)$$
$$= w_2(t) * \bigl(w_1(t) - g_\alpha(t) * Aw_1(t)\bigr)$$
$$= w_2(t) * \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x. \tag{64}$$

From (64) it follows that

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x * \bigl(w_1(t) - w_2(t)\bigr) = 0, \tag{65}$$

by Titchmarsh's theorem (see [15, p. 166]) we get $w_1(t) - w_2(t) = 0$, that is $T_\alpha(t)T_\alpha(s)x = T_\alpha(t)T_\alpha(s)x$. By density of $D(A)$, we obtain

$$T_\alpha(t)T_\alpha(s) = T_\alpha(t)T_\alpha(s) \quad \text{for all } t, s \geqslant 0. \tag{66}$$

By (53), (56), (60), (66) and the fact that $T_\alpha(t)$ is strongly continuous on R_+, we conclude that $A \in \mathcal{C}^\alpha(M,\omega)$. Therefore the proof is complete. □

Theorem 3.12. *Let $1 \leqslant \alpha \leqslant 2$. Then $A \in \mathcal{C}^\alpha(M,\omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there is a strongly continuous vector-valued function $T(t)$ satisfying $\|T(t)\| \leqslant Me^{\omega t}$, $M \geqslant 1$, $t \geqslant 0$, and such that*

$$R\bigl(\lambda^\alpha, A\bigr)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt, \quad \lambda > \omega,\ x \in X. \tag{67}$$

If this is the case, $T(t) = T_\alpha(t)$ is the fractional resolvent generated by A.

Proof. Suppose there exist a function $T(t)$ and a linear operator A satisfying the conditions above. $R(\lambda^\alpha, A)$ is differentiable any number of times for $\lambda > \omega$. Then, by differentiating $R(\lambda^\alpha, A)x$ k-times for λ, we obtain

$$\frac{d^k}{d\lambda^k}\big(R\big(\lambda^\alpha, A\big)\big)x = (-1)^k \int_0^\infty t^k e^{-\lambda t} T(t)x\,dt.$$

Hence, for $\lambda > \omega$, we have

$$\left\| \frac{d^k}{d\lambda^k}\big(R\big(\lambda^\alpha, A\big)\big) \right\| \leqslant \frac{Mk!}{(\lambda-\omega)^{k+1}}, \quad \lambda > \omega,\ k \in N_0.$$

From Theorem 3.11, it follows that $A \in \mathcal{C}^\alpha(M,\omega)$. Let $T_\alpha(t)$ be the corresponding α-order fractional resolvent. Then $T_\alpha(t)$ and $T(t)$ both satisfy (67), by the uniqueness theorem of Laplace transforms, we see that $T_\alpha(t) = T(t)$. The converse has been already proven in Theorem 3.11. □

Remark 3.13. It is clear that for $\alpha = 1$, (67) is consistent with the characterization by Laplace transform of a C_0-semigroup; for $\alpha = 2$, (67) is consistent with the characterization by Laplace transform of a sine function (see [3]).

Proposition 3.14. *Let $1 \leqslant \alpha \leqslant 2$, A is the generator of an exponentially bounded α-times resolvent family $S_\alpha(t)$. Then A is the generator of an exponentially bounded α-order fractional resolvent $T_\alpha(t)$.*

Proof. Assume there are constants $M \geqslant 1$, $\omega \geqslant 0$ such that $\|S_\alpha(t)\| \leqslant Me^{\omega t}$, then by (2.5) and (2.6) in [1] we have

$$\big(\omega^\alpha, \infty\big) \subset \rho(A)$$

and

$$\lambda^{\alpha-1} R\big(\lambda^\alpha, A\big) = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt, \quad \lambda > \omega,\ x \in X. \tag{68}$$

Define

$$T_\alpha(t)x = J_t^{\alpha-1} S_\alpha(t)x, \quad x \in X.$$

It is easy to see that $T_\alpha(t)$ is exponentially bounded. By the convolution property of Laplace transforms and (68), we have

$$\int_0^\infty e^{-\lambda t} T_\alpha(t)x\,dt = \int_0^\infty e^{-\lambda t} (g_{\alpha-1} * S_\alpha)(t)x\,d\tau$$

$$= \frac{1}{\lambda^{\alpha-1}} \int_0^\infty e^{-\lambda t} S_\alpha(t)x\, dt$$

$$= R\big(\lambda^\alpha, A\big).$$

From Theorem 3.12, it follows that A is the generator of the exponentially bounded α-order fractional resolvent $T_\alpha(t)$. □

Example 3.15. Let $\alpha \in (1,2)$, $\theta \in [0,\pi)$. Consider the problem

$$\begin{cases} D_t^\alpha u(t,x) = e^{i\theta} \dfrac{\partial^2}{\partial x^2} u(t,x), \quad 0 < x < 1,\ t > 0, \\ u(t,0) = u(t,1) = 0, \\ \big(g_{2-\alpha}(t) * u(t,x)\big)\big|_{t=0} = 0, \qquad \dfrac{\partial u}{\partial t}\big(g_{2-\alpha}(t) * u(t,x)\big)\Big|_{t=0} = f(x). \end{cases} \tag{69}$$

Take $X = L^2(0,1)$, $A = e^{i\theta}\frac{\partial^2}{\partial x^2}$ with domain $D(A) = \{\varphi \in W^{2,2}(0,1), \varphi(0) = \varphi(1) = 0\}$. It is easy to see that A has eigenvalues $-e^{i\theta} n^2\pi^2$ with eigenfunctions $\sin n\pi x$, $n \in N$. If $f(x) = \sum_{n=1}^\infty c_n \sin n\pi x$, the solution of (69) is

$$u(t,x) = \sum_{n=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta} n^2\pi^2 t^\alpha\big) c_n \sin n\pi x.$$

