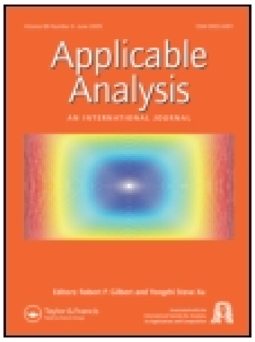

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A class of abstract fractional relaxation equations

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In this paper, we are concerned with a class of abstract fractional relaxation equations. We develop a new notion, named fractional (α, β) resolvent and derive some of its properties. By virtue of the obtained properties and the properties of general Mittag-Leffler function, we present some sufficient conditions to guarantee that the classical solutions of homogeneous and inhomogeneous fractional relaxation equations exist. An illustrative example is presented.

Keywords: fractional relaxation equation; fractional (α, β) resolvent; classical solution

AMS Subject Classifications: Primary: 34A08; Secondary: 47D06

1. Introduction

Recently, fractional differential equations have received increasing attention because the behavior of many physical systems, such as fluid flows, electrical networks, viscoelasticity, chemical physics, electron-analytical chemistry, biology, and control theory, can be properly described by using the fractional order system theory, etc. (see [1–5]). Fractional derivatives appear in the theory of fractional differential equations; they describe the property of memory and heredity of materials, and it is the major advantage of fractional derivatives compared with integer order derivatives.

Many of the references on fractional differential equations were focused on the existence and/or uniqueness of solutions for fractional differential equations.[1,6–9] Lots of fractional differential equations contained only one fractional derivative. However, in the real problems, the equations should be described by more than one fractional derivatives because of the complexity of models.[10–13] In [14,15], Bagley and Nonnenmacher studied fractional ordinary relaxation equations. Lizama and Prado [16] studied abstract fractional relaxation equations described by

$$u'(t) - A\mathbb{D}_t^\alpha u(t) + u(t) = f(t),\ 0 < \alpha < 1,\ t \geq 0,\ u(0) = 0, \tag{1.1}$$

on a Banach space X, where A is a closed linear operator, $\mathbb{D}_t^\alpha$ is the Caputo derivative of fractional α-order, and f is an X-valued function. They gave some existence and unique conditions.

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In this paper, we consider homogeneous abstract fractional relaxation equations described by

$$(AFRE)\begin{cases}\mathbb{D}_t^{\alpha}u(t) - A\mathbb{D}_t^{\beta}u(t) + u(t) = 0, \ \ t > 0,\\ u(0) = x,\end{cases} \tag{1.2}$$

and nonhomogeneous fractional relaxation equations

$$(NAFRE)\begin{cases}\mathbb{D}_t^{\alpha}u(t) - A\mathbb{D}_t^{\beta}u(t) + u(t) = \int_0^t f(s)ds, \ \ t > 0,\\ u(0) = x,\end{cases} \tag{1.3}$$

where $0 < \beta < \alpha \leq 1$, $u(\cdot)$ is the state, $A : D(A) \subset X \to X$ is a closed linear operator, $(X, \|\cdot\|)$ is a Banach space, $D(A)$ is the domain of A endowed with the graph norm $\|\cdot\|_{D(A)} = \|\cdot\| + \|A\cdot\|$, $\mathbb{D}_t^{\alpha}$ and $\mathbb{D}_t^{\beta}$ are, respectively, the α-order and β-order Caputo fractional derivative operators, $f : [0, \infty) \to X$ is locally integrable. Such class of fractional relaxation equations was subsequently generalized in the recent paper,[17] where the authors studied the characterization of periodic solutions.

The aim of this paper is to develop an operator theory to study abstract fractional relaxation equations $(AFRE)$ and $(NARE)$. Besides of the first section, we arrange this paper as follows. Section 2 is to recall some related definitions and preliminaries. In Section 3, we introduce the notion fractional (α, β) resolvent and derive some properties. The classical solutions of system $(AFRE)$ and $(NAFRE)$ are considered, some existence conditions of the classical solutions are obtained, and an illustration example is presented in Section 4.

2. Preliminaries

Let $\gamma > 0$ and denote $m = [\gamma]$ the smallest integer greater than or equal to γ. Denote $\mathbb{C}$ by the set consisting of all complex numbers. For $z \in \mathbb{C}$, Rez denotes the real part of z. Let $(X, \|\cdot\|)$ be a Banach space and A linear operator on X. We denote the resolvent operator of A by $R(\lambda, A) = (\lambda - A)^{-1}$ with λ being in the resolvent set $\rho(A)$. $L^1((0, T); X)$ denotes the space of X-valued Bochner integrable functions $u : (0, T) \to X$ with the norm $\|u\|_{L^1((0,T);X)} = (\int_0^T \|u(t)\|dt)$. The Banach space of k-times continuously differentiable functions $u : [0, T] \to X$ with the norm $\|u\|_{C^k([0,T];X)} = \Sigma_{l=0}^k \sup_{t\in[0,T]} \|u^{(l)}(t)\|$ is denoted by $C^k([0, T], X)$. Obviously, $L^1((0, T); X)$ is a Banach space. We denote the convolution of two functions f and g by $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \ t \geq 0$. The Laplace transform of a function $u \in L^1_{loc}(R^+, X)$ is defined by

$$\hat{u}(\lambda) := \int_0^{\infty} e^{-\lambda t}u(t)dt$$

for suitable λ such that the integral $\int_0^{\infty} e^{-\lambda t}u(t)dt$ is convergent on X.

Let $n \in N$, $1 \leq p < \infty$. Let $I = (0, T)$, or $I = [0, T]$, or $I = (0, \infty)$. The Sobolev spaces $W^{n,p}(I; X)$ is defined as follows ([18, Appendix]):

$$W^{n,p}(I; X) = \left\{u \mid \exists\varphi \in L^p(I; X) : u(t) = \sum_{k=0}^{n-1} c_k\frac{t^k}{k!} + \frac{t^{n-1}}{(n-1)!} * \varphi(t), \ t \in I\right\}.$$

In this case, we have $\varphi(t) = u^{(n)}(t), \ c_k = u^{(k)}(0)$.

For $\beta \geq 0$, let

$$g_\beta(t) = \begin{cases} \dfrac{t^{\beta-1}}{\Gamma(\beta)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

For the convenience of the readers, we shall introduce some definitions and some fundamental properties of fractional calculus theory, which can be found in [2,5,19–23].

Definition 2.1 For any $u \in L^1((0, T); X)$, the α-order Riemann-Liouville fractional integral of u is defined by

$$J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau. \tag{2.2}$$

We denote $J_t^0 u(t) = u(t)$. Obviously, the fractional integral operators $\{J_t^\alpha\}_{\alpha \geq 0}$ satisfies the semigroup property $J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \ \alpha, \beta \geq 0$.

Definition 2.2 Let $\alpha > 0$ and $m = [\alpha]$. The (modified) α-order the Caputo fractional derivative of u is defined by

$$\mathbb{D}_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\sigma)^{m-\alpha-1} \left(u(\sigma) - \Sigma_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0) \right) d\sigma. \tag{2.3}$$

Obviously, the operator J_t^α as well as $\mathbb{D}_t^\alpha$ is linear operator. If $u \in C^m([0, \infty), X)$, then we have the following equivalent related to Caputo fractional derivative of u

$$\mathbb{D}_t^\alpha u(t) = \int_0^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} u^{(m)}(s) ds.$$

In particular, for the function $u(t) \equiv c \in X$, we have $\mathbb{D}_t^\alpha u(t) = 0$.

Definition 2.3 The general Mittag-Leffler function is defined by

$$E_{\alpha,\beta}^k(z) = \sum_{n=0}^\infty \frac{(k)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} (z, \beta, k \in \mathbb{C};\ Re(\alpha) > 0),$$

where

$$(k)_n := \frac{\Gamma(k+n)}{\Gamma(k)} = \begin{cases} 1, & n = 0 \\ k(k+1)\cdots(k+n-1), & n \neq 0. \end{cases}$$

Remark 2.4 In the special case $k = 1$, the general Mittag-Leffler function is equal to the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$. If, in addition, $\beta = 1$, the general Mittag-Leffler function is equal to the one-parameter Mittag-Leffler function $E_\alpha(\cdot)$.

Remark 2.5 The one-parameter Mittag-Leffler function was introduced by Mittag-Leffler [24,25]; Wiman [26,27] defined two-parameter Mittag-Leffler function; while Prabhakar [22] introduced the general Mittag-Leffler function. For more details of Mittag-Leffler function, we refer to [19,21,23,28].

For $\alpha,\ \delta,\ \xi,\ \gamma,\ \mu > 0$, $\beta > \tau > 0$ and $t > 0$, there hold the following three equalities related to Mittag-Lefller functions [19]:

$$J_t^{\xi}(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(wt^{\alpha})) = t^{\beta+\xi-1}E^{\gamma}_{\alpha,\beta+\xi}(wt^{\alpha}), \tag{2.4}$$

$$\int_0^t (t-s)^{\beta-1}s^{\mu-1}E^{\gamma}_{\alpha,\beta}(w(t-s)^{\alpha})E^{\delta}_{\alpha,\mu}(ws^{\alpha})ds = t^{\beta+\mu-1}E^{\gamma+\delta}_{\alpha,\beta+\mu}(wt^{\alpha}), \tag{2.5}$$

and

$$D_t^{\tau}t^{\beta-1}E^{\gamma}_{\alpha,\beta}(wt^{\alpha}) = t^{\beta-\tau-1}E^{\gamma}_{\alpha,\beta-\tau}(wt^{\alpha}). \tag{2.6}$$

The Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$ is related to the Laplace integral

$$\int_0^{\infty} e^{-\lambda t}t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})dt = \frac{\lambda^{\gamma\alpha-\beta}}{(\lambda^{\alpha}-\omega)^{\gamma}},\ \ Re\lambda > |\omega|^{1/\alpha}. \tag{2.7}$$

We denote the two-parameter Mittag-Leffler integral operator by

$$\mathbb{E}_t^{\alpha,\beta}f(t) = \int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}(-(t-s)^{\alpha})f(s)ds,\ \ t > 0. \tag{2.8}$$

3. Fractional (α, β) resolvent

In this section, we shall present the notion fractional (α, β) resolvent and deduce its some properties. In this section, we assume $0 < \beta < \alpha \le 1$.

Definition 3.1 A family $\{T(t)\}_{t\ge 0}$ of bounded linear operators is called a fractional (α, β) resolvent, if it satisfies the following assumptions:

(a) For any $x \in X,\ T(\cdot)x \in C([0,\infty), X)$, and

$$\lim_{t\to 0^+}\frac{T(t)x}{t^{\alpha}} = \frac{x}{\Gamma(\alpha+1)},\quad x \in X; \tag{3.1}$$

(b) $T(s)T(t) = T(t)T(s),\ t, s \ge 0$;

(c) there holds

$$T(s)\mathbb{E}_t^{\alpha,\alpha-\beta,-1}T(t) - \mathbb{E}_s^{\alpha,\alpha-\beta}T(s)T(t) = s^{\alpha}E_{\alpha,\alpha+1}(-s^{\alpha})\mathbb{E}_t^{\alpha,\alpha-\beta}T(t) - t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})\mathbb{E}_s^{\alpha,\alpha-\beta}T(s),\ \ t, s \ge 0. \tag{3.2}$$

Remark 3.2 The integrals in (3.2) are understood strongly in the sense of Bochner.

Remark 3.3 It is noted that the equality (3.2) is indeed the functional Equation (4.1) of [29] with $k(t) = t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})$ and $a(t) = t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^{\alpha})$.

Proposition 3.4 Assume that $\{T(t)\}_{t\ge 0}$ be a fractional resolvent. Then, the function $t \mapsto \frac{T(t)}{t^{\alpha}}$ is bounded in the sense of operator norm over the interval $(0, b]$ with $b > 0$.

Proof Define operator $(PT)(\cdot)$: $x \in X$,

$$(PT)(t)x \triangleq \begin{cases} \frac{\Gamma(\alpha+1)T(t)x}{t^\alpha}, & t > 0, \\ x, & t = 0. \end{cases}$$

Then, for any $x \in X$, (a) of Definition 3.1 implies that $(PT)x(\cdot)$ is strongly continuous on $[0, \infty)$. This means that $(PT)x(t)$ is bounded over then interval $[0, b]$. By uniform boundedness theorem, the function $t \mapsto (PT)(t)$ is bounded over the interval $[0, b]$. The proof is completed. □

Definition 3.5 The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0+} \frac{t^{-\alpha}T(t)x - E_{\alpha,\alpha+1}(-t^\alpha)x}{t^{\alpha-\beta}} \text{ exists} \right\}$$

and

$$Ax = \Gamma(2\alpha - \beta + 1) \lim_{t\to 0+} \frac{t^{-\alpha}T(t)x - E_{\alpha,\alpha+1}(-t^\alpha)x}{t^{\alpha-\beta}} \quad \text{for } x \in D(A)$$

is said to be the generator of the fractional (α, β) resolvent $\{T(t)\}_{t\geq 0}$, where $D(A)$ is the domain of A.

Remark 3.6 We note that the above defined operator A is the same as B in (4.2) and (4.3) of [29] with $k(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)$ and $a(t) = t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^\alpha)$. In fact, it follows from (2.5) that

$$\lim_{t\to 0+} \frac{T(t)x - k(t)x}{(a * k)(t)} = \lim_{t\to 0+} \frac{t^{-\alpha}T(t)x - E_{\alpha,\alpha+1}(-t^\alpha)x}{t^{\alpha-\beta}} \frac{1}{\lim_{t\to 0+} E^2_{\alpha,2\alpha-\beta+1}(-t^\alpha)}.$$

Here the limit $\lim_{t\to 0+} E^2_{\alpha,2\alpha-\beta+1}(-t^\alpha) = \frac{1}{\Gamma(2\alpha-\beta+1)}$ is used to obtain the result.