We shall show that $A \in \mathcal{C}^\alpha(0)$ if and only if $|\theta| < (1 - \frac{\alpha}{2})\pi$. Define a mapping $T_\alpha(t): X \to X$, $T_\alpha(t) f(x) = u(t,x)$, $t \geqslant 0$. Since $\sqrt{2}\sin n\pi x$, $n \in N$, is the orthonormal set of eigenvectors of A, then $\|f\|^2 = \frac{1}{2}\sum_{n=1}^\infty |c_n|^2$. First, we prove that $\|T_\alpha(t)\| \leqslant M$ if $|\theta| < (1 - \frac{\alpha}{2})\pi$, where M is a constant. In fact,

$$\big\|T_\alpha(t) f\big\|^2 = \int_0^1 \left| \sum_{n=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta} n^2\pi^2 t^\alpha\big) c_n \sin n\pi x \right|^2 dx$$

$$= \frac{t^{2(\alpha-1)}}{2} \sum_{n=1}^\infty |c_n|^2 \big|E_{\alpha,\alpha}\big(-e^{i\theta} n^2\pi^2 t^\alpha\big)\big|^2. \tag{70}$$

Since $|E_{\alpha,\alpha}(-e^{i\theta} n^2\pi^2 t^\alpha)| \to \frac{1}{\Gamma(\alpha)}$ as $t \to 0+$, there exists a sufficiently small $\delta > 0$ such that for $t \in (0,\delta)$,

$$\big|E_{\alpha,\alpha}\big(-e^{i\theta} n^2\pi^2 t^\alpha\big)\big| \leqslant 1 + \frac{1}{\Gamma(\alpha)}. \tag{71}$$

By (71), (70),

$$\begin{aligned}\|T_\alpha(t)f\|^2 &\leqslant \frac{1}{2}\left(1+\frac{1}{\Gamma(\alpha)}\right)^2\delta^{2(\alpha-1)}\sum_{n=1}^{\infty}|c_n|^2\\ &= \left(1+\frac{1}{\Gamma(\alpha)}\right)^2\delta^{2(\alpha-1)}\|f\|^2, \quad t\in(0,\delta).\end{aligned} \tag{72}$$

For $|\theta| < (1-\frac{\alpha}{2})\pi$, we have $|\arg(e^{i\theta}n^2\pi^2t^\alpha)| < (1-\frac{\alpha}{2})\pi$. From the asymptotic formulas (15) of the Mittag-Leffler function and noting that $\Gamma(0)=\infty$, it follows that

$$E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big) = O\left(\frac{1}{t^{2\alpha}}\right), \quad t\to\infty. \tag{73}$$

By (73), there is a constant $T>0$, such that for $t>T$,

$$\big|E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big)\big| \leqslant \frac{L_1}{t^{2\alpha}}, \tag{74}$$

where L_1 is a constant.

By (74), (70),

$$\|T_\alpha(t)f\|^2 \leqslant \frac{L_1^2}{2t^{2\alpha+2}}\sum_{n=1}^{\infty}|c_n|^2 \leqslant \frac{L_1^2}{T^{2\alpha+2}}\|f\|^2, \quad t>T. \tag{75}$$

For $t\in[\delta,T]$, $|\theta|<(1-\frac{\alpha}{2})\pi$, we have

$$E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big) = O\left(\frac{1}{n^4}\right), \quad n\to\infty. \tag{76}$$

By (76), there is a constant $N_1>0$, such that for $n>N_1$,

$$\big|E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big)\big| \leqslant L_2, \tag{77}$$

where L_2 is a constant.

By (77), (70),

$$\begin{aligned}\|T_\alpha(t)f\|^2 &= \frac{t^{2(\alpha-1)}}{2}\sum_{n=1}^{\infty}|c_n|^2\big|E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big)\big|^2\\ &\leqslant \frac{T^{2(\alpha-1)}}{2}\left(\sum_{n=1}^{N_1}|c_n|^2\big|E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big)\big|^2 + \sum_{n=N_1+1}^{\infty}|c_n|^2\big|E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2t^\alpha\big)\big|^2\right)\\ &\leqslant \frac{T^{2(\alpha-1)}}{2}\left(\big(E_{\alpha,\alpha}\big(\pi^2N_1^2T^\alpha\big)\big)^2\sum_{n=1}^{N_1}|c_n|^2 + L_2^2\sum_{n=N_1+1}^{\infty}|c_n|^2\right)\\ &\leqslant \frac{T^{2(\alpha-1)}L_3}{2}\|f\|^2, \quad t\in[\delta,T],\end{aligned} \tag{78}$$

where $L_3=\max\{(E_{\alpha,\alpha}(\pi^2N_1^2T^\alpha))^2, L_2^2\}$.

By (72), (75), (78) and noting that $T_\alpha(0) = 0$, we obtain

$$\|T_\alpha(t)\| \leqslant M, \quad t \geqslant 0, \tag{79}$$

where $M = \max\{(1 + \frac{1}{\Gamma(\alpha)})\delta^{\alpha-1}, \frac{L_1}{T^{\alpha+1}}, \sqrt{\frac{T^{2(\alpha-1)}L_3}{2}}\}$.

Next, we prove the strong continuity of $T_\alpha(t)$. For $\Delta t > 0$,

$$\begin{aligned}
&\|T_\alpha(t+\Delta t)f - T_\alpha(t)f\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \big((t+\Delta t)^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2(t+\Delta t)^\alpha\big) - t^{\alpha-1}E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2 t^\alpha\big)\big)c_n \sin n\pi x \right\|^2 \\
&= \int_0^1 \left| \sum_{n=1}^{\infty} \big((t+\Delta t)^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2(t+\Delta t)^\alpha\big) - t^{\alpha-1}E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2 t^\alpha\big)\big)c_n \sin n\pi x \right|^2 \\
&= \frac{1}{2}\sum_{n=1}^{\infty} |c_n|^2 \big|(t+\Delta t)^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2(t+\Delta t)^\alpha\big) - t^{\alpha-1}E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2 t^\alpha\big)\big|^2.
\end{aligned} \tag{80}$$

Applying the dominated convergence theorem to (80), we obtain

$$\|T_\alpha(t+\Delta t)f - T_\alpha(t)f\| \to 0 \quad \text{as } \Delta t \to 0+. \tag{81}$$