Remark 3.7 Suppose that $\{T(t)\}_{t\geq 0}$ is a fractional (α, β) resolvent on Banach space X with generator A. Let $k(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)$ and $a(t) = t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^\alpha)$. Then (a) of Definition (3.1) implies that

$$T(0)x = \lim_{t\to 0^+} t^\alpha \lim_{t\to 0^+} \frac{T(t)}{t^\alpha} x = 0 = k(0)x, \; x \in X.$$

Moreover, by (2.5), we obtain

$$\lim_{t\to 0^+} \frac{(a * (a * k))(t)}{(a * k)(t)} = \lim_{t\to 0^+} t^{\alpha-\beta} \lim_{t\to 0^+} \frac{E^3_{3\alpha-2\beta+1}(-t^\alpha)}{E^2_{2\alpha-\beta+1}(-t^\alpha)} = 0.$$

Therefore, by Theorem 4.3 of [29], it follows that $\{T(t)\}_{t\geq 0}$ is an (a, k)-regularized resolvent with generator A. This indicates that

- $\{T(t)\}_{t\geq 0}$ commutes with A, that is, $T(t)D(A) \subset D(A)$ and $AT(t)x = T(t)Ax$ for each $x \in D(A)$, $t \geq 0$;
- For any $x \in D(A)$, $t \geq 0$,

$$T(t)x = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)x + \mathbb{E}_t^{\alpha,\alpha-\beta} T(t)Ax. \tag{3.3}$$

However, condition (4.4) of ([29]) does not hold. Indeed,

$$\lim_{t\to 0^+} \frac{\int_0^t |a(s)|ds}{(a*k)(t)} = \lim_{t\to 0^+} \frac{E_{\alpha,\alpha-\beta+1}(-t^\alpha)}{t^\alpha E^2_{\alpha,2\alpha-\beta+1}(-t^\alpha)} = +\infty.$$

Hence, we can't obtain the closedness of A and density of $D(A)$ directly from Theorem 4.1 of ([29]). In order to prove such properties, (a) of Definition 3.1 shall be used.

Proposition 3.8 Assume that $\{T(t)\}_{t\geq 0}$ is a fractional (α, β) resolvent on Banach space X with generator A. Then

(a) For any $x \in X$, $t \geq 0$,

$$T(t)x = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)x + A\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)x. \tag{3.4}$$

(b) A is closed and densely defined.

Proof (a) For any $x \in X$, we have

$$\begin{aligned}
&\left\|\Gamma(2\alpha-\beta+1)\frac{\mathbb{E}_s^{\alpha,\alpha-\beta}T(s)x}{s^{2\alpha-\beta}} - x\right\| \\
&\quad= \left\|\Gamma(2\alpha-\beta+1)\int_0^s (s-\sigma)^{\alpha-\beta-1}s^{-2\alpha+\beta}E_{\alpha,\alpha-\beta}(-(s-\sigma)^\alpha)T(\sigma)xd\sigma - x\right\| \\
&\quad= \left\|\Gamma(2\alpha-\beta+1)\int_0^1 (1-\sigma)^{\alpha-\beta-1}s^{-\alpha}E_{\alpha,\alpha-\beta}(-(s-s\sigma)^\alpha)T(s\sigma)xd\sigma - x\right\| \\
&\quad= \left\|\frac{\Gamma(2\alpha-\beta+1)}{\Gamma(\alpha+1)}\int_0^1 (1-\sigma)^{\alpha-\beta-1}\sigma^\alpha E_{\alpha,\alpha-\beta}(-(s-s\sigma)^\alpha)\cdot\right. \\
&\left.\Gamma(\alpha+1)(s\sigma)^{-\alpha}T(s\sigma)xd\sigma - x\right\| \\
&\quad= \left\|\frac{\Gamma(2\alpha-\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta)}\int_0^1 (1-\sigma)^{\alpha-\beta-1}\sigma^\alpha\Gamma(\alpha-\beta)E_{\alpha,\alpha-\beta}(-(s-s\sigma)^\alpha)\cdot\right. \\
&\left.\Gamma(\alpha+1)(s\sigma)^{-\alpha}T(s\sigma)xd\sigma - \frac{\Gamma(2\alpha-\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta)}\int_0^1 (1-\sigma)^{\alpha-\beta-1}\sigma^\alpha xd\sigma\right\| \\
&\quad\leq \frac{\Gamma(2\alpha-\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta)}\int_0^1 (1-\sigma)^{\alpha-\beta-1}\sigma^\alpha d\sigma\cdot \\
&\sup_{\sigma\in(0,1]}\|\Gamma(\alpha-\beta)E_{\alpha,\alpha-\beta}(-(s-s\sigma)^\alpha)\Gamma(\alpha+1)(s\sigma)^{-\alpha}T(s\sigma)x - x\| \\
&\quad= \sup_{\sigma\in(0,1]}\|\Gamma(\alpha-\beta)E_{\alpha,\alpha-\beta}(-(s-s\sigma)^\alpha)\Gamma(\alpha+1)(s\sigma)^{-\alpha}T(s\sigma)x - x\|
\end{aligned} \tag{3.5}$$

The combination of (3.5), (a) of Definition 3.1, and Theorem 3.4 implies that

$$\lim_{s\to 0^+}\Gamma(2\alpha-\beta+1)\frac{\mathbb{E}_s^{\alpha,\alpha-\beta}T(s)x}{s^{2\alpha-\beta}} = x. \tag{3.6}$$

Using (c) of Definition 3.1, we derive

$$\begin{aligned}
&A\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)x \\
&= \Gamma(2\alpha-\beta+1)\lim_{s\to 0+}\frac{s^{-\alpha}T(s)\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)x - E_{\alpha,\alpha+1}(-s^{\alpha})\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)x}{s^{\alpha-\beta}} \\
&= \Gamma(2\alpha-\beta+1)\lim_{s\to 0+}\frac{\mathbb{E}_s^{\alpha,\alpha-\beta}T(s)\left(T(t)x - t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x\right)}{s^{2\alpha-\beta}} \\
&= T(t)x - t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x, \qquad (3.7)
\end{aligned}$$

which implies that (a) holds.

(b) Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From the equality (3.3), we have

$$\begin{aligned}
&T(t)x - t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x \\
&= \lim_{n\to\infty}\left(T(t)x_n - t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x_n\right) \\
&= \lim_{n\to\infty}\int_0^t (t-\sigma)^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-(t-\sigma)^{\alpha})T(\sigma)Ax_n d\tau \\
&= \int_0^t (t-\sigma)^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-(t-\sigma)^{\alpha})T(\sigma)y d\tau = \mathbb{E}_t^{\alpha,\alpha-\beta}T(t)y,\ t \ge 0. \qquad (3.8)
\end{aligned}$$

Using (3.6), we have

$$\begin{aligned}
Ax &= \Gamma(2\alpha-\beta+1)\lim_{t\to 0+}\frac{t^{-\alpha}T(t)x - E_{\alpha,\alpha+1}(-t^{\alpha})x}{t^{\alpha-\beta}} \\
&= \Gamma(2\alpha-\beta+1)\lim_{t\to 0+}\frac{\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)y}{t^{2\alpha-\beta}} = y. \qquad (3.9)
\end{aligned}$$

The closeness of A is obtained.

For every $x \in X$, set $x_t = \mathbb{E}_t^{\alpha,\alpha-\beta}T(t)x$, from (3.7) it follows that $x_t \in D(A)$, and by (3.6) we have $\Gamma(2\alpha-\beta+1)t^{-2\alpha+\beta}x_t \to x$ as $t \to 0+$. Thus $\overline{D(A)} = X$. □

Remark 3.9 By Remark 3.7 and [30, Lemma 2.2], it follows that (3.3) holds provided $\rho(A) \neq \emptyset$. Here we don't know whether $\rho(A)$ is empty or not, hence (a) of Definition 3.1 is essentially important in the proof.

The following theorem indicates that every closed densely defined operator A generates at most one fractional (α, β) resolvent.

THEOREM 3.10 *Assume that $\{T(t)\}_{t\ge 0}$ and $\{S(t)\}_{t\ge 0}$ are fractional (α, β) resolvents on Banach space X generated by A and B, respectively. Then $T(t) = S(t)$ for $t \ge 0$, provided A is equal to B.*

Proof For $x \in D(A)$, the combination of property (c) and property (a) of Proposition 3.8 implies that

$$\begin{aligned} t^\alpha E_{\alpha,\alpha+1}(-t^\alpha) * S(t)x &= (T(t) - \mathbb{E}_t^{\alpha,\alpha-\beta} AT(t)) * S(t)x \\ &= T(t) * S(t)x - t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha) * AT(t) * S(t)x \\ &= T(t) * S(t)x - t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha) * T(t) * AS(t)x \\ &= T(t) * (S(t)x - t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha) * AS(t)x) \\ &= T(t) * t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)x \\ &= t^\alpha E_{\alpha,\alpha+1}(-t^\alpha) * T(t)x, \end{aligned}$$

by Titchmarsh's theorem, $T(t)x = S(t)x,\ x \in D(A),\ t \geq 0$. The density of $D(A)$ indications that $T(t) = S(t),\ t \geq 0$. □

Definition 3.11 Fractional (α, β) resolvent on Banach space $\{T(t)\}_{t\geq 0}$ is called exponentially bounded if there exist constants $M \geq 1$, $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t},\ t \geq 0. \tag{3.10}$$

An operator A is said to belong $\mathcal{C}^{\alpha,\beta}(M, \omega)$ if it is the generator of a Riemann-Liouville fractional (α, β) resolvent $\{T(t)\}_{t\geq 0}$ satisfying (3.10). Denote $\mathcal{C}^{\alpha,\beta}(\omega) = \bigcup\{\mathcal{C}^{\alpha,\beta}(M, \omega);\ M \geq 1\}$.

Now we introduce the following generation theorem of fractional (α, β) resolvent.

THEOREM 3.12 *$A \in \mathcal{C}^{\alpha,\beta}(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there is a family $\{T(t)\}_{t>0}$ of bounded linear operators satisfying*

(1) *for any $x \in X,\ T(\cdot)x \in C([0, \infty), X)$, and*

$$\lim_{t\to 0^+} \frac{T(t)}{t^\alpha} x = \frac{x}{\Gamma(\alpha + 1)} \quad \text{for all } x \in X; \tag{3.11}$$

(2) $\|T(t)\| \leq Me^{\omega t}$, $M \geq 1$, $t \geq 0$.

(3) *there holds*

$$R(\lambda^{-\beta}(\lambda^\alpha + 1), A)x = \lambda^{1+\beta} \int_0^\infty e^{-\lambda t} T(t)x dt,\ Re\lambda > \omega,\ x \in X. \tag{3.12}$$

In the case, $\{T(t)\}_{t\geq 0}$ is the fractional (α, β) resolvent generated by A.

Proof Let $k(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)$, $a(t) = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha)$. Using (2.7), we obtain $\hat{a}(\lambda) = \frac{\lambda^\beta}{\lambda^\alpha+1},\ \hat{k}(\lambda) = \frac{\lambda^{-1}}{\lambda^\alpha+1}$.

(Necessity) By Remark 3.7, $A \in \mathcal{C}^{\alpha,\beta}(M, \omega)$ implies that A generates an (a, k)-regularized resolvent. The necessity is obtained from Proposition 3.1 of [30].

(Sufficiency) By Proposition 3.1 of [30], (3.12) implies that A generates an (a, k)-regularized resolvent $\{T(t)\}_{t\geq 0}$. The closedness of A is derived from the proof of Proposition 3.8. By Theorem 3.1 of [29], it follows that (b) and (c) of Definition 3.1 hold. The proof is completed. □

4. Existence and uniqueness of the classical solutions

This section is devoted to studying the existence and the uniqueness of the classical solution of the homogeneous and inhomogeneous fractional relaxation equations. The properties of fractional (α, β) resolvent obtained in Sec. 3 and the properties of general Mittag-Leffler functions are used. Moreover, the definition of the mild solution of $(NAFRE)$ will be given. In this section, we assume $0 < \beta < \alpha \leq 1$.

We firstly consider homogeneous fractional relaxation equation $(AFRE)$. The definition of classical solution is defined as follows.

Definition 4.1 Assume that X is a Banach space. A function $u \in C([0, \infty), X)$ is called a classical solution of $(AFRE)$, if

(i) $t \mapsto \int_0^t (t-s)^{-\alpha}[u(s) - x]ds$ and $t \mapsto \int_0^t (t-s)^{-\beta}[u(s) - x]ds$ are continuously differentiable on $[0, \infty)$;

(ii) $\frac{d}{dt}\int_0^t (t-s)^{-\beta}[u(s) - x]ds \in D(A)$;

(iii) there holds

$$\mathbb{D}_t^{\alpha} u(t) - A\mathbb{D}_t^{\beta} u(t) + u(t) = 0, \ t \geq 0.$$

Lemma 4.2 *For any $t \geq 0$, one has*

$$E_{\alpha,1}(-t^{\alpha}) + t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}) = 1. \tag{4.1}$$

Proof Obviously, for any λ with large real part, there holds

$$0 = \frac{\lambda^{\alpha-1} + \lambda^{-1} - (\lambda^{\alpha-1} + \lambda^{-1})}{\lambda^{\alpha} + 1} = \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} + 1} + \frac{\lambda^{-1}}{\lambda^{\alpha} + 1} - \lambda^{-1}. \tag{4.2}$$

Taking inverse Laplace transform on both sides of (4.2), using (2.7), we obtain

$$0 = E_{\alpha,1}(-t^{\alpha}) + t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}) - 1.$$

The proof is therefore completed. □

Theorem 4.3 *Assume operator A to generate a fractional (α, β) resolvent $\{T(t)\}_{t\geq 0}$ on Banach space X. Let $K = \left[\frac{\beta}{\alpha-\beta}\right]$. Then, for any $x \in D(A^{K+1})$, $x - T(\cdot)x$ is a classical solution of $(AFRE)$.*

Proof Assume that $x \in D(A^{K+1})$. Then, $x \in D(A^n), \ n = 1, 2, \cdots, K+1$. By Lemma 4.2, it follows that

$$\left(E_{\alpha,1}(-t^{\alpha}) + t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}) - 1\right) * T(t)x = 0 \tag{4.3}$$