Finally, we prove that $T_\alpha(t)$ is the fractional resolvent generated by A. For $\lambda > (n^2\pi^2)^{1/\alpha}$, by (13), we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} T_\alpha(t) f(x)\,dt &= \int_0^\infty e^{-\lambda t} \sum_{n=1}^{\infty} \big(t^{\alpha-1}E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2 t^\alpha\big)c_n \sin n\pi x\big)\,dt \\
&= \sum_{n=1}^{\infty} \int_0^\infty e^{-\lambda t} t^{\alpha-1}E_{\alpha,\alpha}\big(-e^{i\theta}n^2\pi^2 t^\alpha\big)c_n \sin n\pi x\,dt \\
&= \sum_{n=1}^{\infty} \frac{c_n \sin n\pi x}{\lambda^\alpha + e^{i\theta}n^2\pi^2}.
\end{aligned} \tag{82}$$

Since $f(x) \in X$, by (82), we obtain that $\int_0^\infty e^{-\lambda t} T_\alpha(t) f(x)\,dt \in D(A)$ and

$$\begin{aligned}
A\int_0^\infty e^{-\lambda t} T_\alpha(t) f(x)\,dt &= -\sum_{n=1}^{\infty} \frac{c_n e^{i\theta}n^2\pi^2 \sin n\pi x}{\lambda^\alpha + e^{i\theta}n^2\pi^2} \\
&= \sum_{n=1}^{\infty} \frac{c_n \lambda^\alpha \sin n\pi x}{\lambda^\alpha + e^{i\theta}n^2\pi^2} - \sum_{n=1}^{\infty} c_n \sin n\pi x
\end{aligned}$$

$$= \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) f(x)\,dt - f(x). \tag{83}$$

For $f(x) \in D(A)$, we have

$$\begin{aligned}
\int_0^{\infty} e^{-\lambda t} T_{\alpha}(t)\,dt\,Af(x) &= \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) Af(x)\,dt \\
&= \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) \left(-e^{i\theta} \sum_{n=1}^{\infty} c_n n^2 \pi^2 \sin n\pi x \right) dt \\
&= \int_0^{\infty} e^{-\lambda t} \left(\sum_{n=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\big(-e^{i\theta} n^2 \pi^2 t^{\alpha}\big)\big(-e^{i\theta} c_n n^2 \pi^2\big) \sin n\pi x \right) dt \\
&= \int_0^{\infty} e^{-\lambda t} A T_{\alpha}(t) f(x)\,dt \\
&= A \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) f(x)\,dt.
\end{aligned} \tag{84}$$

From (83), (84), it follows that

$$\int_0^{\infty} e^{-\lambda t} T_{\alpha}(t)\,dt\,\big(\lambda^{\alpha} - A\big) f(x) = f(x), \quad f(x) \in D(A). \tag{85}$$

Thus, by (83), (85), we have

$$R\big(\lambda^{\alpha}, A\big) f(x) = \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) f(x)\,dt, \quad \lambda > \big(n^2\pi^2\big)^{1/\alpha},\ f(x) \in X. \tag{86}$$

Therefore, from (80), (79), (86) and Theorem 3.12, it follows that $A \in \mathcal{C}^{\alpha}(0)$ if $|\theta| < (1 - \frac{\alpha}{2})\pi$.

For $|\theta| \geqslant (1 - \frac{\alpha}{2})\pi$, we shall prove that $A \in \mathcal{C}^{\alpha}(0)$ is not valid. Since $\theta \in [0, \pi)$, then $\theta \geqslant (1 - \frac{\alpha}{2})\pi$. For $t > 0$, $\arg(-e^{i\theta} n^2 \pi^2 t^{\alpha}) = -\pi + \theta \in [-\frac{\alpha}{2}\pi, 0)$. Let $\delta_0 > 0$ be a constant. From the asymptotic formulas (14) and noting that $\Gamma(0) = \infty$, it follows that for $t > \delta_0$,

$$\begin{aligned}
E_{\alpha,\alpha}\big(-e^{i\theta} n^2 \pi^2 t^{\alpha}\big) &= \frac{1}{\alpha}\big(-e^{i\theta} n^2 \pi^2 t^{\alpha}\big)^{(1-\alpha)/\alpha} \exp\Big(\big(-e^{i\theta} n^2 \pi^2 t^{\alpha}\big)^{1/\alpha}\Big) \\
&\quad + O\left(\frac{1}{n^4}\right), \quad n \to \infty.
\end{aligned} \tag{87}$$

By (87), there is a constant $N_2 > 0$, such that for $n > N_2$,

$$\left|E_{\alpha,\alpha}\left(-e^{i\theta}n^2\pi^2t^\alpha\right)\right| \geqslant \left|\frac{1}{\alpha}\left(-e^{i\theta}n^2\pi^2t^\alpha\right)^{(1-\alpha)/\alpha}\exp\left(\left(-4e^{i\theta}n^2\pi^2t^\alpha\right)^{1/\alpha}\right)\right| - L_2, \tag{88}$$

where $L_2 > 0$ is a constant.

Since

$$\begin{aligned}\left|\exp\left(\left(-e^{i\theta}n^2\pi^2t^\alpha\right)^{1/\alpha}\right)\right| &= \left|\exp\left(\left(\pi^2t^\alpha\right)^{1/\alpha}n^{2/\alpha}e^{i(-\pi+\theta)/\alpha}\right)\right| \\ &= \exp\left(\pi^{2/\alpha}tn^{2/\alpha}\cos(\pi-\theta)/\alpha\right)\end{aligned} \tag{89}$$

from (88) and (89), it follows that for $t > \delta_0$,

$$\left|E_{\alpha,\alpha}\left(-e^{i\theta}n^2\pi^2t^\alpha\right)\right| \geqslant \frac{1}{\alpha}\left(\pi^2t^\alpha\right)^{(1-\alpha)/\alpha}n^{2(1-\alpha)/\alpha}\exp\left(\pi^{2/\alpha}tn^{2/\alpha}\cos(\pi-\theta)/\alpha\right) - L_2. \tag{90}$$

Since $\theta \in [(1-\frac{\alpha}{2})\pi, \pi)$, then $(\pi-\theta)/\alpha \in (0, \frac{\pi}{2}]$. This together with (90) implies that for $t > \delta_0$,

$$\left|E_{\alpha,\alpha}\left(-e^{i\theta}n^2\pi^2t^\alpha\right)\right| \to \infty \quad \text{as } n \to \infty. \tag{91}$$

Let $f_n(x) = \sqrt{2}\sin n\pi x$, $n \in N$. Then by (91), (70), for $t > \delta_0$,

$$\left\|T_\alpha(t)f_n\right\| \to \infty \quad \text{as } n \to \infty. \tag{92}$$

Therefore, $A \in \mathcal{C}^\alpha(0)$ if and only if $|\theta| < (1-\frac{\alpha}{2})\pi$.