The combination of (3.3) and (4.3) implies that

$$\begin{aligned}&\int_0^t E_{\alpha,1}(-(t-s)^\alpha)T(s)xds+\int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}((t-s)^\alpha)T(s)xds\\&\quad=\int_0^t T(s)xds\\&\quad=\int_0^t (s^\alpha E_{\alpha,\alpha+1}(-s^\alpha)x+\mathbb{E}_s^{\alpha,\alpha-\beta}T(s)Ax)ds\\&\quad=t^{\alpha+1}E_{\alpha,\alpha+2}(-t^\alpha)x+\int_0^t (t-s)^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-(t-s)^\alpha)T(s)Axds.\end{aligned}$$

Using (2.4), we have

$$\begin{aligned}&\int_0^t E_{\alpha,1}(-(t-s)^\alpha)\left[T(s)x+\int_0^s \frac{(s-\sigma)^{\alpha-1}}{\Gamma(\alpha)}T(\sigma)xd\sigma\right]ds\\&=\int_0^t E_{\alpha,1}(-(t-s)^\alpha)\left[\frac{s^\alpha}{\Gamma(\alpha+1)}x+\int_0^s \frac{(s-\sigma)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}T(\sigma)Axd\sigma\right]ds,\ t\geq 0.\end{aligned}$$

By Titchmarsh's theorem, we obtain

$$\begin{aligned}&T(t)x+\int_0^t \frac{(t-\sigma)^{\alpha-1}}{\Gamma(\alpha)}T(\sigma)xd\sigma\\&\quad=\frac{t^\alpha}{\Gamma(\alpha+1)}x+\int_0^t \frac{(t-\sigma)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}T(\sigma)Axd\sigma,\ t\geq 0.\end{aligned}$$

Taking $1-\alpha$ times integral on both sides of the above equality, it follows that

$$\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma+\int_0^t T(\sigma)xd\sigma=tx+\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)Axd\sigma,\ t\geq 0.$$

The closedness of A implies that

$$\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}T(\sigma)xd\sigma+\int_0^t T(\sigma)xd\sigma=tx+A\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)xd\sigma,\ t\geq 0.$$

Obviously, $\int_0^t T(\sigma)xd\sigma$ and tx are continuously differentiable on $[0,\infty)$. By (3.3), we can compute

$$\begin{aligned}&\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}T(\sigma)Axd\sigma\\&\quad=J_t^{1-\beta}\left(t^\alpha E_{\alpha,\alpha+1}(-t^\alpha)Ax+\mathbb{E}_t^{\alpha,\alpha-\beta}T(t)A^2x\right)\\&\quad=t^{\alpha-\beta+1}E_{\alpha,\alpha+2-\beta}(-t^\alpha)Ax+\mathbb{E}_t^{\alpha,1-\beta+(\alpha-\beta)}T(t)A^2x.\end{aligned}$$

Obviously,

$$-\beta+K(\alpha-\beta)\geq 0. \tag{4.4}$$

By induction, we derive

$$\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) Ax d\sigma$$
$$= \sum_{n=1}^{K} t^{n(\alpha-\beta)} E^n_{\alpha,2+n(\alpha-\beta)}(-t^\alpha) A^n x$$
$$+ t^{K(\alpha-\beta)-\beta} E^K_{\alpha,1-\beta+K(\alpha-\beta)}(-t^\alpha) * T(t) A^{K+1} x.$$

Using the property of general Mittag-Leffler functions, we have that

$$\sum_{n=1}^{K} t^{n(\alpha-\beta)} E^n_{\alpha,1+n(\alpha-\beta)}(-t^\alpha) A^n x$$

is continuously differentiable on $[0, \infty)$. By (4.4), it follows that

$$t^{K(\alpha-\beta)-\beta} E^K_{\alpha,1-\beta+K(\alpha-\beta)}(-t^\alpha) * T(t) A^{K+1} x$$

is continuously differentiable and

$$\frac{d}{dt} t^{K(\alpha-\beta)-\beta} E^K_{\alpha,1-\beta+K(\alpha-\beta)}(-t^\alpha) * T(t) A^{K+1} x$$
$$= t^{K(\alpha-\beta)-\beta} \frac{d}{dt} E^K_{\alpha,1-\beta+K(\alpha-\beta)}(-t^\alpha) * T(t) A^{K+1} x, \ t \geq 0.$$

This means that $\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) Ax d\sigma$ is continuously differentiable on $[0, \infty)$. Hence $\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} T(\sigma) x d\sigma$ is continuously differentiable on $[0, \infty)$. Similarly, we obtain

$$\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) x d\sigma$$
$$= \sum_{n=1}^{K} t^{n(\alpha-\beta)} E^n_{\alpha,2+n(\alpha-\beta)}(-t^\alpha) A^{n-1} x + t^{K(\alpha-\beta)-\beta} E^K_{\alpha,1-\beta+K(\alpha-\beta)} * T(t) A^K x.$$

and the function $t \mapsto \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) x d\sigma$ is continuously differentiable on $[0, \infty)$.

$$A \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) x d\sigma = \int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} T(\sigma) x d\sigma + \int_0^t T(\sigma) x d\sigma - tx$$

is continuously differentiable on $[0, \infty)$. Since A is closed, we have

$$A \frac{d}{dt} \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma) x d\sigma = \frac{d}{dt} \int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} T(\sigma) x d\sigma + T(t)x - x, \ t \geq 0.$$

This means that

$$\frac{d}{dt} \int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} [(x - T(\sigma)x) - x] d\sigma - A \frac{d}{dt} \int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)} [(x - T(\sigma)x) - x] d\sigma$$
$$+ x - T(t)x = 0, \ t \geq 0,$$

that is,

$$\mathbb{D}_t^{\alpha}[x - T(t)x] - A\mathbb{D}_t^{\beta}[x - T(t)x] + x - T(t)x = 0,\ t \geq 0.$$

The proof is completed. □

THEOREM 4.4 *Assume A to generate a fractional (α, β) resolvent $\{T(t)\}_{t\geq 0}$ on Banach space X. Let u be a classical solution of $(AFRE)$. Then, $u(t) = x - T(t)x,\ t \geq 0$.*

Proof Suppose that u is a classical solution of $(AFRE)$. Then, $u(\cdot) \in C([0,\infty), X)$, $u(0) = x$,

$$\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma) - x]d\sigma \in D(A),\ t \geq 0,$$

and

$$\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma) - x]d\sigma - A\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma) - x]d\sigma + u(t) = 0. \quad (4.5)$$

The combination of (4.5) and closedness of A implies that $t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}$ $[u(\sigma) - x]d\sigma \in D(A)$ and

$$\begin{aligned} &t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * A\frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma) - x]d\sigma \\ &= A\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma) - x]d\sigma\right]. \end{aligned}$$

Then, we have

$$\begin{aligned} &t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma) - x]d\sigma \\ &= A\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma) - x]d\sigma\right] - t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * u(t). \end{aligned} \quad (4.6)$$