Example 3.16. Consider the problem

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x), \quad 0 \leqslant x \leqslant \pi,\ t \in R, \\ u(t,0) = u(t,\pi) = 0, \\ u(x,0) = f(x), \qquad \left.\dfrac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x), \quad 0 \leqslant x \leqslant \pi.\end{cases} \tag{93}$$

Take $X = L^2[0,\pi]$, and A is the operator from X to X defined by

$$D(A) = \left\{f \in X:\ f, f' \text{ are absolutely continuous},\ f'' \in X,\ f(0) = f(\pi) = 0\right\}$$

and

$$Af = f''.$$

Then, we have

$$Af = -\sum_{n=1}^{\infty} n^2(f, f_n)f_n, \quad f \in D(A),$$

where $f_n(s) = \sqrt{2/\pi}\sin ns$, $n \in N$, is the orthonormal set of eigenvectors of A. Since $-A$ is positive and self-adjoint in X, it is easy to show that A is the generator of a strongly continuous cosine function $C(t)$, $t \in R$, in X and

$$C(t)f = \sum_{n=1}^{\infty} \cos nt(f, f_n)f_n, \quad f \in X. \tag{94}$$

Thus, by Theorem 3.1 (Subordination principle) in [1], for $\alpha \in (1,2)$, A generates an exponentially bounded α-times resolvent family $S_\alpha(t)$ and

$$S_\alpha(t) = \int_0^{\infty} \varphi_{t,\alpha/2}(s)C(s)\,ds, \quad t > 0, \tag{95}$$

where

$$\varphi_{t,\alpha/2}(s) = t^{-\alpha/2}\sum_{n=0}^{\infty} \frac{(-st^{-\alpha/2})^n}{n!\Gamma(-n\alpha/2 + 1 - \alpha/2)}.$$

It follows from Proposition 3.14 that A generates an exponentially bounded α-order fractional resolvent $T_\alpha(t) = J_t^{\alpha-1}S_\alpha(t)$.

4. Well-posedness of a homogeneous α-order Cauchy problem

This section is devoted to studying the well-posedness of problem (1). Besides the discussion on the uniqueness of solutions to problem (1), it is shown that problem (1) is well posed if and only if its coefficient operator A generates an α-order fractional resolvent.

Definition 4.1. A function $u \in C(R_+; X)$ is called a weak solution of (1) if

$$J_t^\alpha u(t) \in D(A) \quad \text{and} \quad u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AJ_t^\alpha u(t), \quad t \in R_+. \tag{96}$$

Definition 4.2. A function $u \in C(R_+; X)$ is called a strong solution of (1) if $u(t) \in D(A)$ for all $t \geqslant 0$, $D_t^\alpha u(t)$ is continuous on R_+ and (1) holds.

Lemma 4.3. *Let A be the generator of an α-order fractional resolvent $T_\alpha(t)$. Let $x \in X$ and $u \in C(R_+; X)$ be a weak solution of* (1)*, then $u(t) = T_\alpha(t)x$ for all $t \geqslant 0$.*

Proof. From property (b) of Proposition 3.7 and the closedness of A, it follows that

$$\begin{aligned}\frac{t^{\alpha-1}}{\Gamma(\alpha)} * u(t) &= \big(T_\alpha(t) - AJ_t^\alpha T_\alpha(t)\big) * u(t) \\ &= T_\alpha(t) * u(t) - A\big(g_\alpha(t) * T_\alpha(t)\big) * u(t) \\ &= T_\alpha(t) * u(t) - AT_\alpha(t) * \big(g_\alpha(t) * u(t)\big)\end{aligned}$$

$$\begin{aligned}
&= T_\alpha(t) * u(t) - T_\alpha(t) * \big(Ag_\alpha(t) * u(t)\big) \\
&= T_\alpha(t) * \big(u(t) - Ag_\alpha(t) * u(t)\big) \\
&= T_\alpha(t) * \big(u(t) - AJ_t^\alpha u(t)\big) \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)} * T_\alpha(t)x,
\end{aligned}$$

by Titchmarsh's theorem, we have $u(t) = T_\alpha(t)x$, $t \geqslant 0$. □

Remark 4.4. From Lemma 4.3, we see that weak solutions of (1) are unique. It is clear that every strong solution of (1) is a weak solution, therefore strong solutions of (1) are unique.

Lemma 4.5. *Let A be the generator of an α-order fractional resolvent $T_\alpha(t)$, then $T_\alpha(t)x$ is the unique strong solution of* (1) *for every $x \in D(A)$.*

Proof. From property (a) of Proposition 3.7 it follows that

$$u(t) = T_\alpha(\cdot)x \in C\big(R_+; D(A)\big), \quad x \in D(A). \tag{97}$$

Since $T_\alpha(t)$ is strongly continuous on R_+, then

$$\begin{aligned}
g_{2-\alpha}(t) * T_\alpha(t)x|_{t=0} &= \frac{1}{\Gamma(2-\alpha)} \lim_{t\to0+} \int_0^t (t-\tau)^{1-\alpha} T_\alpha(\tau)x\,d\tau \\
&= \frac{1}{\Gamma(2-\alpha)} \lim_{t\to0+} t^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} T_\alpha(t\tau)x\,d\tau \\
&= 0.
\end{aligned} \tag{98}$$

By (24), we have

$$\begin{aligned}
\frac{d}{dt}\big(g_{2-\alpha}(t) * T_\alpha(t)x\big)\Big|_{t=0} &= \frac{d}{dt}\big(g_{2-\alpha}(t) * \big(g_\alpha(t)x + J_t^\alpha T_\alpha(t)Ax\big)\big)\Big|_{t=0} \\
&= \frac{d}{dt}\big(g_{2-\alpha}(t) * g_\alpha(t)x\big)\Big|_{t=0} + \frac{d}{dt}\big(g_{2-\alpha}(t) * g_\alpha(t) * T_\alpha(t)x\big)\Big|_{t=0} \\
&= \left(\frac{d}{dt} g_2(t)x\right)\Big|_{t=0} + \frac{d}{dt}\big(g_2(t) * T_\alpha(t)x\big)\Big|_{t=0} \\
&= x + \left(\int_0^t T_\alpha(\tau)Ax\,d\tau\right)\Big|_{t=0} \\
&= x.
\end{aligned} \tag{99}$$

Using properties (c), (a) of Proposition 3.7, we have

$$\begin{aligned}
D_t^\alpha T_\alpha(t)x &= D_t^\alpha\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}x + J_t^\alpha T_\alpha(t)Ax\right)\\
&= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-\tau)^{1-\alpha}\tau^{\alpha-1}x\,d\tau + D_t^\alpha J_t^\alpha T_\alpha(t)Ax\\
&= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}tx\,d\tau + AT_\alpha(t)x\\
&= \frac{d^2}{dt^2}(tx) + AT_\alpha(t)x\\
&= AT_\alpha(t)x. \qquad (100)
\end{aligned}$$

From (97), (98), (99), (100) and Remark 4.4 it follows that $T_\alpha(t)x$ is the unique strong solution of (1) for all $x \in D(A)$ and $t \geqslant 0$. □

Definition 4.6. The fractional abstract Cauchy problem (1) is said to be well posed if for any $x \in D(A)$ there exists a unique strong solution $u(t,x)$, and $D(A) \ni x_n \to 0$ implies that $u(t,x_n) \to 0$ as $n \to \infty$ in X, uniformly on any compact subinterval of R_+.

Theorem 4.7. *The fractional abstract Cauchy problem* (1) *is well posed for every initial value $x \in D(A)$ if and only if its coefficient operator A is the generator of an α-order fractional resolvent $\{T_\alpha(t)\}_{t\geqslant 0}$.*

Proof. If A is the generator of an α-order fractional resolvent $\{T_\alpha(t)\}_{t\geqslant 0}$, then from Lemma 4.5, it follows that for every $x \in D(A)$, $T_\alpha(t)x$ is the unique strong solution of (1). Moreover, since $T_\alpha(t)$ is strongly continuous on R_+, by the uniform boundedness principle, $T_\alpha(t)$ is uniform bounded on compact subinterval of R_+.

On the other hand, we need to prove that well-posedness of (1) implies that A is the generator of an α-order fractional resolvent $T_\alpha(t)$. We assume that problem (1) is well posed for every $x \in D(A)$ and $u(t;x)$ is the unique strong solution of (1). Set $T_\alpha(t)x = u(t;x)$, $x \in D(A)$, $t \geqslant 0$. From the linearity of (1) and the uniqueness of solutions, it is clear that $T_\alpha(t)$ is linear for every $t \geqslant 0$. The fact that $u(t;x)$ is the solution of (1) implies

$$(g_{2-\alpha} * u)(0) = 0, \qquad (g_{2-\alpha} * u)'(0) = x. \qquad (101)$$

Applying (10) to (1), we obtain

$$u(t;x) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x = J_t^\alpha Au(t;x), \qquad (102)$$

then

$$\begin{aligned}
\lim_{t\to 0+}\frac{T_\alpha(t)}{t^{\alpha-1}}x &= \lim_{t\to 0+}\frac{u(t;x)}{t^{\alpha-1}}x\\
&= \frac{1}{\Gamma(\alpha)}x + \lim_{t\to 0+} t^{1-\alpha}J_t^\alpha Au(t;x)
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)} \lim_{t\to 0+} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} Au(\tau; x)\,d\tau$$

$$= \frac{1}{\Gamma(\alpha)}x + \frac{1}{\Gamma(\alpha)} \lim_{t\to 0+} t \int_0^1 (1-\tau)^{\alpha-1} Au(t\tau; x)\,d\tau. \tag{103}$$

Since A is closed, $D(A)$ is dense in X and $u(t;x)$ is continuous on R_+, we apply the dominated convergence theorem to (103) to conclude that

$$\lim_{t\to 0+} \frac{T_\alpha(t)}{t^{\alpha-1}}x = \frac{1}{\Gamma(\alpha)}x, \quad x \in X. \tag{104}$$

For any compact subinterval of R_+, we claim that $T_\alpha(t)$ is uniformly bounded. By contradiction, assume that there exists a sequence $\{t_n\} \subset [0, T]$ such that $\|T_\alpha(t_n)\| \to \infty$ as $n \to \infty$. We choose $x_n \in D(A)$ such that $\lim_{n\to\infty} x_n = 0$ and $\|T_\alpha(t_n)x_n\| \geqslant 1$. This contradicts Definition 4.6 since $u(t_n, x_n) = T_\alpha(t_n)x_n$. So $T_\alpha(t)$ is uniformly bounded compact intervals of R_+. This implies that $T_\alpha(t)$ can be extended to all of X, $T_\alpha(t)x$ is continuous for every $x \in X$. By the closedness of A, $T_\alpha(t)x = u(t;x)$ and (102), we have

$$T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x = J_t^\alpha A T_\alpha(t)x = A J_t^\alpha T_\alpha(t)x, \quad x \in D(A),\ t \geqslant 0. \tag{105}$$

For every $x \in X$, since $T_\alpha(t)$ is bounded and A is a closed densely defined operator, we get $J_t^\alpha T(t)x \in D(A)$ and $A J_t^\alpha T_\alpha(t) = T_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is strongly continuous for $t \geqslant 0$, hence $u(t;x) = T_\alpha(t)x$ is a weak solution of (1). Next, we show that the weak solutions of (1) are unique. Let $u_1, u_2 \in C(R_+; X)$ be two weak solutions of (1), then $u = u_1 - u_2 \in C(R_+; X)$, $(g_{2-\alpha} * u)(0) = 0$, $(g_{2-\alpha} * u)'(0) = 0$, and $u(t) = J_t^\alpha Au(t) = A J_t^\alpha u(t)$. Let $v(t) = J_t^\alpha u(t)$, then $v(t)$ is a strong solution of (1) with initial values $(g_{2-\alpha} * v)(0) = 0$, $(g_{2-\alpha} * v)'(0) = 0$. It is clear that $u = 0$ is a strong solution of (1) with initial values $(g_{2-\alpha} * 0)(0) = 0$, $(g_{2-\alpha} * 0)'(0) = 0$. By uniqueness of the strong solutions, we have $v(t) = 0$. Hence $u(t) = D_t^\alpha v(t) = 0$. For $x \in D(A)$, both $u(t; Ax)$ and $Au(t;x)$ are weak solutions of (1) with $\frac{t^{\alpha-1}}{\Gamma(\alpha)}x$ replaced by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}Ax$, therefore

$$T_\alpha(t)Ax = u(t; Ax) = Au(t;x) = AT_\alpha(t)x, \quad t \geqslant 0. \tag{106}$$