Obviously, we have the following two equalities

$$\begin{aligned} &\frac{d}{dt}\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma) - x]d\sigma\right] \\ &= t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) * \frac{d}{dt}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma) - x]d\sigma \\ &\quad + t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) \lim_{t\to 0^+}\int_0^t \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma) - x]d\sigma. \end{aligned} \quad (4.7)$$

and

$$\begin{aligned}\frac{d}{dt}\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*\int_0^t\frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma)-x]d\sigma\right]\\ =t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*\frac{d}{dt}\int_0^t\frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma)-x]d\sigma\\ +t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})\lim_{t\to 0^+}\int_0^t\frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma)-x]d\sigma.\end{aligned}\tag{4.8}$$

Using the inequality

$$\begin{aligned}\left\|\int_0^t\frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma)-x]d\sigma\right\|&\le\int_0^t\frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}\|u(\sigma)-x\|d\sigma\\ &\le\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\max_{0\le\sigma\le t}\|u(\sigma)-x\|\end{aligned}$$

and the fact $u\in([0,\infty),X)$ with $u(0)=x$, we obtain that

$$\lim_{t\to 0^+}\int_0^t\frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma)-x]d\sigma=0.\tag{4.9}$$

Similarly, we have

$$\lim_{t\to 0^+}\int_0^t\frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma)-x]d\sigma=0.\tag{4.10}$$

Put (4.6)–(4.10) to get

$$\begin{aligned}&\frac{d}{dt}\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*\int_0^t\frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}[u(\sigma)-x]d\sigma\right]\\ &=A\frac{d}{dt}\left[t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*\int_0^t\frac{(t-\sigma)^{-\beta}}{\Gamma(1-\beta)}[u(\sigma)-x]d\sigma\right]-t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*u(t).\end{aligned}$$

By virtue of (2.4) and (2.6), there holds

$$\begin{aligned}&\frac{d}{dt}\left[E_{\alpha,1}(-t^{\alpha})*(u(t)-x)\right]\\ &=A\left[E_{\alpha,\alpha-\beta}(-t^{\alpha})*(u(t)-x)\right]\\ &\quad-t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*(u(t)-x)-t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x,\end{aligned}$$

that is,

$$\begin{aligned}&A\left[E_{\alpha,\alpha-\beta}(-t^{\alpha})*(x-u(t))\right]+t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x\\ &=\frac{d}{dt}\left[E_{\alpha,1}(-t^{\alpha})*(x-u(t))\right]+t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})*(x-u(t))\\ &=\frac{d}{dt}\left[\left(E_{\alpha,1}(-t^{\alpha})+t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})\right)*(x-u(t))\right].\end{aligned}$$

By equality (4.1), it follows that

$$A\left[E_{\alpha,\alpha-\beta}(-t^{\alpha}) * (x - u(t))\right] + t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha})x = x - u(t). \tag{4.11}$$

By (3.4) and (4.11), for any $t \geq 0$, we have

$$\begin{aligned}
&t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha}) * [u(t) - x] \\
&= \left(T(t) - t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^{\alpha}) * AT(t)\right) * [u(t) - x] \\
&= T(t) * [u(t) - x] - t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^{\alpha}) * AT(t) * [u(t) - x] \\
&= T(t) * \left(u(t) - x - t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}(-t^{\alpha}) * A[u(t) - x]\right) \\
&= -t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha}) * T(t)x.
\end{aligned}$$

By Titchmarsh's theorem, we have $u(t) = x - T(t)x,\ t \geq 0$. □

Remark 4.5 Theorem 4.4 implies that fractional relaxation equation $(AFRE)$ has at most one classical solution.

Now we consider system $(NAFRE)$. The definition of the classical solution is defined as follows.

Definition 4.6 Assume that X is a Banach space. A function $u \in C([0, \infty), X)$ is called a classical solution of $(AFRE)$, if

(i) $t \mapsto \int_0^t (t-s)^{-\alpha}[u(s) - x]ds$ and $t \mapsto \int_0^t (t-s)^{-\beta}[u(s) - x]ds$ are continuously differentiable on $[0, \infty)$;
(ii) $\frac{d}{dt}\int_0^t (t-s)^{-\beta}[u(s) - x]ds \in D(A)$;
(iii) there holds

$$\mathbb{D}_t^{\alpha}u(t) - A\mathbb{D}_t^{\beta}u(t) + u(t) = \int_0^t f(s)ds,\ t \geq 0.$$

THEOREM 4.7 *Let A be the generator of fractional (α, β) resolvent $\{T(t)\}_{t\geq 0}$ on Banach space X. Let K be defined as in Theorem 4.3. Assume that either of the following two conditions holds:*

(i) $f \in C([0, \infty), D(A^{K+1}))$.
(ii) $f = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^{\alpha})g(s)ds$ *and* $\int_0^t E_{\alpha}(-(t-s)^{\alpha})g(s)ds \in W^{1,1}([0, T]; X)$,
Then for every $x \in D(A^{K+1})$, system $(AFRE)$ has a unique classical solution u given by

$$u(t) = x - T(t)x + \int_0^t T(t-s)f(s)ds, \quad t \in [0, T]. \tag{4.12}$$

Proof Uniqueness: Let $u_1,\ u_2$ be two classical solutions of $(NAFRE)$. Then $w := u_1 - u_2$, $w(0) = 0$, and $\mathbb{D}_t^\alpha w(t) - A\mathbb{D}_t^\beta w(t) + w(t) = 0$ for all $t \in [0, T]$. It follows from Theorem 4.4 that $w \equiv 0$.

Existence: By Theorems 4.3 and 4.4, it follows that $T(t)x$ is the unique classical solution of system $(AFRE)$. Therefore, we only need to verify that $v(t)$ defined by

$$v(t) = \int_0^t T(t-s)f(s)ds,\ t \in [0, T] \tag{4.13}$$

is a strong solution of system $(NAFRE)$ with $x = 0$.

Case (I): Assume (i) holds. By Proposition 1.3.4 in [31], it follows that $v(\cdot) \in C([0, T]; X)$. It is easy to obtain that $v(0) = 0$. The combination of the strong continuousness of $T(\cdot)$ and uniform boundedness theorem implies that $T(\cdot)$ is bounded over any bounded interval. For any $0 < s < t$, we have

$$\begin{aligned}\left\| \int_0^{t-s} \frac{(t-s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} T(\sigma)f(s)d\sigma \right\| &\le \int_0^{t-s} \frac{(t-s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} \|T(\sigma)\|\|f(s)\|d\sigma \\ &\le \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} \max_{0\le\sigma\le t} \|T(\sigma)\|\|f(s)\|.\end{aligned}$$

Hence,

$$\lim_{s\to t^-} \int_0^{t-s} \frac{(t-s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} T(\sigma)f(s)d\sigma \to 0. \tag{4.14}$$

Similarly, we obtain that

$$\lim_{s\to t^-} \int_0^{t-s} \frac{(t-s-\sigma)^{-\beta}}{\Gamma(1-\beta)} T(\sigma)f(s)d\sigma \to 0. \tag{4.15}$$