For all $x \in D(A)$, $t, s \geqslant 0$, from (105) and (106) it follows that

$$T_\alpha(t)T_\alpha(s)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + J_t^\alpha A T_\alpha(t)T_\alpha(s)x$$

$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + \int_0^t g_\alpha(t-\tau) A T_\alpha(\tau)T_\alpha(s)x\,d\tau, \tag{107}$$

and

$$\begin{aligned} T_\alpha(s)T_\alpha(t)x &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + T_\alpha(s)J_t^\alpha A T_\alpha(t)x \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + \int_0^t g_\alpha(t-\tau)T_\alpha(s)AT_\alpha(\tau)x\,d\tau \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}T_\alpha(s)x + \int_0^t g_\alpha(t-\tau)AT_\alpha(s)T_\alpha(\tau)x\,d\tau. \end{aligned} \tag{108}$$

Note that $D_t^\alpha t^{\alpha-1} = 0$ and $D_t^{\alpha-1}(\frac{t^{\alpha-1}}{\Gamma(\alpha)}) = 1$ for $1 < \alpha < 2$, by Definition 4.2, (107), (108), we obtain that both $T(\cdot)T(s)x$ and $T(s)T(\cdot)x$ are strong solutions of (1) with initial values $g_{2-\alpha}(t) * T_\alpha(t)T(s)x|_{t=0} = g_{2-\alpha}(t) * T_\alpha(s)T_\alpha(t)x|_{t=0} = 0$, $\frac{d}{dt}(g_{2-\alpha}(t) * T_\alpha(t)T(s)x)|_{t=0} = \frac{d}{dt}(g_{2-\alpha}(t) * T_\alpha(s)T_\alpha(t))x|_{t=0} = T_\alpha(s)x$. Therefore $T_\alpha(t)T_\alpha(s)x = T_\alpha(s)T_\alpha(t)x$ for every $x \in D(A)$ by the well-posedness of (1). Since $D(A)$ is dense in X, it follows that

$$T_\alpha(t)T_\alpha(s) = T_\alpha(s)T_\alpha(t), \quad t, s \geqslant 0. \tag{109}$$

Finally, we show that $T_\alpha(t)$ satisfies (c) of Definition 3.1. For $x \in D(A)$, from (105) and (106) it follows that

$$\begin{aligned} T_\alpha(s)J_t^\alpha T_\alpha(t)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)}J_t^\alpha T_\alpha(t)x &= AJ_s^\alpha T_\alpha(s)J_t^\alpha T_\alpha(t)x \\ &= J_s^\alpha T_\alpha(s)AJ_t^\alpha T_\alpha(t)x \\ &= J_s^\alpha T_\alpha(s)T_\alpha(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T_\alpha(s)x. \end{aligned} \tag{110}$$

Density of $D(A)$ implies that

$$T_\alpha(s)J_t^\alpha T_\alpha(t) - \frac{s^{\alpha-1}}{\Gamma(\alpha)}J_t^\alpha T_\alpha(t) = J_s^\alpha T_\alpha(s)T_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T_\alpha(s), \quad t \geqslant 0,\ s \geqslant 0. \tag{111}$$

According to (104), (109) and (111) we see that the well-posedness of (1) implies that A is the generator of an α-order fractional resolvent $T_\alpha(t)$. Therefore the proof is complete. □

5. Existence and uniqueness of an inhomogeneous α-order Cauchy problem

In this section we shall prove the existence and uniqueness of the weak solutions and strong solutions of problem (2) under some general conditions.

Definition 5.1. A function $u \in C([0,T]; X)$ is called a weak solution of (2) if

$$J_t^\alpha u(t) \in D(A) \quad \text{and} \quad u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AJ_t^\alpha u(t) + J_t^\alpha f(t), \quad t \in [0,T].$$

Definition 5.2. A function $u \in C([0,T]; X)$ is called a strong solution of (2) if $u(t) \in D(A)$ for all $t \in [0,T]$, $D_t^\alpha u(t)$ is continuous on $[0,T]$ and (2) holds.

Theorem 5.3. *Let A be the generator of an α-order fractional resolvent $\{T_\alpha(t)\}_{t\geqslant 0}$ on X. Let $f\in L^1([0,T];X)$, then for every $x\in X$, problem* (2) *has a unique weak solution u given by*

$$u(t)=T(t)x+\int_0^t T(t-s)f(s)\,ds,\quad t\in[0,T]. \tag{112}$$

Proof. Uniqueness: Let $u_1,u_2\in C([0,T];X)$ be two weak solutions of (2). Then $w:=u_1-u_2\in C([0,T];X)$, and $AJ_t^\alpha w(t)=w(t)$ for all $t\geqslant 0$. It follows from Lemma 4.3 that $w\equiv 0$.