Observe that $f(t) \in D(A^{k+1})$ for $t \ge 0$. Since A is closed, combining (4.14), (4.15) and Theorem 4.3, it follows that

$$\begin{aligned}
&\mathbb{D}_t^\alpha v(t) - A\mathbb{D}_t^\beta v(t) + v(t) \\
&\quad = \frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}[v(s) - v(0)]ds - A\frac{d}{dt}\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}[v(s) - v(0)]ds + v(t) \\
&\quad = \frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds - A\frac{d}{dt}\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}v(s)ds + v(t) \\
&\quad = \frac{d}{dt}J_t^{1-\alpha}(T * f)(t) - A\frac{d}{dt}J_t^{1-\beta}(T * f)(t) + v(t) \\
&\quad = \int_0^t \frac{d}{dt}(g_{1-\alpha} * T)(t-s)f(s)ds - A\int_0^t \frac{d}{dt}(g_{1-\beta} * T)(t-s)f(s)ds \\
&\qquad + \lim_{s\to t^-}(g_{1-\alpha} * T)(t-s)f(s) - \lim_{s\to t^-}(g_{1-\beta} * T)(t-s)Af(s) + v(t) \\
&\quad = \int_0^t \left(\frac{d}{dt}(g_{1-\alpha} * T)(t-s)f(s) - A\frac{d}{dt}(g_{1-\beta} * T)(t-s)f(s)\right)ds + v(t) \\
&\quad = -\int_0^t \left(\mathbb{D}_r^\alpha[f(s) - T(r)|_{r=t-s}f(s)] - A\mathbb{D}_r^\beta[f(s) - T(r)|_{r=t-s}f(s)]\right)ds \\
&\qquad + v(t)
\end{aligned}$$

$$= \int_0^t \left(f(s) - T(t-s)f(s) \right) ds + v(t)$$
$$= \int_0^t f(s)ds. \tag{4.16}$$

This indicates that v is a classical solution of $(NAFRE)$ with $x = 0$.

Case (II): Assume (ii) to hold. Let

$$v(t) = \int_0^t T(t-\tau)f(\tau)d\tau, \ t \in [0, T]. \tag{4.17}$$

Since $f \in L([0, T]; X)$ and $T(t)$ is strongly continuous on $[0, \infty)$, by Proposition 1.3.4 in [31], we have $v(t) \in C([0, T]; X)$ with $v(0) = 0$. It is not difficult to obtain that

$$\lim_{t\to 0^+} J_t^{1-\alpha} f(t) = 0 \tag{4.18}$$

Using Fubini theorem and (4.18), we derive that

$$\begin{aligned} \mathbb{D}_t^\alpha v(t) =& \frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} [(T * f)(s) - v(0)]ds \\ =& \frac{d}{dt} J_t^{1-\alpha} (T * f)(t) \\ =& \frac{d}{dt} (T * J_t^{1-\alpha} f)(t) \\ =& T(t) * \frac{d}{dt} J_t^{1-\alpha} f(t) + T(t) \lim_{t\to 0^+} J_t^{1-\alpha} f(t). \end{aligned} \tag{4.19}$$

Combining (2.3), (2.6), (4.18) and (4.19), we obtain

$$\begin{aligned} \mathbb{D}_t^\beta v(t) =& \frac{d}{dt} \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(\beta)} [(T * f)(s) - v(0)]ds \\ =& \frac{d}{dt} J_t^{1-\beta} (T * f)(t) \\ =& T(t) * \frac{d}{dt} J_t^{1-\beta} f(t) + T(t) \lim_{t\to 0^+} J_t^{\alpha-\beta} J_t^{1-\alpha} f(t) \\ =& T(t) * D_t^\beta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) g(s)ds \\ & + T(t) \lim_{t\to 0^+} J_t^{\alpha-\beta} J_t^{1-\alpha} f(t) \\ =& T(t) * t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha) * g(t). \end{aligned}$$

The closedness of A and property (a) of Proposition 3.8 imply that $\mathbb{D}_t^\beta v(t) \in D(A)$ and

$$\begin{aligned} A\mathbb{D}_t^\beta v(t) =& AT(t) * t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-t^\alpha) * g(t) \\ =& \left(T(t) - t^\alpha E_{\alpha,\alpha+1}(-t^\alpha) \right) * g(t) \end{aligned}$$

$$= \int_0^t \left[T(t-s)g(s) - (t-s)^\alpha E_{\alpha,\alpha+1}(-(t-s)^\alpha)g(s) \right] ds$$

$$= \int_0^t T(t-s)g(s)ds - \int_0^t f(s)ds. \tag{4.20}$$

Therefore, by (4.19) and (4.20), we have

$$\mathbb{D}_t^\alpha v(t) - A\mathbb{D}_t^\beta v(t) + v(t)$$
$$= T(t) * \left[f(t) + \frac{d}{dt} J_t^{1-\alpha} f(t) - g(t) \right] + T(t) \lim_{t\to 0^+} J_t^{1-\alpha} f(t) + \int_0^t f(s)ds. \tag{4.21}$$

In order to complete the proof, we only need to show that

$$T(t) * \left[f(t) + \frac{d}{dt} J_t^{1-\alpha} f(t) - g(t) \right] + T(t) \lim_{t\to 0^+} J_t^{1-\alpha} f(t) = 0, \ t \geq 0.$$

Denote $g_b(\cdot)$ by the truncation of $g(\cdot)$ and $T_b(\cdot)$ by the truncation of $T(\cdot)$, that is,

$$g_b(t) = \begin{cases} g(t), & t \leq b, \\ 0, & t > b, \end{cases} \quad \text{and } T_b(t) = \begin{cases} T(t), & t \leq b, \\ 0, & t > b. \end{cases}$$

Obviously,

$$\frac{\hat{g}_b(\lambda)}{\lambda^\alpha + 1} + \frac{\lambda^\alpha \hat{g}_b(\lambda)}{\lambda^\alpha + 1} - \hat{g}_b(\lambda) = 0,$$

which implies that

$$\hat{T}_b(\lambda) \left[\frac{\hat{g}_b(\lambda)}{\lambda^\alpha + 1} + \frac{\lambda^\alpha \hat{g}_b(\lambda)}{\lambda^\alpha + 1} - \lim_{t\to 0^+} J_t^{1-\alpha} f(t) - \hat{g}_b(\lambda) + \lim_{t\to 0^+} J_t^{1-\alpha} f(t) \right] = 0.$$

By virtue of Laplace transform, we derive

$$T(t) * \left[f(t) + \frac{d}{dt} J_t^{1-\alpha} f(t) - g(t) \right] + T(t) \lim_{t\to 0^+} J_t^{1-\alpha} f(t) = 0, \ t \in [0, b].$$

The arbitrariness of b implies that

$$T(t) * \left[f(t) + \frac{d}{dt} J_t^{1-\alpha} f(t) - g(t) \right] + T(t) \lim_{t\to 0^+} J_t^{1-\alpha} f(t) = 0, \ t \geq 0.$$

The proof is therefore completed. □

The mild solution of $(NAFRE)$ can be defined as follows.