Existence: Let

$$v(t)=\int_0^t T_\alpha(t-s)f(s)\,ds,\quad t\in[0,T] \tag{113}$$

and let

$$u(t)=T_\alpha(t)x+\int_0^t T_\alpha(t-s)f(s)\,ds. \tag{114}$$

Since $T_\alpha(t)$ is strongly continuous on R_+ and $f\in L^1([0,T];X)$, Proposition 1.3.4 in [3] shows that $v\in C([0,T];X)$. Then

$$u\in C\big([0,T];X\big). \tag{115}$$

Using property (b) of Proposition 3.7, the closedness of A, we obtain $J_t^\alpha v(t)\in D(A)$, and

$$\begin{aligned}AJ_t^\alpha v(t)&=A(g_\alpha*T_\alpha*f)\\&=A(g_\alpha*T_\alpha)*f\\&=(T_\alpha*f)(t)-(g_\alpha*f)(t)\\&=v(t)-J_t^\alpha f(t).\end{aligned} \tag{116}$$

$J_t^\alpha v(t)\in D(A)$ implies

$$J_t^\alpha u(t)\in D(A). \tag{117}$$

By (116), (114) and property (b) of Proposition 3.7,

$$\begin{aligned}AJ_t^\alpha u(t)&=AJ_t^\alpha T_\alpha(t)x+AJ_t^\alpha v(t)\\&=T_\alpha(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x+v(t)-J_t^\alpha f(t)\\&=u(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x-J_t^\alpha f(t).\end{aligned} \tag{118}$$

The proof is therefore complete. □

Theorem 5.4. *Let A be the generator of an α-order fractional resolvent $\{T_\alpha(t)\}_{t\geqslant 0}$ on X. Assume either*

(i) $f \in C([0,T]; D(A))$ *and* $Af \in C([0,T]; X)$, *or*

(ii) $f \in C([0,T]; X)$ *and* $g_{2-\alpha} * f \in W^{2,1}[0,T]$.

Then for every $x \in D(A)$, problem (2) *has a unique strong solution u defined by*

$$u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-s)f(s)\,ds, \quad t \in [0,T]. \tag{119}$$

Proof. Uniqueness: Let $u_1, u_2 \in C([0,T]; D(A))$ be two strong solutions of (2). Then $w := u_1 - u_2 \in C([0,T]; D(A))$, $(g_{2-\alpha} * w)(t) = 0$, $(g_{2-\alpha} * w)'(t) = 0$, and $D_t^\alpha w(t) = Aw(t)$ for all $t \in [0,T]$. By Remark 4.4, $w \equiv 0$.

Existence: It follows from Lemma 4.5 that $T_\alpha(t)x$ is a unique strong solution of the homogeneous Cauchy problem (1). Let

$$v(t) = \int_0^t T_\alpha(t-s)f(s)\,ds, \quad t \in [0,T]. \tag{120}$$

Then we only need to show that $v(t)$ is a strong solution of problem (2) with initial values $(g_{2-\alpha} * v)(0) = 0$, $(g_{2-\alpha} * v)'(0) = 0$.

Case (I): Assume (i) holds. By Proposition 1.3.4 in [3], it follows that $v(t) \in C([0,T]; X)$, then

$$(g_{2-\alpha} * v)(0) = 0. \tag{121}$$

$f \in D(A)$ and the closedness of A imply $v \in D(A)$ and

$$Av(t) = \int_0^t T_\alpha(t-s)Af(s)\,ds, \quad t \in [0,T]. \tag{122}$$

Also, $g_{2-\alpha} * v$ is differentiable for $t \in [0,T]$ and

$$\begin{aligned}
(g_{2-\alpha} * v)'(t) &= \frac{d}{dt}(g_{2-\alpha} * T_\alpha * f)(t) \\
&= \int_0^t \frac{d}{dt}(g_{2-\alpha} * T_\alpha)(t-s)f(s)\,ds + \lim_{s\to t-0}(g_{2-\alpha} * T_\alpha)(t-s)f(s) \\
&= \int_0^t \left(D_r^{\alpha-1}T_\alpha(r)f(s)\big|_{r=t-s}\right)ds + \lim_{s\to 0+}(g_{2-\alpha} * T_\alpha)(s)f(t-s) \\
&= \int_0^t \left(D_r^{\alpha-1}T_\alpha(r)f(s)\big|_{r=t-s}\right)ds + \lim_{s\to 0+} J_s^{2-\alpha}T_\alpha(s)f(t)
\end{aligned}$$

$$= \int_0^t \left(D_r^{\alpha-1} T_\alpha(r) f(s)\big|_{r=t-s} \right) ds. \tag{123}$$

From property (b) of Proposition 3.7, (123), it follows that

$$\begin{aligned} (g_{2-\alpha} * v)'(t) &= \int_0^t \left(D_r^{\alpha-1} \left(\frac{r^{\alpha-1}}{\Gamma(\alpha)} f(s) + J_r^\alpha T_\alpha(r) A f(s) \right) \bigg|_{r=t-s} \right) ds \\ &= \int_0^t f(s)\, ds + \int_0^t \int_0^{t-s} T_\alpha(\tau) A f(s)\, d\tau\, ds. \end{aligned} \tag{124}$$

Then

$$(g_{2-\alpha} * v)'(0) = 0. \tag{125}$$

By (124), (122),

$$\begin{aligned} D_t^\alpha v(t) &= \frac{d^2}{dt^2} J_t^{2-\alpha} v(t) \\ &= \frac{d}{dt} \left(\int_0^t f(s)\, ds + \int_0^t \int_0^{t-s} T_\alpha(\tau) A f(s)\, d\tau\, ds \right) \\ &= f(t) + \int_0^t \left(\frac{d}{dt} \int_0^{t-s} T_\alpha(\tau) A f(s)\, d\tau \right) ds \\ &= f(t) + \int_0^t T_\alpha(t-s) A f(s)\, ds \\ &= f(t) + A \int_0^t T_\alpha(t-s) f(s)\, ds \\ &= f(t) + A v(t). \end{aligned} \tag{126}$$

Therefore v is a strong solution of (2) with initial values $(g_{2-\alpha} * v)(0) = 0$, $(g_{2-\alpha} * v)'(0) = 0$.

Case (II): Assume (ii) holds. Let

$$v(t) = \int_0^t T_\alpha(t-\tau) f(\tau)\, d\tau, \quad t \in [0, T]. \tag{127}$$

Since $f \in C([0,T];X)$ and $T_\alpha(t)$ is strongly continuous on R_+, by Proposition 1.3.4 in [3], we have

$$v(t) \in C\big([0,T];X\big). \tag{128}$$

Since $f \in C([0,T];X)$, $g_{2-\alpha} * f \in W^{2,1}([0,T];X)$, it is easy to see that $(g_{2-\alpha} * f)(0) = 0$, then it follows from (10) that

$$\begin{aligned} f(s) &= J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)(0) g_{\alpha-1}(s) + (g_{2-\alpha} * f)'(0) g_\alpha(s) \\ &= J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)'(0) g_\alpha(s). \end{aligned}$$

Then

$$\begin{aligned} v(t) &= \int_0^t T_\alpha(t-s) f(s)\, ds \\ &= \int_0^t T_\alpha(t-s)\big(J_s^\alpha D_s^\alpha f(s) + (g_{2-\alpha} * f)'(0) g_\alpha(s)\big)\, ds \\ &= \int_0^t T_\alpha(t-s) J_s^\alpha D_s^\alpha f(s)\, ds + \int_0^t g_\alpha(s) T(t-s)(g_{2-\alpha} * f)'(0)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \int_\tau^t (s-\tau)^{\alpha-1} T_\alpha(t-s) D_\tau^\alpha f(\tau)\, ds\, d\tau + J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^{t-\tau} (t-\tau-r)^{\alpha-1} T_\alpha(r) D_\tau^\alpha f(\tau)\, dr\, d\tau + J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0). \end{aligned}$$

The closedness of A and property (b) of Proposition 3.7 imply that $v(t) \in D(A)$ and

$$\begin{aligned} Av(t) &= \frac{1}{\Gamma(\alpha)} A \int_0^t \int_0^{t-\tau} (t-\tau-r)^{\alpha-1} T_\alpha(r) D_\tau^\alpha f(\tau)\, dr\, d\tau + A J_t^\alpha T_\alpha(t)(g_{2-\alpha} * f)'(0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t A \int_0^{t-\tau} (t-\tau-r)^{\alpha-1} T_\alpha(r) D_\tau^\alpha f(\tau)\, dr\, d\tau + T_\alpha(t)(g_{2-\alpha} * f)'(0) \\ &\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)} (g_{2-\alpha} * f)'(0) \\ &= \int_0^t \left(T_\alpha(t-\tau) D_\tau^\alpha f(\tau) - \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} D_\tau^\alpha f(\tau) \right) d\tau + T_\alpha(t)(g_{2-\alpha} * f)'(0) \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau)\,d\tau - J_t^\alpha D_t^\alpha f(t) + T_\alpha(t)(g_{2-\alpha} * f)'(0) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau)\,d\tau - f(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}(g_{2-\alpha} * f)'(0) + T_\alpha(t)(g_{2-\alpha} * f)'(0) \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(g_{2-\alpha} * f)'(0) \\
&= \int_0^t T_\alpha(t-\tau) D_\tau^\alpha f(\tau)\,d\tau - f(t) + T_\alpha(t)(g_{2-\alpha} * f)'(0). \qquad (129)
\end{aligned}
$$

By Definition 2.2,

$$
\begin{aligned}
D_t^{\alpha-1} v(t) &= \frac{d}{dt} J_t^{2-\alpha} v(t) \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t \int_0^r (t-r)^{1-\alpha} T_\alpha(s) f(r-s)\,ds\,dr. \qquad (130)
\end{aligned}
$$

Using Fubini's theorem, we obtain

$$
\begin{aligned}
D_t^{\alpha-1} v(t) &= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t \int_s^t (t-r)^{1-\alpha} T_\alpha(s) f(r-s)\,dr\,ds \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t T_\alpha(s) \int_0^{t-s} (t-s-r)^{1-\alpha} f(r)\,dr\,ds \\
&= \frac{1}{\Gamma(2-\alpha)} \int_0^t T_\alpha(s) \frac{d}{dt} \int_0^{t-s} (t-s-r)^{1-\alpha} f(r)\,dr\,ds \\
&\quad + \frac{1}{\Gamma(2-\alpha)} T_\alpha(s) \lim_{s\to t-0} \int_0^{t-s} (t-s-r)^{1-\alpha} f(r)\,dr \\
&= \frac{1}{\Gamma(2-\alpha)} \int_0^t T_\alpha(s) \frac{d}{dt} \int_0^{t-s} (t-s-r)^{1-\alpha} f(r)\,dr\,ds
\end{aligned}
$$

$$+\frac{1}{\Gamma(2-\alpha)}T_\alpha(s)\lim_{s\to t-0}(t-s)^{2-\alpha}\int_0^1(1-r)^{1-\alpha}f\big((t-s)r\big)\,dr$$

$$=\frac{1}{\Gamma(2-\alpha)}\int_0^t T_\alpha(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)\,dr\,ds. \tag{131}$$

Then

$$(g_{2-\alpha}*v)'(0)=D_t^{\alpha-1}v(t)\big|_{t=0}=0. \tag{132}$$

From Definition 2.2, (130), (131), we see that

$$\begin{aligned}
D_t^\alpha v(t) &= \frac{d^2}{dt^2}J_t^{2-\alpha}v(t)\\
&= \frac{d}{dt}D_t^{\alpha-1}v(t)\\
&= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t T(s)\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)\,dr\,ds\\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T(s)\frac{d^2}{dt^2}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)\,dr\,ds\\
&\quad+\frac{1}{\Gamma(2-\alpha)}T(t)\lim_{s\to t-0}\frac{d}{dt}\int_0^{t-s}(t-s-r)^{1-\alpha}f(r)\,dr\\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t T(t-\tau)\frac{d^2}{d\tau^2}\int_0^{\tau}(\tau-r)^{1-\alpha}f(r)\,dr\,d\tau\\
&\quad+\frac{1}{\Gamma(2-\alpha)}T(t)\lim_{s\to 0+}\frac{d}{ds}\int_0^{s}(s-r)^{1-\alpha}f(r)\,dr\\
&= \int_0^t T(t-\tau)D_\tau^\alpha f(\tau)\,d\tau+T_\alpha(t)(g_{2-\alpha}*f)'(0).
\end{aligned} \tag{133}$$

Therefore, by (129) and (133), we have

$$D_t^\alpha v(t)=Av(t)+f(t). \tag{134}$$

Since $Av \in C([0,T];X)$, $f \in C([0,T];X)$, then $D_t^\alpha v(t)$ is continuous on $[0,T]$. From (128), we have

$$(g_{2-\alpha} * v)(0) = 0. \tag{135}$$

Therefore v is a strong solution of (2) with initial values $(g_{2-\alpha} * v)(0) = 0$, $(g_{2-\alpha} * v)'(0) = 0$. The proof is complete. □

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