Definition 4.8 Let A be the generator a fractional (α, β) resolvent $\{T(t)\}_{t>0}$ on Banach space X. For any $x \in X$, $f \in L^1_{loc}([0, \infty), X)$, the mild solution of $(NAFRE)$ is defined by

$$u(t) = x - T(t)x + \int_0^t T(t-s)f(s)ds, \ t > 0.$$

Remark 4.9 We can see from the proof of Theorem 4.4 that the mild solution of $(NAFRE)$ is unique.

Example 4.10 As an application, we consider the following fractional differential equations

$$\begin{cases} \mathbb{D}_t^\alpha(t,x) = \mu^2 \frac{\partial^2}{\partial x^2} \mathbb{D}_t^\beta u(t) - u(t,x) + \int_0^t f(s,x)ds, t \in (0,T], x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ \lim_{t\to 0^+} J_t^{1-\alpha} u(t,x) = p(x). \end{cases} \tag{4.22}$$

Denote $X = L^2(0,\pi)$, and $A = \mu^2 \frac{\partial^2}{\partial x^2}$ with domain $D(A) = \{g \in W^{2,2}(0,1) : g(0) = g(1) = 0\}$. Then, system (4.22) can be convert to abstract fractional relaxation equation of the form $(NAFRE)$ with p replacing x.

Observe that A is closed, densely defined and has eigenvalues $\{\lambda_n = -\mu^2 n^2 \pi^2\}_{n\in N}$ with eigenfunctions $\{sin(n\pi x)\}_{n\in N}$. Moreover, we have $\rho(A) = \mathbb{C}/\{sin(n\pi x)\}_{n\in N}$. For $g(x) = \sum_{n=1}^\infty g_n sin(n\pi x)$, we define the family $\{T(t)\}_{t\geq 0}$ by

$$(T(t)g)(x) = \sum_{n=1}^\infty \left(\sum_{k=0}^\infty (-1)^k (\mu n\pi)^{2k} t^{(\alpha-\beta)k+\alpha} E^{k+1}_{\alpha,(\alpha-\beta)k+1+\alpha}(-t^\alpha) \right) g_n \sin(n\pi x).$$

We shall show that $\{T(t)\}_{t\geq 0}$ is a fractional (α,β) resolvent by verifying the following three conditions. (1) Obversely, $T(\cdot)g \in C([0,\infty), X)$, and

$$\begin{aligned} &\lim_{t\to 0^+} \frac{T(t)g}{t^\alpha} \\ &= \lim_{t\to 0^+} \sum_{n=1}^\infty \left(\sum_{k=0}^\infty (-1)^k (\mu n\pi)^{2k} t^{(\alpha-\beta)k} E^{k+1}_{\alpha,(\alpha-\beta)k+1+\alpha}(-t^\alpha) \right) g_n \sin(n\pi x) \\ &= \frac{g}{\Gamma(\alpha+1)}. \end{aligned}$$

(2) The commutativity of $T(t)$ and $T(s)$, $t, s \geq 0$ is obtained directly by the definition of $\{T(t)\}_{t\geq 0}$.
c) By (2.7), the Laplace transform of $\{T(t)\}_{t\geq 0}$ is obtained by

$$\begin{aligned} \hat{T}(\lambda)g &= \sum_{n=1}^\infty \left(\sum_{k=0}^\infty (-1)^k (\mu n\pi)^{2k} \frac{\lambda^{\alpha(k+1)-[(\alpha-\beta)k+1+\alpha]}}{(\lambda^\alpha+1)^{k+1}} \right) g_n \sin(n\pi\cdot) \\ &= \sum_{n=1}^\infty \frac{\lambda^{-1}}{\lambda^\alpha+1} \left(\sum_{k=0}^\infty \frac{(-1)^k (\mu n\pi)^{2k} \lambda^{\beta k}}{(\lambda^\alpha+1)^k} \right) g_n \sin(n\pi\cdot) \\ &= \sum_{n=1}^\infty \frac{\lambda^{-1}}{\lambda^\alpha+1} \frac{1}{1 - \frac{-\mu^2 n^2 \pi^2 \lambda^\beta}{\lambda^\alpha+1}} g_n \sin(n\pi\cdot) \\ &= \sum_{n=1}^\infty \frac{\lambda^{-1-\beta}}{\lambda^{-\beta}(\lambda^\alpha+1) + \mu^2 n^2 \pi} g_n \sin(n\pi\cdot). \end{aligned}$$

Assume that $f \in D(A)$ such that

$$\lambda f - Af = g, \ \lambda \in \rho(A).$$

Then, $f = \sum_{n=0}^{\infty} f_n sin(n\pi \cdot)$, and

$$\sum_{n=0}^{\infty} \left(\lambda + \mu^2 n^2 \pi^2\right) f_n sin(n\pi \cdot) = \sum_{n=0}^{\infty} g_n sin(n\pi \cdot), \ \lambda \in \rho(A).$$

This indicates that

$$R(\lambda, A)g = \sum_{n=0}^{\infty} \frac{g_n}{\lambda + \mu^2 n^2 \pi^2} sin(n \cdot), \ \lambda \in \rho(A)$$

and

$$\lambda^{-1-\beta} R(\lambda^{-\beta}(\lambda^{\alpha} + 1), A)g = \sum_{n=0}^{\infty} \frac{\lambda^{-1-\beta} g_n}{\lambda^{-\beta}(\lambda^{\alpha} + 1) + \mu^2 n^2 \pi^2} sin(n\pi \cdot), \tag{4.23}$$

for any $\lambda > 0, \ x \in X$. Hence,

$$\lambda^{-1-\beta} R(\lambda^{-\beta}(\lambda^{\alpha} + 1), A)g = \int_0^{\infty} e^{-\lambda t} T(t) g dt. \tag{4.24}$$

Since $D(A)$ is dense in X, by equality (4.24) and Theorem 3.12, it follows that $\{T(t)\}_{t \geq 0}$ is fractional (α, β) resolvent generated by A. Let K be defined as in Theorem 4.3. Then, for any $p \in D(A^{K+1})$, the homogeneous equation

$$\begin{cases} D_t^{\alpha} u(t, x) = \frac{\partial^2}{\partial x^2} D_t^{\beta} u(t) - u(t, x), t \in (0, T], x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0, \\ \lim_{t \to 0^+} J_t^{1-\alpha} u(t, x) = p(x), \end{cases}$$

has a unique classical solution. Moreover, by Theorem 4.4, for any $p \in X$, $(T(t)p)(x)$ is the unique solution of $(AFRE)$ with p replacing x.

Below we consider the inhomogeneous Equation (4.22). If $p \in D(A^{K+1})$,

$$f = \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) g(s) ds$$

and

$$\int_0^t E_{\alpha}(-(t - s)^{\alpha}) g(s) ds \in W^{1,1}([0, T]; X),$$

then it follows from Theorem 4.7 that

$$u(t, x) = (T(t)p)(x) + \int_0^t T(t - s) f(s, x) ds \tag{4.25}$$

is the unique mild solution of $(NAFRE)$ with p replacing x. Moreover, for any $p \in X$ and $f \in L^1_{loc}([0, \infty), X)$, (4.25) is the unique mild solution of $(NAFRE)$ with p replacing x.

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